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Hom-Lie 2-superalgebras

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Abstract: Hom-Lie 2-superalgebras can be considered as the categorification of Hom-Lie superalgebras. We give the definition of Hom-Lie 2-superalgebras and study their superderivations. We obtain the representation, deformation, and abelian extensions related to the 2-cocycle and Hom-Nijenhuis operators. Moreover, we also construct a skeletal (strict) Hom-Lie 2-superalgebra from a Hom-associative Rota–Baxter superalgebra.

Key words: Hom-Lie 2-superalgebras, superderivations, representations, deformations, abelian extensions, Hom-associative Rota–Baxter superalgebras

1. Introduction

Higher categorical structures play an important role in both string theory [2] and physics [9,15]. Some higher categorical structures are obtained by categorifying existing mathematical concepts. One of the simplest higher structures is a categorical vector space, that is, a 2-vector space. A categorical Lie algebra introduced by Baez and Crans [3], which is called a Lie 2-algebra, is a 2-vector space equipped with a skew-symmetric bilinear functor, whose Jacobi identity is replaced by the Jacobiator satisfying some coherence laws of its own. Baez and Crans [3] showed that the category of Lie 2-algebras is equivalent to the category of 2-term L_∞-algebras, so a Lie 2-algebra is often defined by a 2-term L_∞-algebra. Recently, Lie 2-algebra theories have been widely developed [4,5,10,12,14,16–19]. In particular, Lie 2-superalgebras were studied in [7,25].

Hom-Lie algebras were initially introduced by Hartwig et al. [6] to study the deformations of the Witt and the Virasoro algebras. A Hom-algebra is also connected with deformed vector fields, so many results about Hom-algebra structures have been investigated [1,8,13,20,22–24]. The categorification of Hom-Lie algebras, which is called a Hom-Lie 2-algebra, was given in [21].

In this paper, we generalize Hom-Lie 2-algebras to Hom-Lie 2-superalgebras, which are regarded as the deformation and categorification of Lie superalgebras. It was proved that the category of Hom-Lie 2-algebras and the category of 2-term HL_∞-algebras are equivalent in [21]. An analogous result is obtained in the case of Hom-Lie 2-superalgebras, so we define Hom-Lie 2-superalgebras by 2-term Hom-L_∞-algebras. Motivated by deformations of Lie 2-algebras [11], we give notions of representations and 2-cocycles of Hom-Lie 2-superalgebras, and we prove that a 1-parameter infinitesimal deformation is related to a 2-cocycle with coefficients in adjoint representations. Furthermore, we study Hom-Nijenhuis operators and abelian extensions

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connected with representations and 2-cocycles. In particular, we show that the superderivation of idempotent Hom-Lie 2-superalgebras under a commutator is a strict Lie 2-superalgebra.

The paper is organized as follows. In Section 2, we give notions of Hom-Lie 2 superalgebras and their homomorphisms. In Section 3, we give the definition of superderivations of Hom-Lie 2-superalgebras, and we prove that the superderivation of degree 0 of idempotent Hom-Lie 2-superalgebras is a Lie superalgebra. In Section 4, we show the relation between 1-parameter infinitesimal deformations and 2-cocycles of Hom-Lie 2-superalgebras. In Section 5, the Hom-Nijenhuis operators of Hom-Lie 2-superalgebras are studied. In Section 6, we show that there exists a representation and a 2-cocycle associated to any abelian extensions. Finally, we construct a skeletal (strict) Hom-Lie 2-superalgebra from a Hom-associative Rota–Baxter superalgebra.

The parity of the homogeneous element x in superalgebras (super vector spaces) is denoted by $|x|$. The set of all homogeneous elements of Hom-Lie 2-superalgebras $\mathbb{M}$ is denoted by $hg(\mathbb{M})$.

2. Preliminaries

In this section, we first give the notion of Hom-Lie 2-superalgebras, and then we study some properties of the homomorphism of Hom-Lie 2-superalgebras.

Definition 2.1 *A Hom-Lie 2-superalgebra consists of the following data:*

- *two super vector spaces M_0 and M_1 together with an even linear map $d : M_1 \to M_0$,*
- *an even bilinear map $[\cdot,\cdot] : M_i \times M_j \to M_{i+j}$ $(0 \le i+j \le 1)$,*
- *two even linear maps $\tau_0 : M_0 \to M_0$ and $\tau_1 : M_1 \to M_1$ satisfying $\tau_0 \circ d = d \circ \tau_1$,*
- *an even skew-symmetric trilinear map $l_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $l_3 \circ \tau_0 = \tau_1 \circ l_3$, such that for any $x, y, z, t \in hg(M_0)$, $a, b \in hg(M_1)$, the following equalities are satisfied:*

(1) $[x, y] = -(-1)^{|x||y|}[y, x]$,

(2) $[x, a] = -(-1)^{|x||a|}[a, x]$,

(3) $[a, b] = 0$,

(4) $d([x, a]) = [x, da]$,

(5) $[da, b] = [a, db]$,

(6) $\tau_0([x, y]) = [\tau_0(x), \tau_0(y)]$,

(7) $\tau_1([x, a]) = [\tau_0(x), \tau_1(a)]$,

(8) $dl_3(x, y, z) = [\tau_0(x), [y, z]] + (-1)^{|x|(|y|+|z|)}[\tau_0(y), [z, x]] + (-1)^{(|x|+|y|)|z|}[\tau_0(z), [x, y]]$,

(9) $l_3(x, y, da) = [\tau_0(x), [y, a]] + (-1)^{|x|(|y|+|a|)}[\tau_0(y), [a, x]] + (-1)^{(|x|+|y|)|a|}[\tau_1(a), [x, y]]$,

(10)
$$\begin{aligned} &l_3([t, x], \tau_0(y), \tau_0(z)) + (-1)^{|z|(|x|+|y|)}l_3([t, z], \tau_0(x), \tau_0(y)) + (-1)^{|t|(|x|+|y|)}l_3([x, y], \tau_0(t), \tau_0(z)) \\ &+ (-1)^{(|x|+|t|)(|y|+|z|)}l_3([y, z], \tau_0(t), \tau_0(x)) + (-1)^{|t|(|x|+|y|+|z|)}[l_3(x, y, z), \tau_0^2(t)] \\ &= [l_3(t, x, y), \tau_0^2(z)] + (-1)^{|x||y|}l_3([t, y], \tau_0(x), \tau_0(z)) + (-1)^{|y||z|+|t|(|x|+|z|)}l_3([x, z], \tau_0(t), \tau_0(y)) \\ &+ (-1)^{|x|(|y|+|z|)}[l_3(t, y, z), \tau_0^2(x)] - (-1)^{|y||z|}[l_3(t, x, z), \tau_0^2(y)]. \end{aligned}$$

A Hom-Lie 2-superalgebra is denoted by $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot], l_3, \tau_0, \tau_1)$, simply denoted by $\mathbb{M}$.

A Hom-Lie 2-superalgebra is called skeletal if $d = 0$ or strict if $l_3 = 0$. A Hom-Lie 2-superalgebra is called idempotent if $\tau_0^2 = \tau_0$, $\tau_1^2 = \tau_1$.

Example 2.2 *Let $(M, [\cdot,\cdot]_M, \beta, B)$ be a multiplicative quadratic Hom-Lie superalgebra. It gives a Hom-Lie*

2-superalgebra on the super vector space $M\oplus\mathbb{R}$*, denoted by* $(M\oplus\mathbb{R}:\mathbb{R}\xrightarrow{d=0}M,[\cdot,\cdot],l_3,\beta,I_{\mathbb{R}})$*, where* M *is of degree* 0*,* $\mathbb{R}$ *is of degree* -1*, an even linear map* d *is defined by* $0=d:\mathbb{R}\to M$*, an even bilinear map* $[\cdot,\cdot]:(M\oplus\mathbb{R})\times(M\oplus\mathbb{R})\to M\oplus\mathbb{R}$ *is defined by* $[x+a,y+b]=[x,y]_M$*, and an even trilinear map* $l_3:M\times M\times M\to\mathbb{R}$ *is defined by* $l_3(x,y,z)=B([x,y]_M,z)$*.*

Definition 2.3 *Let* $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$ *and* $(\mathbb{M}':M_1'\xrightarrow{d'}M_0',[\cdot,\cdot]_{\mathbb{M}'},l_3',\tau_0',\tau_1')$ *be two Hom-Lie 2-superalgebras. A Hom-Lie 2-superalgebra homomorphism* $g:\mathbb{M}\to\mathbb{M}'$ *consists of*

- *an even linear map* $g_0:M_0\to M_0'$ *satisfying* $g_0\circ\tau_0=\tau_0'\circ g_0$*,*
- *an even linear map* $g_1:M_1\to M_1'$ *satisfying* $g_1\circ\tau_1=\tau_1'\circ g_1$*,*
- *an even skew supersymmetry bilinear map* $g_2:M_0\times M_0\to M_1'$ *satisfying* $g_2(\tau_0(x),\tau_0(y))=\tau_1'(g_2(x,y))$

such that the following equalities hold for any $x,y,z\in hg(M_0),a\in hg(M_1)$*:*

(1) $g_0\circ d=d'\circ g_1$,

(2) $g_0([x,y]_{\mathbb{M}})-[g_0(x),g_0(y)]_{\mathbb{M}'}=d'(g_2(x,y))$,

(3) $g_1([x,a]_{\mathbb{M}})-[g_0(x),g_1(a)]_{\mathbb{M}'}=g_2(x,da)$,

(4) $g_2([x,y]_{\mathbb{M}},\tau_0(z))+(-1)^{|x|(|y|+|z|)}g_2([y,z]_{\mathbb{M}},\tau_0(x))+(-1)^{(|x|+|y|)|z|}g_2([z,x]_{\mathbb{M}},\tau_0(y))$
$+g_1(l_3(x,y,z))-[g_0(\tau_0(x)),g_2(y,z)]_{\mathbb{M}'}-(-1)^{|x|(|y|+|z|)}[g_0(\tau_0(y)),g_2(z,x)]_{\mathbb{M}'}$
$=(-1)^{(|x|+|y|)|z|}[g_0(\tau_0(z)),g_2(x,y)]_{\mathbb{M}'}+l_3'(g_0(x),g_0(y),g_0(z))$.

The homomorphism of Hom-Lie 2-superalgebras is denoted by $g=(g_0,g_1,g_2)$*.*

The homomorphism g *is called strict if* $g_2=0$*. The identity homomorphism* $I_{\mathbb{M}}:\mathbb{M}\to\mathbb{M}$ *is defined by* $I_0:M_0\to M_0$*,* $I_1:M_1\to M_1$*, and* $I_2=0$*, denoted by* $I_{\mathbb{M}}=(I_0,I_1,0)$*.*

Let $g:\mathbb{M}\to\mathbb{M}'$ and $g':\mathbb{M}'\to\mathbb{M}''$ be two homomorphisms of Hom-Lie 2-superalgebras. Their composition $g'g=((g'g)_0,(g'g)_1,(g'g)_2):\mathbb{M}\to\mathbb{M}''$ is defined by $(g'g)_0=g_0'\circ g_0:M_0\to M_0''$, $(g'g)_1=g_1'\circ g_1:M_1\to M_1''$, and $(g'g)_2=g_2'\circ(g_0\times g_0)+g_1'\circ g_2:M_0\times M_0\to M_1''$. It is clear that $g'g=((g'g)_0,(g'g)_1,(g'g)_2)$ is a homomorphism of Hom-Lie 2-superalgebras.

Definition 2.4 *A homomorphism of Hom-Lie 2-superalgebras* $g:\mathbb{M}\to\mathbb{M}'$ *is called an isomorphism if there exists a homomorphism of Hom-Lie 2-superalgebras* $h:\mathbb{M}'\to\mathbb{M}$ *such that* $hg:\mathbb{M}\to\mathbb{M}$ *and* $gh:\mathbb{M}'\to\mathbb{M}'$ *are both identity homomorphisms.*

Proposition 2.5 *Let* $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$ *and* $(\mathbb{M}':M_1'\xrightarrow{d'}M_0',[\cdot,\cdot]_{\mathbb{M}'},l_3',\tau_0',\tau_1')$ *be two Hom-Lie 2-superalgebras. Let* $g=(g_0,g_1,g_2):\mathbb{M}\to\mathbb{M}'$ *be a homomorphism of Hom-Lie 2-superalgebras. If* g_0,g_1 *are invertible, then there exists a map* $g^{-1}=(g_0^{-1},g_1^{-1},-g_1^{-1}g_2(g_0^{-1}\times g_0^{-1}))$ *such that* g *is an isomorphism of Hom-Lie 2-superalgebras.*

Proof For any $x',y',z'\in hg(M_0)$, we have

$$[g_0^{-1}(\tau_0'(x')),-g_1^{-1}(g_2(g_0^{-1}(y'),g_0^{-1}(z')))]_{\mathbb{M}}+(-1)^{|x|(|y|+|z|)}[g_0^{-1}(\tau_0'(y')),-g_1^{-1}(g_2(g_0^{-1}(z'),g_0^{-1}(x')))]_{\mathbb{M}}$$

$$+(-1)^{(|x|+|y|)|z|}[g_0^{-1}(\tau_0'(z')),-g_1^{-1}(g_2(g_0^{-1}(x'),g_0^{-1}(y')))]_{\mathbb{M}}+l_3(g_0^{-1}(x'),g_0^{-1}(y'),g_0^{-1}(z'))$$

$$=-(-1)^{|x|(|y|+|z|)}g_1^{-1}g_2(g_0^{-1}[y',z']_{\mathbb{M}'},\tau_0'(g_0^{-1}(x')))-(-1)^{|z|(|x|+|y|)}g_1^{-1}g_2(g_0^{-1}[z',x']_{\mathbb{M}'},\tau_0'(g_0^{-1}(y')))$$

$-g_1^{-1}g_2(g_0^{-1}[x',y']_{\mathbb{M}'},\tau_0'(g_0^{-1}(z'))) + g_1^{-1}l_3'(x',y',z').$ □

Proposition 2.6 *Let* $(\mathbb{M}: M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *be a Hom-Lie 2-superalgebra. For a graded super vector space* $\mathbb{M}' = M_0' \oplus M_1'$ *with two invertible even linear maps* $g_0 : M_0' \to M_0$, $g_1 : M_1' \to M_1$, *and an even skew supersymmetry bilinear map* $g_2 : M_0' \times M_0' \to M_1$, *define*

(1) $d' \triangleq g_0^{-1} \circ d \circ g_1$,

(2) $[x,y]_{\mathbb{M}'} \triangleq g_0^{-1}([g_0(x), g_0(y)]_{\mathbb{M}} + d(g_2(x,y)))$,

(3) $[x,a]_{\mathbb{M}'} \triangleq g_1^{-1}([g_0(x), g_1(a)]_{\mathbb{M}} + g_2(x, d'a))$,

(4) $[a,b]_{\mathbb{M}'} \triangleq 0$,

(5) $\tau_0' \triangleq g_0^{-1} \circ \tau_0 \circ g_0 : M_0' \to M_0'$, $\tau_1' \triangleq g_1^{-1} \circ \tau_1 \circ g_1 : M_1' \to M_1'$ *satisfying*

$$g_2(\tau_0'(x), \tau_0'(y)) = \tau_1(g_2(x,y)),$$

(6) $$\begin{aligned} l_3'(x,y,z) \triangleq\ & g_1^{-1}([g_0(\tau_0'(x)), g_2(y,z)]_{\mathbb{M}} - g_2([x,y]_{\mathbb{M}'}, \tau_0'(z)) - (-1)^{|x|(|y|+|z|)} g_2([y,z]_{\mathbb{M}'}, \tau_0'(x)) \\ & - (-1)^{|z|(|x|+|y|)} g_2([z,x]_{\mathbb{M}'}, \tau_0'(y)) + l_3(g_0(x), g_0(y), g_0(z)) \\ & + (-1)^{|x|(|y|+|z|)} [g_0(\tau_0'(y)), g_2(z,x)]_{\mathbb{M}} + (-1)^{|z|(|x|+|y|)} [g_0(\tau_0'(z)), g_2(x,y)]_{\mathbb{M}}). \end{aligned}$$

Then $(\mathbb{M}' : M_1' \xrightarrow{d'} M_0', [\cdot,\cdot]_{\mathbb{M}'}, l_3', \tau_0', \tau_1')$ *is a Hom-Lie 2-superalgebra. Furthermore,* $g = (g_0, g_1, g_2) : \mathbb{M}' \to \mathbb{M}$ *is an isomorphism of Hom-Lie 2-superalgebras.*

Proof For any $x, y, z, t \in hg(M_0)$, since

$$\begin{aligned} & l_3([g_0(t), g_0(x)]_{\mathbb{M}}, \tau_0(g_0(y)), \tau_0(g_0(z))) + (-1)^{|z|(|x|+|y|)} l_3([g_0(t), g_0(z)]_{\mathbb{M}}, \tau_0(g_0(x)), \tau_0(g_0(y))) \\ & + (-1)^{|t|(|x|+|y|)} l_3([g_0(x), g_0(y)]_{\mathbb{M}}, \tau_0(g_0(t)), \tau_0(g_0(z))) \\ & + (-1)^{(|x|+|t|)(|y|+|z|)} l_3([g_0(y), g_0(z)]_{\mathbb{M}}, \tau_0(g_0(t)), \tau_0(g_0(x))) \\ & + (-1)^{|t|(|x|+|y|+|z|)} [l_3(g_0(x), g_0(y), g_0(z)), \tau_0^2(g_0(t))]_{\mathbb{M}} + (-1)^{|y||z|} [l_3(g_0(t), g_0(x), g_0(z)), \tau_0^2(g_0(y))]_{\mathbb{M}} \\ & = [l_3(g_0(t), g_0(x), g_0(y)), \tau_0^2(g_0(z))]_{\mathbb{M}} + (-1)^{|x||y|} l_3([g_0(t), g_0(y)]_{\mathbb{M}}, \tau_0(g_0(x)), \tau_0(g_0(z))) \\ & + (-1)^{|y||z|+|t|(|x|+|z|)} l_3([g_0(x), g_0(z)]_{\mathbb{M}}, \tau_0(g_0(t)), \tau_0(g_0(y))) \\ & + (-1)^{|x|(|y|+|z|)} [l_3(g_0(t), g_0(y), g_0(z)), \tau_0^2(g_0(x))]_{\mathbb{M}}, \end{aligned}$$

we have

$$\begin{aligned} & l_3'([t,x]_{\mathbb{M}'}, \tau_0'(y), \tau_0'(z)) + (-1)^{|z|(|x|+|y|)} l_3'([t,z]_{\mathbb{M}'}, \tau_0'(x), \tau_0'(y)) \\ & + (-1)^{|t|(|x|+|y|)} l_3'([x,y]_{\mathbb{M}'}, \tau_0'(t), \tau_0'(z)) + (-1)^{|y||z|} [l_3'(t,x,z), \tau_0'^2(y)]_{\mathbb{M}'} \\ & + (-1)^{(|x|+|t|)(|y|+|z|)} l_3'([y,z]_{\mathbb{M}'}, \tau_0'(t), \tau_0'(x)) + (-1)^{|t|(|x|+|y|+|z|)} [l_3'(x,y,z), \tau_0'^2(t)]_{\mathbb{M}'} \\ & = [l_3'(t,x,y), \tau_0'^2(z)]_{\mathbb{M}'} + (-1)^{|x||y|} l_3'([t,y]_{\mathbb{M}'}, \tau_0'(x), \tau_0'(z)) \\ & + (-1)^{|y||z|+|t|(|x|+|z|)} l_3'([x,z]_{\mathbb{M}'}, \tau_0'(t), \tau_0'(y)) + (-1)^{|x|(|y|+|z|)} [l_3'(t,y,z), \tau_0'^2(x)]_{\mathbb{M}'}. \end{aligned}$$

□

Let $\mathbb{V}: V_1 \xrightarrow{d} V_0$ be a 2-term complex of super vector spaces with an even linear map d. In the following, we can construct a new 2-term complex of super vector spaces $\mathrm{End}(\mathbb{V}) : \mathrm{End}^1(\mathbb{V}) \xrightarrow{\delta} \mathrm{End}^0_d(\mathbb{V})$. Define an even linear map δ by

$$\delta(F) = d \circ F + F \circ d$$

for any $F \in \mathrm{End}^1(\mathbb{V})$, where

$$\mathrm{End}^1(\mathbb{V}) = \mathrm{Hom}(V_0, V_1),$$

$$\mathrm{End}^0_d(\mathbb{V}) = \{G = (G_0, G_1) \in \mathrm{End}(V_0, V_0) \oplus \mathrm{End}(V_1, V_1) | G_0 \circ d = d \circ G_1\},$$

$|G| = |G_0| = |G_1|$. Define an even bilinear map $l_2 : \mathrm{End}(\mathbb{V}) \times \mathrm{End}(\mathbb{V}) \to \mathrm{End}(\mathbb{V})$ by setting:

$$\begin{cases} l_2(G, G') &= [G, G']_C, \\ l_2(G, F) &= [G, F]_C, \\ l_2(F, F') &= 0, \end{cases}$$

for any $G, G' \in \mathrm{hg}(\mathrm{End}^0_d(\mathbb{V})), F, F' \in \mathrm{hg}(\mathrm{End}^1(\mathbb{V}))$, where $[\cdot,\cdot]_C$ is the graded commutator. It is easy to show that:

Theorem 2.7 $(\mathrm{End}(\mathbb{V}), \delta, l_2)$ *is a strict Lie 2-superalgebra.*

Proof It is a straightforward calculation. □

3. Derivations of Hom-Lie 2-superalgebras

In this section, we will give the notion of superderivations and obtain some properties of superderivations. A new 2-term complex of super vector spaces will be formed by the superderivation of Hom-Lie 2 superalgebras.

Definition 3.1 *Let* $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *be a Hom-Lie 2-superalgebra. A homogeneous superderivation of degree* 0 *of* $\mathbb{M}$ *consists of*

- *a homogeneous element* $D = (D_0, D_1) \in hg(\mathrm{End}^0_d(\mathbb{M}))$ *satisfying*

$$D_0 \circ \tau_0 = \tau_0 \circ D_0, \qquad D_1 \circ \tau_1 = \tau_1 \circ D_1,$$

- *a skew-supersymmetric bilinear map* $l_D : M_0 \times M_0 \to M_1$ *satisfying*

$$l_D(\tau_0(x), \tau_0(y)) = \tau_1(l_D(x, y))$$

such that the following equations hold for any $x, y, , z \in hg(M_0)$, $a \in hg(M_1)$:

(1) $D[x, y]_{\mathbb{M}} - [Dx, \tau_0(y)]_{\mathbb{M}} - (-1)^{|D||x|}[\tau_0(x), Dy]_{\mathbb{M}} = dl_D(x, y)$,

(2) $D[x, a]_{\mathbb{M}} - [Dx, \tau_1(a)]_{\mathbb{M}} - (-1)^{|D||x|}[\tau_0(x), Da]_{\mathbb{M}} = l_D(x, da)$,

(3) $l_D(\tau_0(x), [y, z]_{\mathbb{M}}) + (-1)^{|D||x|}[\tau_0^2(x), l_D(y, z)]_{\mathbb{M}} + l_3(Dx, \tau_0(y), \tau_0(z))$
$+ (-1)^{|D||x|} l_3(\tau_0(x), Dy, \tau_0(z)) + (-1)^{|D|(|x|+|y|)} l_3(\tau_0(x), \tau_0(y), Dz)$
$= Dl_3(x, y, z) + l_D([x, y]_{\mathbb{M}}, \tau_0(z)) + (-1)^{|x||y|} l_D(\tau_0(y), [x, z]_{\mathbb{M}}) + [l_D(x, y), \tau_0^2(z)]_{\mathbb{M}}$
$+ (-1)^{|y|(|D|+|x|)}[\tau_0^2(y), l_D(x, z)]_{\mathbb{M}}$,

where $|D| = |l_D|$.

A homogeneous superderivation of degree 0 of $\mathbb{M}$ is denoted by (D, l_D) and the set of all homogeneous superderivations of degree 0 of $\mathbb{M}$ by $\mathrm{Der}^0(\mathbb{M})$.

Proposition 3.2 *Let* $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *be a Hom-Lie 2-superalgebra. For any* $x \in hg(M_0)$ *satisfying* $\tau_0(x) = x$, *define a homogeneous linear map* ad_x *by* $ad_x(y+a) = [x, y+a]$ *for any* $y \in hg(M_0), a \in hg(M_1)$, *and then* $(ad_x, l_{ad_x} = l_3(x,\cdot,\cdot)) \in \mathrm{Der}^0(\mathbb{L})$, *where* $|ad_x| = |l_{ad_x}| = |x|$, *which is called an inner derivation.*

Proof It is a straightforward calculation by Definition 2.1. □

Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be an idempotent Hom-Lie 2-superalgebra. For any $(D, l_D), (D', l_{D'}) \in hg(\mathrm{Der}^0(\mathbb{M}))$, $x, y \in hg(M_0)$, we obtain

$$\begin{aligned}
&[D, D']_C([x,y]_{\mathbb{M}}) - [[D, D']_C(x), \tau_0(y)]_{\mathbb{M}} - (-1)^{|x|(|D|+|D'|)}[\tau_0(x), [D, D']_C(y)]_{\mathbb{M}} \\
&= d(l_D(D'x, \tau_0(y)) + (-1)^{|D'||x|} l_D(\tau_0(x), Dy) + Dl_{D'}(x,y) \\
&- (-1)^{|D||D'|} l_{D'}(Dx, \tau_0(y)) - (-1)^{|D||D'|+|D||x|} l_{D'}(\tau_0(x), Dy) - (-1)^{|D||D'|} D'(l_D(x,y)).
\end{aligned}$$

Define

$$\begin{aligned}
l_{[D,D']_C}(x,y) \triangleq\ & l_D(D'x, \tau_0(y)) + (-1)^{|D'||x|} l_D(\tau_0(x), D'y) + Dl_{D'}(x,y) - (-1)^{|D||D'|} l_{D'}(Dx, \tau_0(y)) \\
& - (-1)^{|D||D'|+|D||x|} l_{D'}(\tau_0(x), Dy) - (-1)^{|D||D'|} D' l_D(x,y).
\end{aligned}$$

For any $a \in hg(M_1)$, we have

$$[D, D']_C([x,a]_{\mathbb{M}}) - [[D, D']_C(x), \tau_1(a)]_{\mathbb{M}} - (-1)^{|x|(|D|+|D'|)}[\tau_0(x), [D, D']_C(a)]_{\mathbb{M}} = l_{[D,D']_C}(x, da).$$

Since $\mathbb{M}$ is idempotent and $l_D, l_{D'}$ satisfy equation (3) in Definition 3.1, we obtain that $l_{[D,D']}$ satisfies equation (3) in Definition 3.1. Define an even skew-supersymmetric bilinear map on $\mathrm{Der}^0(\mathbb{M})$ by

$$\begin{aligned}
&[\cdot,\cdot]_{\mathrm{Der}} : \mathrm{Der}^0(\mathbb{M}) \times \mathrm{Der}^0(\mathbb{M}) \to \mathrm{Der}^0(\mathbb{M}) \\
&[(D, l_D), (D', l_{D'})]_{\mathrm{Der}} \triangleq ([D, D']_C, l_{[D,D']_C}).
\end{aligned} \tag{1}$$

We obtain the following theorem:

Theorem 3.3 *Let* $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *be an idempotent Hom-Lie 2-superalgebra. Then* $(\mathrm{Der}^0(\mathbb{M}), [\cdot,\cdot]_{\mathrm{Der}})$ *is a Lie superalgebra.*

Proof We only need to verify

$$\circlearrowleft_{D_1, D_2, D_3} (-1)^{|D_1||D_3|} l_{[[D_1,D_2]_C, D_3]_C} = 0.$$

For any $(D_1, l_{D_1}), (D_2, l_{D_2}), (D_3, l_{D_3}) \in \mathrm{Der}^0(\mathbb{M})$, $x, y \in hg(M_0)$, we have

$$\begin{aligned}
&\circlearrowleft_{D_1,D_2,D_3} (-1)^{|D_1||D_3|} l_{[[D_1,D_2]_C,D_3]_C}(x,y)\\
&= (-1)^{|D_1||D_3|} l_{D_1}(D_2D_3x, \tau_0^2(y)) + (-1)^{|D_1||D_3|+|D_2|(|D_3|+|x|)} l_{D_1}(\tau_0(D_3x), D_2\tau_0(y))\\
&+ (-1)^{|D_1||D_3|} D_1 l_{D_2}(D_3x, \tau_0(y)) - (-1)^{|D_1||D_3|+|D_1||D_2|} l_{D_2}(D_1D_3x, \tau_0^2(y))\\
&- (-1)^{|D_1||D_3|+|D_1||D_2|+|D_1|(|D_3|+|x|)} l_{D_2}(\tau_0(D_3x), D_1\tau_0(y)) - (-1)^{|D_1||D_3|+|D_1||D_2|} D_2 l_{D_1}(D_3x, \tau_0(y))\\
&+ (-1)^{|D_1||D_3|+|D_3||x|} l_{D_1}(D_2\tau_0(x), \tau_0(D_3y)) + (-1)^{|D_1||D_3|+|x|(|D_2|+|D_3|)} l_{D_1}(\tau_0^2(x), D_2D_3y)\\
&+ (-1)^{|D_1||D_3|+|D_3||x|} D_1 l_{D_2}(\tau_0(x), D_3y) - (-1)^{|D_1||D_3|+|D_3||x|+|D_2||D_1|} l_{D_2}(D_1\tau_0(x), \tau_0(D_3y))\\
&- (-1)^{|D_1||D_3|+|D_3||x|+|D_2||D_1|+|D_1||x|} l_{D_2}(\tau_0^2(x), D_3D_1y) - (-1)^{|D_1||D_3|+|D_3||x|+|D_2||D_1|} D_3 l_{D_1}(\tau_0(x), D_3y)\\
&+ (-1)^{|D_1||D_3|} D_1D_2 l_{D_3}(x,y) - (-1)^{|D_1||D_3|+|D_1||D_2|} D_2D_1 l_{D_3}(x,y)\\
&- (-1)^{|D_1||D_3|+(|D_1|+|D_2|)|D_3|} l_{D_3}(D_1D_2x, \tau_0(y)) + (-1)^{|D_1||D_3|+(|D_1|+|D_2|)|D_3|+|D_1||D_2|} l_{D_3}(D_2D_1x, \tau_0(y))\\
&- (-1)^{|D_1||D_3|+(|D_1|+|D_2|)|D_3|+(|D_1|+|D_2|)|x|} l_{D_3}(\tau_0(x), D_1D_2y)\\
&+ (-1)^{|D_1||D_3|+(|D_1|+|D_2|)|D_3|+(|D_1|+|D_2|)|x|+|D_1||D_2|} l_{D_3}(\tau_0(x), D_2D_1y)\\
&- (-1)^{|D_2||D_3|} D_3 l_{D_1}(D_2x, \tau_0(y)) - (-1)^{|D_2||D_3|+|D_2||x|} D_3 l_{D_1}(\tau_0(x), D_2y)\\
&- (-1)^{|D_2||D_3|} D_3D_1 l_{D_2}(x,y) + (-1)^{|D_2||D_3|+|D_2||D_1|} D_3 l_{D_2}(D_1x, \tau_0(y))\\
&+ (-1)^{|D_2||D_3|+|D_1||x|+|D_2||D_1|} D_3 l_{D_2}(\tau_0(x), D_1y) + (-1)^{|D_2||D_3|+|D_2||D_1|} D_3D_2 l_{D_1}(x,y)\\
&= 0,
\end{aligned}$$

where $\circlearrowleft_{D_1,D_2,D_3}$ denotes summation over the cyclic permutation on D_1, D_2, D_3. □

Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. We consider the complex $\mathrm{End}^1(\mathbb{M}) \xrightarrow{\overline{\delta}} \mathrm{End}^0_d(\mathbb{M}) \oplus \mathrm{Hom}(M_0 \times M_0, M_1)$, where $\overline{\delta}$ is given by

$$\overline{\delta}(G) = (\delta(G), l_{\delta(G)}), \tag{2}$$

in which $l_{\delta(G)} : M_0 \times M_0 \to M_1$ is given by

$$l_{\delta(G)}(x,y) = G([x,y]_{\mathbb{M}}) - (-1)^{|G||x|}[\tau_0(x), G(y)]_{\mathbb{M}} - [G(x), \tau_0(y)]_{\mathbb{M}}. \tag{3}$$

Lemma 3.4 *Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. Then $\overline{\delta}(G) \in \mathrm{Der}^0(\mathbb{M})$.*

Proof For any $x, y, z \in hg(M_0)$, $a \in hg(M_1)$, we have

$$\begin{aligned}
&\delta(G)[x,y]_{\mathbb{M}} - [\delta(G)(x), \tau_0(y)]_{\mathbb{M}} - (-1)^{|G||x|}[\tau_0(x), \delta(G)(y)]_{\mathbb{M}}\\
&= d(l_{\delta(G)}(x,y)) + (-1)^{|G||x|} d([\tau_0(x), G(y)]_{\mathbb{M}}) + d([G(x), \tau_0(y)]_{\mathbb{M}})\\
&- [d(G(x)), \tau_0(y)]_{\mathbb{M}} - (-1)^{|G||x|}[\tau_0(x), d(G(y))]_{\mathbb{M}}\\
&= dl_{\delta(G)}(x,y).
\end{aligned}$$

Similarly, we have

$$\delta(G)[x,a]_{\mathbb{M}} - [\delta(G)(x),\tau_1(a)]_{\mathbb{M}} - (-1)^{|G||x|}[\tau_0(x),\delta(G)(a)]_{\mathbb{M}} = l_{\delta(G)}(x,da).$$

Finally, we obtain

$$\begin{aligned}
&l_{\delta(G)}(\tau_0(x),[y,z]_{\mathbb{M}}) + (-1)^{|G||x|}[\tau_0^2(x),l_{\delta(G)}(y,z)]_{\mathbb{M}} + l_3(\delta(G)(x),\tau_0(y),\tau_0(z))\\
&+ (-1)^{|G||x|}l_3(\tau_0(x),\delta(G)(y),\tau_0(z)) + (-1)^{|G|(|x|+|y|)}l_3(\tau_0(x),\tau_0(y),\delta(G)(z))\\
&- \delta(G)(l_3(x,y,z)) - l_{\delta(G)}([x,y]_{\mathbb{M}},\tau_0(z)) - (-1)^{|x||y|}l_{\delta(G)}(\tau_0(y),[x,z]_{\mathbb{M}})\\
&- [l_{\delta(G)}(x,y),\tau_0^2(z)]_{\mathbb{M}} - (-1)^{|y|(|G|+|x|)}[\tau_0^2(y),l_{\delta(G)}(x,z)]_{\mathbb{M}}\\
&= G[\tau_0(x),[y,z]_{\mathbb{M}}]_{\mathbb{M}} - (-1)^{|G||x|}[\tau_0^2(x),G[y,z]_{\mathbb{M}}]_{\mathbb{M}} - [G(\tau_0(x)),\tau_0([y,z]_{\mathbb{M}})]_{\mathbb{M}}\\
&+ (-1)^{|G||x|}[\tau_0^2(x),G[y,z]_{\mathbb{M}}]_{\mathbb{M}} - (-1)^{|G|(|x|+|y|)}[\tau_0^2(x),[\tau_0(y),G(z)]_{\mathbb{M}}]_{\mathbb{M}}\\
&- (-1)^{|G||x|}[\tau_0^2(x),[G(y),\tau_0(z)]_{\mathbb{M}}]_{\mathbb{M}} + l_3(\delta(G)(x),\tau_0(y),\tau_0(z))\\
&+ (-1)^{|G||x|}l_3(\tau_0(x),\delta(G)(y),\tau_0(z)) + (-1)^{|G|(|x|+|y|)}l_3(\tau_0(x),\tau_0(y),\delta(G)(z))\\
&- \delta(G)l_3(x,y,z) - G([[x,y]_{\mathbb{M}},\tau_0(z)]_{\mathbb{M}}) + (-1)^{|G|(|x|+|y|)}[\tau_0([x,y]_{\mathbb{M}}),G(\tau_0(z))]_{\mathbb{M}}\\
&+ [G([x,y]_{\mathbb{M}}),\tau_0^2(z)]_{\mathbb{M}} - (-1)^{|x||y|}G([\tau_0(y),[x,z]_{\mathbb{M}}]_{\mathbb{M}}) + (-1)^{|y|(|x|+|G|)}[\tau_0^2(y),G([x,z]_{\mathbb{M}})]_{\mathbb{M}}\\
&+ (-1)^{|x||y|}[G(\tau_0(y)),\tau_0([x,z]_{\mathbb{M}})]_{\mathbb{M}} - [G([x,y]_{\mathbb{M}}),\tau_0^2(z)]_{\mathbb{M}} + (-1)^{|G||x|}[[\tau_0(x),G(y)]_{\mathbb{M}},\tau_0^2(z)]_{\mathbb{M}}\\
&+ [[G(x),\tau_0(y)]_{\mathbb{M}},\tau_0^2(z)]_{\mathbb{M}} - (-1)^{|y|(|x|+|G|)}[\tau_0^2(y),G([x,z]_{\mathbb{M}})]_{\mathbb{M}}\\
&+ (-1)^{|y|(|x|+|G|)+|G||x|}[\tau_0^2(y),[\tau_0(x),G(z)]_{\mathbb{M}}]_{\mathbb{M}} + (-1)^{|y|(|x|+|G|)}[\tau_0^2(y),[G(x),\tau_0(z)]_{\mathbb{M}}]_{\mathbb{M}}\\
&= 0.
\end{aligned}$$

□

From Lemma 3.4, there exists a complex

$$\mathrm{Der}(\mathbb{M}) : \mathrm{Der}^1(\mathbb{M}) \triangleq \mathrm{End}^1(\mathbb{M}) \xrightarrow{\overline{\delta}} \mathrm{Der}^0(\mathbb{M}), \tag{4}$$

where $\mathrm{End}^1(\mathbb{M}) = \{G \in \mathrm{Hom}(M_0,M_1) | G\circ\tau_0 = \tau_1\circ G\}$.

Define an even skew-supersymmetric bilinear map $[\cdot,\cdot]_{\mathrm{Der}} : \mathrm{Der}^0(\mathbb{M}) \times \mathrm{Der}^1(\mathbb{M}) \to \mathrm{Der}^1(\mathbb{M})$ by

$$[(D,l_D),G]_{\mathrm{Der}} \triangleq [D,G]_C. \tag{5}$$

Theorem 3.5 *Let* $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *be an idempotent Hom-Lie 2-superalgebra. Then* $(\mathrm{Der}(\mathbb{M}) : \mathrm{Der}^1(\mathbb{M}) \xrightarrow{\overline{\delta}} \mathrm{Der}^0(\mathbb{M}), [\cdot,\cdot]_{\mathrm{Der}})$ *is a strict Lie 2-superalgebra, where the complex* $\mathrm{Der}(\mathbb{M})$ *is given by* (4)*, the differential* $\overline{\delta}$ *is given by* (2)*, and the bracket is given by* (1) *and* (5)*.*

Proof We only need to show that $l_{\delta[D,G]_C} = l_{[D,\delta(G)]_C}$. For any $x, y \in hg(M_0)$, we have

$$\begin{aligned} l_{\delta[D,G]_C}(x,y) &= Dl_{\delta(G)}(x,y) + (-1)^{|G||x|} l_D(\tau_0(x), d(G(y))) \\ &+ (-1)^{|G||x|+|D||x|}[\tau_0^2(x), DG(y)]_{\mathbb{M}} + (-1)^{|G||x|}[D\tau_0(x), \tau_1 G(y)]_{\mathbb{M}} \\ &+ l_D(d(G(x)), \tau_0(y)) + [DG(x), \tau_0^2(y)]_{\mathbb{M}} \\ &+ (-1)^{|D|(|G|+|x|)}[\tau_1(G(x)), D\tau_0(y)]_{\mathbb{M}} - (-1)^{|D||G|} G(d(l_D(x,y))) \\ &- (-1)^{|D||G|+|D||x|} G[\tau_0(x), Dy]_{\mathbb{M}} - (-1)^{|D||G|} G[Dx, \tau_0(y)]_{\mathbb{M}} \\ &- (-1)^{|x|(|D|+|G|)}[\tau_0(x), DG(y)]_{\mathbb{M}} + (-1)^{|x|(|D|+|G|)+|D||G|}[\tau_0(x), G(Dy)]_{\mathbb{M}} \\ &- [DG(x), \tau_0(y)]_{\mathbb{M}} + (-1)^{|D||G|}[G(Dx), \tau_0(y)]_{\mathbb{M}}. \end{aligned}$$

Similarly,

$$\begin{aligned} l_{[D,\delta(G)]_C}(x,y) &= l_D(d(G(x)), \tau_0(y)) + (-1)^{|G||x|} l_D(\tau_0(x), d(G(y))) \\ &+ Dl_{\delta(G)}(x,y) - (-1)^{|D||G|} G[Dx, \tau_0(y)]_{\mathbb{M}} \\ &+ (-1)^{|G||x|}[D\tau_0(x), \tau_1 G(y)]_{\mathbb{M}} + (-1)^{|D||G|}[G(Dx), \tau_0^2(y)]_{\mathbb{M}} \\ &- (-1)^{|D||G|+|D||x|} G[\tau_0(x), Dy]_{\mathbb{M}} + (-1)^{|x|(|D|+|G|)+|D||G|}[\tau_0^2(x), G(Dy)]_{\mathbb{M}} \\ &+ (-1)^{|D|(|G|+|x|)}[\tau_1(G(x)), D\tau_0(y)]_{\mathbb{M}} - (-1)^{|D||G|} G(d(l_D(x,y))). \end{aligned}$$

□

4. 2-cocycles of Hom-Lie 2-superalgebras

In this section, we will give notions of representations and 2-cocycles of Hom-Lie 2 superalgebras and show the relation between 1-parameter infinitesimal deformations and 2-cocycles of Hom-Lie 2-superalgebras.

Definition 4.1 *A representation $\rho = (\rho_0, \rho_1, \rho_2)$ of a Hom-Lie 2-superalgebra $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ on 2-term complex $\mathbb{V}$ with respect to an even linear map $\varphi_V = (\varphi_{V_0}, \varphi_{V_1}) : \mathbb{V} \to \mathbb{V}$, where $\varphi_{V_0} : V_0 \to V_0$, $\varphi_{V_1} : V_1 \to V_1$, consists of:*

- *an even linear map $\rho_0 : M_0 \to \mathrm{End}_0^d(\mathbb{V})$ satisfying $\rho_0(\tau_0(x))\varphi_{\mathbb{V}} = \varphi_{\mathbb{V}}\rho_0(x)$,*
- *an even linear map $\rho_1 : M_1 \to \mathrm{End}^1(\mathbb{V})$ satisfying $\rho_1(\tau_1(a))\varphi_{V_0} = \varphi_{V_1}\rho_1(a)$,*
- *an even bilinear map $\rho_2 : M_0 \times M_0 \to \mathrm{End}^1(\mathbb{V})$ satisfying $\rho_2(\tau_0(x), \tau_0(y))\varphi_{V_0} = \varphi_{V_1}\rho_2(x,y)$ such that*

for any $x, y, z \in hg(M_0)$, $a \in hg(M_1)$, the following equations are satisfied:

(1) $\rho_0 \circ d = \delta \circ \rho_1$,

(2) $\rho_0([x,y]_{\mathbb{M}})\varphi_{\mathbb{V}} - \rho_0(\tau_0(x))\rho_0(y) + (-1)^{|x||y|}\rho_0(\tau_0(y))\rho_0(x) = \delta(\rho_2(x,y))$,

(3) $\rho_1([x,a]_{\mathbb{M}})\varphi_{V_0} - \rho_0(\tau_0(x))\rho_1(a) + (-1)^{|x||a|}\rho_0(\tau_1(a))\rho_0(x) = \rho_2(x, da)$,

(4) $(-1)^{|x||z|}\rho_2([x,y]_{\mathbb{M}}, \tau_0(z))\varphi_{V_0} + (-1)^{|x||y|}\rho_2([y,z]_{\mathbb{M}}, \tau_0(x))\varphi_{V_0}$
$+ (-1)^{|y||z|}\rho_2([z,x]_{\mathbb{M}}, \tau_0(y))\varphi_{V_0} + (-1)^{|x||z|}\rho_1(l_3(x,y,z))\varphi_{V_0}^2$
$= (-1)^{|x||z|}\rho_0(\tau_0^2(x))\rho_2(y,z) - (-1)^{|x||y|}\rho_2(\tau_0(y), \tau_0(z))\rho_0(x)$

$+ (-1)^{|x||y|}\rho_0(\tau_0^2(y))\rho_2(z,x) - (-1)^{|y||z|}\rho_2(\tau_0(z),\tau_0(x))\rho_0(y)$
$+ (-1)^{|y||z|}\rho_0(\tau_0^2(z))\rho_2(x,y) - (-1)^{|x||z|}\rho_2(\tau_0(x),\tau_0(y))\rho_0(z)\,.$

For any $x,y,z \in M_0, a \in M_1$, define even linear maps $ad^0 : M_0 \to \mathrm{End}_0^d(\mathbb{M})$ by $ad_x^0(y+a) = [x,y]_{\mathbb{M}} + [x,a]_{\mathbb{M}}$, $ad^1 : M_1 \to \mathrm{End}^1(\mathbb{M})$ by $ad_b^1 x = [b,x]_{\mathbb{M}}$, and an even bilinear map $ad^2 : M_0 \times M_0 \to \mathrm{End}^1(\mathbb{V})$ by $ad_{x,y}^2 z = -l_3(x,y,z)$. Then $ad = (ad^0, ad^1, ad^2)$ is a representation on $\mathbb{M}$ with respect to τ_0, τ_1, which is called an adjoint representation of Hom-Lie 2-superalgebras.

Definition 4.2 *Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. A 2-cocycle of $\mathbb{M}$ with coefficients in the representation $\rho = (\rho_0, \rho_1, \rho_2)$ consists of:*

- *an even linear map $\chi_1 : M_1 \to M_0$ satisfying $\tau_0 \circ \chi_1 = \chi_1 \circ \tau_1$,*
- *an even skew-supersymmetric bilinear map $\chi_2^0 : M_0 \times M_0 \to M_0$ satisfying $\tau_0(\chi_2^0(x,y)) = \chi_2^0(\tau_0(x),\tau_0(y))$,*
- *an even skew-supersymmetric bilinear map $\chi_2^1 : M_0 \times M_1 \to M_1$ satisfying $\tau_1(\chi_2^1(x,a)) = \chi_2^1(\tau_0(x),\tau_1(a))$,*
- *an even skew-supersymmetric trilinear map $\chi_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $\chi_3 \circ \tau_0 = \tau_1 \circ \chi_3$,*

such that the following equations hold for any $x,y,z,t \in hg(M_0)$, $a,b \in hg(M_1)$:

(1) $\rho_0(x)\chi_1(a) + \chi_2^0(x,da) - \chi_1([x,a]_{\mathbb{M}}) - d\chi_2^1(x,a) = 0$,

(2) $\rho_1(a)\chi_1(b) + \chi_2^1(a,db) + (-1)^{|a||b|}\rho_1(b)(\chi_1(a)) - \chi_2^1(da,b) = 0$,

(3) $\rho_0(\tau_0(x))\chi_2^0(y,z) + (-1)^{|x|(|y|+|z|)}\rho_0(\tau_0(y))\chi_2^0(z,x) + (-1)^{|z|(|x|+|y|)}\rho_0(\tau_0(z))\chi_2^0(x,y)$
$+ \chi_2^0(\tau_0(x),[y,z]_{\mathbb{M}}) + (-1)^{|x|(|y|+|z|)}\chi_2^0(\tau_0(y),[z,x]_{\mathbb{M}}) + (-1)^{|z|(|x|+|y|)}\chi_2^0(\tau_0(z),[x,y]_{\mathbb{M}})$
$- d\chi_3(x,y,z) - \chi_1 l_3(x,y,z) = 0$,

(4) $\chi_3(x,y,da) - \rho_2(x,y)\chi_1(a) - \chi_2^1(\tau_0(x),[y,a]_{\mathbb{M}})$
$- (-1)^{|x|(|y|+|a|)}\chi_2^1(\tau_0(y),[a,x]_{\mathbb{M}}) - (-1)^{|a|(|x|+|y|)}\chi_2^1(\tau_1(a),[x,y]_{\mathbb{M}}) - \rho_0(\tau_0(x))\chi_2^1(y,a)$
$- (-1)^{|x|(|y|+|a|)}\rho_0(\tau_0(y))\chi_2^1(a,x) - (-1)^{|a|(|x|+|y|)}\rho_1(\tau_1(a))\chi_2^0(x,y) = 0$,

(5) $\chi_3([t,x]_{\mathbb{M}},\tau_0(y),\tau_0(z)) - (-1)^{(|t|+|x|)(|y|+|z|)}\rho_2(\tau_0(y),\tau_0(z))\chi_2^0(t,x)$
$+ (-1)^{|z|(|x|+|y|)}\chi_3([t,z]_{\mathbb{M}},\tau_0(x),\tau_0(y)) - (-1)^{|t|(|x|+|y|)}\rho_2(\tau_0(x),\tau_0(y))\chi_2^0(t,z)$
$+ (-1)^{|t|(|x|+|y|)}\chi_3([x,y]_{\mathbb{M}},\tau_0(t),\tau_0(z)) - (-1)^{|z|(|x|+|y|)}\rho_2(\tau_0(t),\tau_0(z))\chi_2^0(x,y)$
$+ (-1)^{(|t|+|x|)(|y|+|z|)}\chi_3([y,z]_{\mathbb{M}},\tau_0(t),\tau_0(x)) - \rho_2(\tau_0(t),\tau_0(x))\chi_2^0(y,z)$
$+ (-1)^{|y||z|}\chi_2^1(l_3(t,x,z),\tau_0^2(y)) - (-1)^{|y|(|x|+|t|)}\rho_0(\tau_0^2(y))\chi_3(t,x,z)$
$+ (-1)^{|t|(|x|+|y|+|z|)}\chi_2^1(l_3(x,y,z),\tau_0^2(t)) - \rho_0(\tau_0^2(t))\chi_3(x,y,z)$
$- \chi_2^1(l_3(t,x,y),\tau_0^2(z)) + (-1)^{|z|(|t|+|x|+|y|)}\rho_0(\tau_0^2(z))\chi_3(t,x,y)$
$- (-1)^{|x||y|}\chi_3([t,y]_{\mathbb{M}},\tau_0(x),\tau_0(z)) + (-1)^{|z||y|+|z||t|+|x||t|}\rho_2(\tau_0(x),\tau_0(z))\chi_2^0(t,y)$
$- (-1)^{|y||z|+|t|(|x|+|z|)}\chi_3([x,z]_{\mathbb{M}},\tau_0(t),\tau_0(y)) + (-1)^{|y||x|}\rho_2(\tau_0(t),\tau_0(y))\chi_2^0(x,z)$
$- (-1)^{|x|(|y|+|z|)}\chi_2^1(l_3(t,y,z),\tau_0^2(x)) + (-1)^{|x||t|}\rho_0(\tau_0^2(x))\chi_3(t,y,z) = 0$.

Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra, $\chi_1 : M_1 \to M_0$ satisfying $\tau_0 \circ \chi_1 = \chi_1 \circ \tau_1$ be an even linear map, $\chi_2^0 : M_0 \times M_0 \to M_0$ satisfying $\tau_0(\chi_2^0(x,y)) = \chi_2^0(\tau_0(x),\tau_0(y))$ and $\chi_2^1 : M_0 \times M_1 \to M_1$ satisfying $\tau_1(\chi_2^1(x,a)) = \tau_2^1(\tau_0(x),\tau_1(a))$ be two even skew-supersymmetric bilinear maps respectively, and $\chi_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $\chi_3 \circ \tau_0 = \tau_1 \circ \chi_3$ be an even skew-supersymmetric trilinear map. In the following, we consider a λ-parameterized family of even linear maps:

(1) $d^{\lambda}(a) \triangleq da + \lambda\chi_1(a)$,

(2) $[x,y]_{\lambda} \triangleq [x,y]_{\mathbb{M}} + \lambda\chi_2^0(x,y)$,

(3) $[x,a]_{\lambda} \triangleq [x,a]_{\mathbb{M}} + \lambda\chi_2^1(x,a)$,

(4) $[a,b]_{\lambda} \triangleq [a,b]_{\mathbb{M}} = 0$,

(5) $l_3^{\lambda}(x,y,z) \triangleq l_3(x,y,z) + \lambda\chi_3(x,y,z)$.

With the above notations, if $(\mathbb{M} : M_1 \xrightarrow{d_\lambda} M_0, [\cdot,\cdot]_{\lambda}, l_3^{\lambda}, \tau_0, \tau_1)$ is a Hom-Lie 2-superalgebra, then $(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ generates a 1-parameter infinitesimal deformation of the Hom-Lie 2 superalgebra $\mathbb{M}$.

Theorem 4.3 *Let $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. $(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ generates a 1-parameter infinitesimal deformation of the Lie 2-superalgebra $\mathbb{M}$ if and only if the following conditions hold:*

(1) *$(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ is a 2-cocycle of $\mathbb{M}$ with coefficients in the adjoint representation,*

(2) *$(\mathbb{M} = M_0 \oplus M_1, \chi_1, \chi_2^0, \chi_2^1, \chi_3, \tau_0, \tau_1)$ is a Hom-Lie 2-superalgebra.*

Proof It is clear that $[\cdot,\cdot]_{\lambda}$ is skew-supersymmetric.

For all $x,y,z,t \in hg(M_0)$, $a,b \in hg(M_1)$, equation (4) in Definition 4.1 holds if and only if

$$d\chi_2^1(x,a) + \chi_1([x,a]_{\mathbb{M}}) - \chi_2^0(x,da) - [x,\chi_1(a)]_{\mathbb{M}} = 0, \tag{6}$$

and

$$\chi_1(\chi_2^1(x,a)) - \chi_2^0(x,\chi_1(a)) = 0. \tag{7}$$

Equation (5) in Definition 4.1 holds if and only if

$$\chi_2^1(da,b) + [\chi_1(a),b]_{\mathbb{M}} - \chi_2^1(a,db) - [a,\chi_1(b)]_{\mathbb{M}} = 0, \tag{8}$$

and

$$\chi_2^1(\chi_1(a),b) - \chi_2^1(a,\chi_1(b)) = 0. \tag{9}$$

Equation (6) in Definition 4.1 holds if and only if

$$\tau_0\chi_2^0(x,y) - \chi_2^0(\tau_0(x),\tau_0(y)) = 0. \tag{10}$$

Equation (7) in Definition 4.1 holds if and only if

$$\tau_1\chi_2^1(x,a) - \chi_2^1(\tau_0(x),\tau_1(a)) = 0. \tag{11}$$

Equation (8) in Definition 4.1 holds if and only if

$$\begin{aligned} &d(\chi_3(x,y,z)) + \chi_1(l_3(x,y,z)) - \chi_2^0(\tau(x),[y,z]_{\mathbb{M}}) \\ &- (-1)^{|x|(|y|+|z|)}\chi_2^0(\tau_0(y),[z,x]_{\mathbb{M}}) - (-1)^{|z|(|y|+|x|)}\chi_2^0(\tau_0(z),[x,y]_{\mathbb{M}}) \\ &- [\tau_0 x,\chi_2^0(y,z)]_{\mathbb{M}} - (-1)^{|x|(|y|+|z|)}[\tau_0(y),\chi_2^0(z,x)]_{\mathbb{M}} \\ &- (-1)^{|z|(|y|+|x|)}[\tau_0(z),\chi_2^0(x,y)]_{\mathbb{M}} = 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\chi_1(\chi_3(x,y,z)) - \chi_2^0(\tau_0(x),\chi_2^0(y,z))\\
&-(-1)^{|x|(|y|+|z|)}\chi_2^0(\tau_0(y),\chi_2^0(z,x)) - (-1)^{|z|(|y|+|x|)}\chi_2^0(\phi_0(z),\chi_2^0(x,y))\\
&=0.
\end{aligned}\tag{13}$$

Equation (9) in Definition 4.1 holds if and only if

$$\begin{aligned}
&\chi_3(x,y,da) - l_3(x,y,\chi_1(a)) - \chi_2^1(\tau_0(x),[y,a]_{\mathbb{M}}) - (-1)^{|x|(|y|+|a|)}\chi_2^1(\tau_0(y),[a,x]_{\mathbb{M}})\\
&-(-1)^{|a|(|x|+|y|)}\chi_2^1(\tau_1(a),[x,y]_{\mathbb{M}}) - [\tau_0(x),\chi_2^1(y,a)]_{\mathbb{M}}\\
&-(-1)^{|x|(|y|+|a|)}[\tau_0(y),\chi_2^1(a,x)]_{\mathbb{M}} - (-1)^{|a|(|x|+|y|)}[\tau_1(a),\chi_2^0(x,y)]_{\mathbb{M}}\\
&=0,
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&\chi_3(x,y,\chi_1(a)) - \chi_2^1(\tau_0(x),\chi_2^1(y,a))\\
&-(-1)^{|x|(|y|+|a|)}\chi_2^1(\tau_0(y),\chi_2^1(a,x)) - (-1)^{|a|(|x|+|y|)}\chi_2^1(\tau_1(a),\chi_2^0(x,y))\\
&=0.
\end{aligned}\tag{15}$$

Equation (10) in Definition 4.1 holds if and only if

$$\begin{aligned}
&\chi_3([t,x]_{\mathbb{M}},\tau_0(y),\tau_0(z)) + l_3(\chi_2^0(t,x),\tau_0(y),\tau_0(z))\\
&+(-1)^{|z|(|x|+|y|)}\chi_3([t,z]_{\mathbb{M}},\tau_0(x),\tau_0(y)) + (-1)^{|z|(|x|+|y|)}l_3(\chi_2^0(t,z),\tau_0(x),\tau_0(y))\\
&+(-1)^{|t|(|x|+|y|)}\chi_3([x,y]_{\mathbb{M}},\tau_0(t),\tau_0(z)) + (-1)^{|t|(|x|+|y|)}l_3(\chi_2^0(x,y),\tau_0(t),\tau_0(z))\\
&+(-1)^{(|x|+|t|)(|y|+|z|)}\chi_3([y,z]_{\mathbb{M}},\tau_0(t),\tau_0(x)) + (-1)^{(|x|+|t|)(|y|+|z|)}l_3(\chi_2^0(y,z),\tau_0(t),\tau_0(x))\\
&+(-1)^{|y||z|}\chi_2^1(l_3(t,x,z),\tau_0^2(y)) + (-1)^{|y||z|}[\chi_3(t,x,z),\tau_0^2(y)]_{\mathbb{M}}\\
&+(-1)^{|t|(|x|+|y|+|z|)}\chi_2^1(l_3(x,y,z),\tau_0^2(t)) + (-1)^{|t|(|x|+|y|+|z|)}[\chi_3(x,y,z),\tau_0^2(t)]_{\mathbb{M}}\\
&-\chi_2^1(l_3(t,x,y),\tau_0^2(z)) - [\chi_3(t,x,y),\tau_0^2(z)]_{\mathbb{M}}\\
&-(-1)^{|x||y|}\chi_3([t,y]_{\mathbb{M}},\tau_0(x),\tau_0(z)) - (-1)^{|x||y|}l_3(\chi_2^0(t,y),\tau_0(x),\tau_0(z))\\
&-(-1)^{|y||z|+|t|(|x|+|z|)}\chi_3([x,z]_{\mathbb{M}},\tau_0(t),\tau_0(y)) - (-1)^{|y||z|+|t|(|x|+|z|)}l_3(\chi_2^0(x,z),\tau_0(t),\tau_0(y))\\
&-(-1)^{|x|(|y|+|z|)}\chi_2^1(l_3(t,y,z),\tau_0^2(x)) - (-1)^{|x|(|y|+|z|)}[\chi_3(t,y,z),\tau_0^2(x)]_{\mathbb{M}}\\
&=0,
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&\chi_3(\chi_2^0(t,x),\tau_0(y),\tau_0(z)) + (-1)^{|z|(|x|+|y|)}\chi_3(\chi_2^0(t,z),\tau_0(x),\tau_0(y))\\
&+(-1)^{|t|(|x|+|y|)}\chi_3(\chi_2^0(x,y),\tau_0(t),\tau_0(z)) + (-1)^{|t|(|x|+|y|)}\chi_3(\chi_2^0(x,y),\tau_0(t),\tau_0(z))
\end{aligned}$$

$$\begin{aligned}
&+(-1)^{(|x|+|t|)(|y|+|z|)}\chi_3(\chi_2^0(y,z),\tau_0(t),\tau_0(x))+(-1)^{|y||z|}\chi_2^1(\chi_3(t,x,z),\tau_0^2(y))\\
&+(-1)^{|t|(|x|+|y|+|z|)}\chi_2^1(\chi_3(x,y,z),\tau_0^2(t))-\chi_2^1(\chi_3(t,x,y),\tau_0^2(z))\\
&-(-1)^{|x||y|}\chi_3(\chi_2^0(t,y),\tau_0(x),\tau_0(z))-(-1)^{|y||z|+|t|(|x|+|z|)}\chi_3(\chi_2^0(x,z),\tau_0(t),\tau_0(y))\\
&-(-1)^{|x|(|y|+|z|)}\chi_2^1(\chi_3(t,y,z),\tau_0^2(x))=0.
\end{aligned} \tag{17}$$

From equations (6), (8), (12), (14), and (16), we show that $(\chi_1,\chi_2^0,\chi_2^1,\chi_3)$ is a 2-cocycle of $\mathbb{M}$ with the coefficients in the adjoint representation. Moreover, by equations (7), (9), (10), (11), (13), (15), and (17), $(\mathbb{M}=M_0\oplus M_1,\chi_1,\chi_2^0,\chi_2^1,\chi_3,\tau_0,\tau_1)$ is a Hom-Lie 2-superalgebra. □

5. Hom-Nijenhuis operators on Hom-Lie 2-superalgebras

In this section, we introduce the notion of Hom-Nijenhuis operators and study trivial deformations of Hom-Lie 2-superalgebras.

Let $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$ be a Hom-Lie 2-superalgebra, and $N_0:M_0\to M_0$ and $N_1:M_1\to M_1$ be two even linear maps satisfying $N_0\circ\tau_0=\tau_0\circ N_0$ and $N_1\circ\tau_1=\tau_1\circ N_1$. For any $x,y,z\in hg(M_0)$, $a\in hg(M_1)$, define

$$\begin{aligned}
d_N &= d\circ N_1-N_0\circ d=0,\\
[x,y]_N &= [N_0x,y]_{\mathbb{M}}+[x,N_0y]_{\mathbb{M}}-N_0[x,y]_{\mathbb{M}},\\
[x,a]_N &= [N_0x,a]_{\mathbb{M}}+[x,N_1a]_{\mathbb{M}}-N_1[x,a]_{\mathbb{M}},\\
l_3^N(x,y,z) &= l_3(N_0x,y,z)+l_3(x,N_0y,z)+l_3(x,y,N_0z)-N_1^2l_3(x,y,z).
\end{aligned}$$

Definition 5.1 *An even linear map $N=(N_0,N_1)$ is called a Hom-Nijenhuis operator on Hom-Lie 2-superalgebras if for any $x,y,z\in hg(M_0)$, $a\in hg(M_1)$, the following conditions are satisfied:*

(1) $d\circ N_1=N_0\circ d=0$,

(2) $N_0[x,y]_N=[N_0x,N_0y]_{\mathbb{M}}$,

(3) $N_1[x,a]_N=[N_0x,N_1a]_{\mathbb{M}}$,

(4) $N_1l_3^N(x,y,z)=0$,

(5) $l_3(N_0x,N_0y,N_0z)=0$,

(6) $l_3(N_0x,N_0y,z)+l_3(N_0x,y,N_0z)+l_3(x,N_0y,N_0z)=0$.

Proposition 5.2 *Let $N=(N_0,N_1)$ be a Hom-Nijenhuis operator, then for any $\lambda\in\mathbb{R}$, $\lambda N=(\lambda N_0,\lambda N_1)$ is also a Hom-Nijenhuis operator. Furthermore, $(\mathbb{M}:M_1\xrightarrow{d_{\lambda N}=0}M_0,[\cdot,\cdot]_{\lambda N},l_3^{\lambda N},\tau_0,\tau_1)$ is a skeletal Hom-Lie 2-superalgebra and*

$$\lambda N:(\mathbb{M}:M_1\xrightarrow{d_{\lambda N}=0}M_0,[\cdot,\cdot]_{\lambda N},l_3^{\lambda N},\tau_0,\tau_1)\to(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$$

is a homomorphism of Hom-Lie 2-superalgebras.

Proof It is a straightforward calculation. □

Let $(M\oplus\mathbb{R}:\mathbb{R}\xrightarrow{d=0}M,[\cdot,\cdot],l_3,\beta,I_{\mathbb{R}})$ be a Hom-Lie 2-superalgebra in Example 2.2. We define even operators $N_0:M\to M$ and $N_1=0:\mathbb{R}\to\mathbb{R}$. We can see that $N=(N_0,0)$ is a Hom-Nijenhuis operator if and only if

$$N_0\circ\beta-\beta\circ N_0=0, \tag{18}$$

$$N_0[N_0x,y]_M+N_0[x,N_0y]_M-N_0^2[x,y]_M-[N_0x,N_0y]_M=0, \tag{19}$$

$$B([N_0x,N_0y]_M,N_0z)=0, \tag{20}$$

$$B([N_0x,N_0y]_M,z)+B([N_0x,y]_M,N_0z)+B([x,N_0y]_M,N_0z)=0. \tag{21}$$

Proposition 5.3 *Let $(M\oplus\mathbb{R}:\mathbb{R}\xrightarrow{d=0}M,[\cdot,\cdot],l_3,\beta,I_{\mathbb{R}})$ be a Hom-Lie 2-superalgebra in Example 2.2. If the even linear map $N_0:M\to M$ satisfies equations (18) and (19), bilinear form B satisfies $B(G_\lambda x,G_\lambda y)=B(x,y)$, where $G_\lambda\triangleq I_M+\lambda N_0$, $\lambda\in\mathbb{R}$ is a parameter, and then $N=(N_0,0)$ is a Hom-Nijenhuis operator on the Hom-Lie 2-superalgebra $(M\oplus\mathbb{R}:\mathbb{R}\xrightarrow{d=0}M,[\cdot,\cdot],l_3,\beta,I_{\mathbb{R}})$.*

Proof We only need to show that $N=(N_0,0)$ satisfies equations (20) and (21). By

$$B(G_\lambda x,G_\lambda y)=B(x,y),$$

we have

$$B(x,N_0y)=-B(N_0x,y),\qquad B(N_0x,N_0y)=0.$$

Since B is nondegenerate, we obtain $N_0^2=0$ and

$$\begin{aligned}&B([N_0x,N_0y]_M,N_0z)\\=&B(N_0[N_0x,y]_M,N_0z)+B(N_0[x,N_0y]_M,N_0z)-B(N_0^2[x,y]_M,N_0z)\\=&-B([N_0x,y]_M,N_0^2z)-B([x,N_0y],N_0^2z)=0,\end{aligned}$$

and

$$\begin{aligned}&B([N_0x,N_0y]_M,z)+B([N_0x,y]_M,N_0z)+B([x,N_0y]_M,N_0z)\\=&B([N_0x,N_0y]_M,z)-B(N_0[N_0x,y],z)-B(N_0[x,N_0y]_M,z)\\=&-B(N_0^2[x,y]_M,z)=0.\end{aligned}$$

□

Definition 5.4 *Let $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$ be a Hom-Lie 2-superalgebra. A deformation of $\mathbb{M}$ is called trivial if there exist even linear maps $N_0:M_0\to M_0$, $N_1:M_1\to M_1$ and an even bilinear map $N_2:M_0\times M_0\to M_1$ such that $G=(G_0,G_1,G_2)$ is a homomorphism from the Hom-Lie 2-superalgebra $(\mathbb{M}^\lambda:M_1\xrightarrow{d^\lambda}M_0,[\cdot,\cdot]_\lambda,l_3^\lambda,\tau_0,\tau_1)$ to the Hom-Lie 2-superalgebra $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$, where $G_0=I_{M_0}+\lambda N_0$, $G_1=I_{M_1}+\lambda N_1$, $G_2=\lambda N_2$.*

Theorem 5.5 *A deformation of the Hom-Lie 2-superalgebra* $(\mathbb{M} : M_1 \xrightarrow{d} M_0, [\cdot,\cdot]_{\mathbb{M}}, l_3, \tau_0, \tau_1)$ *is trivial if and only if there exist even linear maps* $N_0 : M_0 \to M_0$, $N_1 : M_1 \to M_1$ *and an even bilinear map* $N_2 : M_0 \times M_0 \to M_1$ *such that for any* $x, y, z, t \in hg(M_0)$, $a \in hg(M_1)$, *the following equalities are satisfied:*

(1) $N_0 \circ \tau_0 = \tau_0 \circ N_0$,

(2) $N_1 \circ \tau_1 = \tau_1 \circ N_1$,

(3) $N_2(\tau_0(x), \tau_0(y)) = \tau_1(N_2(x,y))$,

(4) $N_0(d(N_1 a) - N_0(da)) = 0$,

(5) $N_0(dN_2(x,y)) + N_0[N_0x, y]_{\mathbb{M}} + N_0[x, N_0y]_{\mathbb{M}} - N_0^2[x,y]_{\mathbb{M}} = [N_0x, N_0y]_{\mathbb{M}}$,

(6) $N_1N_2(x,da) + N_1[N_0x, a]_{\mathbb{M}} + N_1[x, N_1a]_{\mathbb{M}} - N_1^2[x,a]_{\mathbb{M}} - [N_0x, N_1a]_{\mathbb{M}} = N_2(x, \chi_1(a))$,

(7)
$$
\begin{aligned}
&(-1)^{|x||z|}N_1l_3(N_0x,y,z) + (-1)^{|x||z|}N_1l_3(x,N_0y,z) + (-1)^{|x||z|}N_1l_3(x,y,N_0z)\\
&+ (-1)^{|y||z|}N_1[\tau_0(z), N_2(x,y)]_{\mathbb{M}} + (-1)^{|x||y|}N_1[\tau_0(y), N_2(z,x)]_{\mathbb{M}} + (-1)^{|x||z|}N_1[\tau_0(x), N_2(y,z)]_{\mathbb{M}}\\
&- (-1)^{|x||z|}N_1^2l_3(x,y,z) - (-1)^{|y||z|}N_1N_2([z,x]_{\mathbb{M}}, \tau_0(y)) - (-1)^{|x||y|}N_1N_2([y,z]_{\mathbb{M}}, \tau_0(x))\\
&- (-1)^{|x||z|}N_1N_2([x,y]_{\mathbb{M}}, \tau_0(z)) + (-1)^{|x||z|}N_2(\chi_2^0(x,y), \tau_0(z)) + (-1)^{|x||y|}N_2(\chi_2^0(y,z), \tau_0(x))\\
&+ (-1)^{|y||z|}N_2(\chi_2^0(z,x), \tau_0(y)) - (-1)^{|x||z|}[N_0\tau_0(x), N_2(y,z)]_{\mathbb{M}} - (-1)^{|x||y|}[N_0\tau_0(y), N_2(z,x)]_{\mathbb{M}}\\
&- (-1)^{|y||z|}[N_0\tau_0(z), N_2(x,y)]_{\mathbb{M}} - (-1)^{|x||z|}l_3(x,N_0y,N_0z) - (-1)^{|x||z|}l_3(N_0x,y,N_0z)\\
&- (-1)^{|x||z|}l_3(N_0x,N_0y,z) = 0,
\end{aligned}
$$

(8) $l_3(N_0x, N_0y, N_0z) = 0$.

Proof We only need to show that $G = (G_0, G_1, G_2)$ is a homomorphism of Hom-Lie 2-superalgebras. Since $G_0 d^\lambda(a) = dG_1(a)$, $d^\lambda(a) = da + \lambda\chi_1(a)$, we have

$$da + \lambda\chi_1(a) + \lambda N_0 da + \lambda^2 N_0\chi_1(a) = da + \lambda d(N_1a),$$

which implies that

$$\chi_1(a) + N_0(da) = d(N_1a), \quad N_0(\chi_1(a)) = 0.$$

From equation (2) in Definition 2.3, we have

$$
\begin{aligned}
&\lambda\chi_2^0(x,y) + \lambda N_0[x,y]_{\mathbb{M}} + \lambda^2 N_0\chi_2^0(x,y) - \lambda[x, N_0y]_{\mathbb{M}} - \lambda[N_0x, y]_{\mathbb{M}} - \lambda^2[N_0x, N_0y]_{\mathbb{M}}\\
&= \lambda dN_2(x,y),
\end{aligned}
$$

which means that

$$\chi_2^0(x,y) + N_0[x,y]_{\mathbb{M}} - [x, N_0y]_{\mathbb{M}} - [N_0x, y]_{\mathbb{M}} = dN_2(x,y), \quad N_0\chi_2^0(x,y) = [N_0x, N_0y]_{\mathbb{M}}.$$

From equation (3) in Definition 2.3, we obtain

$$
\begin{aligned}
&\lambda\chi_2^1(x,a) + \lambda N_1[x,a]_{\mathbb{M}} + \lambda^2 N_1\chi_2^1(x,a) - \lambda[x, N_1a]_{\mathbb{M}} - \lambda[N_0x, a]_{\mathbb{M}} - \lambda^2[N_0x, N_1a]_{\mathbb{M}}\\
&= \lambda N_2(x, da) + \lambda^2 N_2(x, \chi_1(a)),
\end{aligned}
$$

which yields that

$$
\begin{aligned}
\chi_2^1(x,a) + N_1[x,a]_{\mathbb{M}} - [x, N_1a]_{\mathbb{M}} - [N_0x, a]_{\mathbb{M}} &= N_2(x, da),\\
N_1\chi_2^1(x,a) - [N_0x, N_1a]_{\mathbb{M}} &= N_2(x, \chi_1(a)).
\end{aligned}
$$

From equation (4) in Definition 2.3, we have

$$\begin{aligned}&(-1)^{|x||z|}N_2([x,y]_{\mathbb{M}},\tau_0(z))+(-1)^{|x||y|}N_2([y,z]_{\mathbb{M}},\tau_0(x))+(-1)^{|y||z|}N_2([z,x]_{\mathbb{M}},\tau_0(y))\\&+(-1)^{|x||z|}\chi_3(x,y,z)+(-1)^{|x||z|}N_1l_3(x,y,z)-(-1)^{|x||z|}[\tau_0(x),N_2(y,z)]_{\mathbb{M}}\\&-(-1)^{|x||y|}[\tau_0(y),N_2(z,x)]_{\mathbb{M}}-(-1)^{|y||z|}[\tau_0(z),N_2(x,y)]_{\mathbb{M}}-(-1)^{|x||z|}l_3(x,y,N_0z)\\&-(-1)^{|x||z|}l_3(x,N_0y,z)-(-1)^{|x||z|}l_3(N_0x,y,z)=0,\end{aligned}$$

and

$$\begin{aligned}&(-1)^{|x||z|}N_2(\chi_2^0(x,y),\tau_0(z))+(-1)^{|x||y|}N_2(\chi_2^0(y,z),\tau_0(x))+(-1)^{|y||z|}N_2(\chi_2^0(z,x),\tau_0(y))\\&+(-1)^{|x||z|}N_1\chi_3(x,y,z)-(-1)^{|x||z|}[N_0\tau_0(x),N_2(y,z)]_{\mathbb{M}}-(-1)^{|x||y|}[N_0\tau_0(y),N_2(z,x)]_{\mathbb{M}}\\&-(-1)^{|z||y|}[N_0\tau_0(z),N_2(x,y)]_{\mathbb{M}}-(-1)^{|x||z|}l_3(x,N_0y,N_0z)-(-1)^{|x||z|}l_3(N_0x,y,N_0z)\\&-(-1)^{|x||z|}l_3(N_0x,N_0y,z)=0,\end{aligned}$$

and

$$l_3(N_0x,N_0y,N_0z)=0.$$

Thus, $G=(G_0,G_1,G_2)$ is a homomorphism of Hom-Lie 2-superalgebra if and only if equations (1)−(8) in Theorem 5.5 hold. □

Remark 5.6 *$N=(N_0,N_1,N_2)$ is not a Hom-Nijenhuis operator in Theorem 5.5.*

6. Abelian extensions of Hom-Lie 2-superalgebras

In this section, we will study abelian extensions of Hom-Lie 2-superalgebras and show that there exists a representation and a 2-cocycle by means of abelian extensions.

Definition 6.1 *Let $(\mathbb{M}:M_1\xrightarrow{d}M_0,[\cdot,\cdot]_{\mathbb{M}},l_3,\tau_0,\tau_1)$, $(\mathbb{M}':M_1'\xrightarrow{d'}M_0',[\cdot,\cdot]_{\mathbb{M}'},l_3',\tau_0',\tau_1')$ and $(\tilde{\mathbb{M}}:\tilde{M}_1\xrightarrow{\tilde{d}}\tilde{M}_0,[\cdot,\cdot]_{\tilde{\mathbb{M}}},\tilde{l_3},\tilde{\tau_0},\tilde{\tau_1})$ be Hom-Lie 2-superalgebras, and $i=(i_0,i_1):\mathbb{M}'\to\tilde{\mathbb{M}}$, $p=(p_0,p_1):\tilde{\mathbb{M}}\to\mathbb{M}$ be strict homomorphisms. The following sequence is called a short exact sequence if $\mathrm{Im}(i)=\mathrm{Ker}(p)$.*

$$\begin{array}{ccccccccc}0&\to&M_1'&\xrightarrow{i_1}&\tilde{M}_1&\xrightarrow{p_1}&M_1&\to&0\\&&d\downarrow&&\tilde{d}\downarrow&&d'\downarrow&&\\0&\to&M_0'&\xrightarrow{i_0}&\tilde{M}_0&\xrightarrow{p_0}&M_0&\to&0\end{array}\qquad(22)$$

$\tilde{\mathbb{M}}$ is called an extension of $\mathbb{M}$ by $\mathbb{M}'$, denoted by $E_{\tilde{\mathbb{M}}}$. The extension $E_{\tilde{\mathbb{M}}}$ is called an abelian extension if $[\cdot,\cdot]_{\mathbb{M}'}=0$ and $l_3'(\cdot,\cdot,\cdot)=0$.

A splitting of an extension is an even linear map $\varphi=(\varphi_0,\varphi_1):\mathbb{M}\to\tilde{\mathbb{M}}$ such that $p_0\circ\varphi_0=I_{M_0}$ and $p_1\circ\varphi_1=I_{M_1}$, where $\varphi_0:M_0\to\tilde{M}_0$ and $\varphi_1:M_1\to\tilde{M}_1$.

Theorem 6.2 *Let $\tilde{\mathbb{M}}$ be an abelian extension of $\mathbb{M}$ by $\mathbb{M}'$ given by (22), and let $\varphi=(\varphi_0,\varphi_1):\mathbb{M}\to\tilde{\mathbb{M}}$*

be a splitting. For any $x,y\in hg(M_0)$, $a\in hg(M_1)$, $s\in hg(M_0')$, $t\in hg(M_1')$, *define an even linear map* $\rho=(\rho_0,\rho_1,\rho_2)$ *by*

$$\begin{cases}\rho_0: M_0 & \to \mathrm{End}^0_{d'}(\mathbb{M}'), & \rho_0(x)(s+t) & \triangleq & [\varphi(x),s+t]_{\tilde{\mathbb{M}}},\\ \rho_1: M_1 & \to \mathrm{End}^1(\mathbb{M}'), & \rho_1(a)(s) & \triangleq & [\varphi(a),s]_{\tilde{\mathbb{M}}},\\ \rho_2: M_0\times M_0 & \to \mathrm{End}^1(\mathbb{M}'), & \rho_2(x,y)(s) & \triangleq & \tilde{l_3}(\varphi(x),\varphi(y),s),\end{cases} \tag{23}$$

and then $\rho=(\rho_0,\rho_1,\rho_2)$ *is a representation of* $\mathbb{M}$ *on* $\mathbb{M}'$ *with respect to* τ_0',τ_1'.

Proof It is a straightforward calculation by Definition 4.1. □

Theorem 6.3 *Let* $\tilde{\mathbb{M}}$ *be an abelian extension of* $\mathbb{M}$ *by* $\mathbb{M}'$ *given by* (22) *and* $\varphi=(\varphi_0,\varphi_1):\mathbb{M}\to\tilde{\mathbb{M}}$ *be a splitting. For any* $x,y,z\in hg(M_0)$, $a,b\in hg(M_1)$, $s\in hg(M_0')$, $t\in hg(M_1')$, *define an even linear map* $\chi=(\chi_1,\chi_2^0,\chi_2^1,\chi_3)$ *by*

$$\begin{cases}\chi_1: M_1 & \to M_0', & \chi_1(a) & = & \tilde{d}\varphi_1(a)-\varphi_0(da),\\ \chi_2^0: M_0\times M_0 & \to M_0', & \chi_2^0(x,y) & = & [\varphi_0(x),\varphi_0(y)]_{\tilde{\mathbb{M}}}-\varphi_0[x,y]_{\mathbb{M}},\\ \chi_2^1: M_0\times M_1 & \to M_1', & \chi_2^1(x,a) & = & [\varphi_0(x),\varphi_1(a)]_{\tilde{\mathbb{M}}}-\varphi_0[x,a]_{\mathbb{M}},\\ \chi_3: M_0\times M_0\times M_0 & \to M_1', & \chi_3(x,y,z) & = & \tilde{l_3}(\varphi_0(x),\varphi_0(y),\varphi_0(z))-\varphi_1(l_3(x,y,z)),\end{cases}$$

and then $\chi=(\chi_1,\chi_2^0,\chi_2^1,\chi_3)$ *is a 2-cocycle of* $\mathbb{M}$ *with coefficients in* $\mathbb{M}'$, *where* $\rho=(\rho_0,\rho_1,\rho_2)$ *is a representation of* $\mathbb{M}$ *on* $\mathbb{M}'$.

Proof It is easy to show that

$$\rho_0(x)\chi_1(a)+\chi_2^0(x,da)-\chi_1([x,a]_{\mathbb{M}})-\tilde{d}\chi_2^1(x,a)=0,$$

$$\rho_1(a)\chi_1(b)+\chi_2^1(a,db)+(-1)^{|a||b|}\rho_1(b)(\chi_1(a))-\chi_2^1(da,b)=0.$$

Since $\tilde{\mathbb{M}}$ is a Hom-Lie 2-superalgebra, we have

$$\begin{aligned}
&\rho_0(\tau_0(x))\chi_2^0(y,z)+(-1)^{|x|(|y|+|z|)}\rho_0(\tau_0(y))\chi_2^0(z,x)+(-1)^{|z|(|x|+|y|)}\rho_0(\tau_0(z))\chi_2^0(x,y)\\
&+\chi_2^0(\tau_0(x),[y,z]_{\mathbb{M}})+(-1)^{|x|(|y|+|z|)}\chi_2^0(\tau_0(y),[z,x]_{\mathbb{M}})+(-1)^{|z|(|x|+|y|)}\chi_2^0(\tau_0(z),[x,y]_{\mathbb{M}})\\
&-\tilde{d}\chi_3(x,y,z)-\chi_1 l_3(x,y,z)\\
&=[\varphi_0(\tau_0(x)),[\varphi_0(y),\varphi_0(z)]_{\tilde{\mathbb{M}}}]_{\tilde{\mathbb{M}}}-[\varphi_0(\varphi_0(x)),\varphi_0[y,z]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}\\
&+(-1)^{|x|(|y|+|z|)}[\varphi_0(\tau_0(y)),[\varphi_0(z),\varphi_0(x)]_{\tilde{\mathbb{M}}}]_{\tilde{\mathbb{M}}}-(-1)^{|x|(|y|+|z|)}[\varphi_0(\tau_0(y)),\varphi_0[z,x]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}\\
&+(-1)^{|z|(|x|+|y|)}[\varphi_0(\tau_0(z)),[\varphi_0(x),\varphi_0(y)]_{\tilde{\mathbb{M}}}]_{\tilde{\mathbb{M}}}-(-1)^{|z|(|x|+|y|)}[\varphi_0(\tau_0(z)),\varphi_0[x,y]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}\\
&+[\varphi_0(\tau_0(x)),\varphi_0[y,z]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}-\varphi_0[\tau_0(x),[y,z]_{\mathbb{M}}]_{\mathbb{M}}\\
&+(-1)^{|x|(|y|+|z|)}[\varphi_0(\tau_0(y)),\varphi_0[z,x]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}-(-1)^{|x|(|y|+|z|)}\varphi_0[\tau_0(y),[z,x]_{\mathbb{M}}]_{\mathbb{M}}\\
&+(-1)^{|z|(|x|+|y|)}[\varphi_0(\tau_0(z)),\varphi_0[x,y]_{\mathbb{M}}]_{\tilde{\mathbb{M}}}-(-1)^{|z|(|x|+|y|)}\chi_0[\tau_0(z),[x,y]_{\mathbb{M}}]_{\mathbb{M}}\\
&-\tilde{d}\tilde{l_3}(\varphi_0(x),\varphi_0(y),\varphi_0(z))+\tilde{d}\varphi_1 l_3(x,y,z)-\tilde{d}\varphi_1 l_3(x,y,z)+\varphi_0 dl_3(x,y,z)\\
&=0.
\end{aligned}$$

Similar to the above proof, equations (4) and (5) in Definition 4.2 can be obtained. Thus, $\chi = (\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ is a 2-cocycle of $\mathbb{M}$ with coefficients in $\mathbb{M}'$ □

7. The construction of Hom-Lie 2-superalgebras

In this section, we will construct a strict Hom-Lie 2-superalgebra and a skeletal Hom-Lie 2-superalgebra from Hom-associative Rota-Baxter superalgebras.

Definition 7.1 [1] *A Hom-associative superalgebra is a triple $(A, \cdot, \tau)$ consisting of a super vector space A, an even bilinear map $\cdot : A \times A \to A$, and an even homomorphism $\tau : A \to A$ satisfying*

$$(x \circ y) \circ \phi(z) = \phi(x) \circ (y \circ z).$$

Definition 7.2 *A Hom-associative Rota–Baxter superalgebra $(M, \cdot, \tau, R)$ is a Hom-associative superalgebra $(M, \cdot, \tau)$ with an even linear map $R : M \to M$ satisfying*

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + \theta x \cdot y), \tag{24}$$

where $\theta \in \mathbb{R}$. The even linear map R is called a Rota–Baxter operator of weight θ, and the identity (24) is called a Rota–Baxter identity.

A Hom-associative Rota–Baxter superalgebra $(M, \cdot, \tau, R)$ is called multiplicative if $\tau(x \cdot y) = \tau(x) \cdot \tau(y)$.

Theorem 7.3 *Let $(M, \cdot, \tau, R)$ be a multiplicative Hom-associative Rota–Baxter superalgebra with a Rota–Baxter operator of weight 0. Assume that even linear maps $\phi_0 = \tau$, $\phi_1 = \tau$, and even linear map $d : M = M_1 \to M_0 = M$ satisfies*

$$\begin{cases} d \circ \tau = \tau \circ d, & \\ d(R(x) \cdot a) = R(x) \cdot da + x \cdot R(da) & x \in hg(M_0), a \in hg(M_1), \\ d(a \cdot R(x)) = da \cdot R(x) + R(da) \cdot x & x \in hg(M_0), a \in hg(M_1), \\ R(da) \cdot b = a \cdot R(db) & a, b \in hg(M_1), \\ b \cdot R(da) = R(db) \cdot a & a, b \in hg(M_1). \end{cases}$$

Define an even bilinear map $l_2 : M_i \times M_j \to M_{i+j}$ $(0 \le i + j \le 1)$ by

$$\begin{cases} l_2(x, y) = R(x) \cdot y + x \cdot R(y) - (-1)^{|x||y|}(y \cdot R(x) + R(y) \cdot x) & x, y \in hg(M_0), \\ l_2(x, a) = -(-1)^{|x||a|} l_2(a, x) = R(x) \cdot a - (-1)^{|x||a|} a \cdot R(x) & x \in hg(M_0), a \in hg(M_1), \\ l_2(a, b) = 0 & a, b \in hg(M_1). \end{cases}$$

If $R \circ \tau = \tau \circ R$, then $(\mathbb{M} : M_1 \xrightarrow{d} M_0, l_2, \phi_0, \phi_1)$ is a strict Hom-Lie 2-superalgebra.

Proof For any $x, y \in hg(M_0)$, we have

$$\begin{aligned} \phi_0(l_2(x, y)) &= R(\tau(x)) \cdot \tau(y) + \tau(x) \cdot R(\tau(y)) - (-1)^{|x||y|} \tau(y) \cdot R(\tau(x)) - (-1)^{|x||y|} R(\tau(y)) \cdot \tau(x) \\ &= \phi_0(l_2(x, y)). \end{aligned}$$

Similarly, we obtain $\phi_1(l_2(x, a)) = l_2(\phi_0(x), \phi_1(a))$. By the Rota–Baxter identity (24), we deduce that equations (8) and (9) in Definition 2.1 hold. □

Definition 7.4 *Let $(M,\cdot,\tau,R)$ be a Hom-associative Rota–Baxter superalgebra and $B: M\times M\to\mathbb{R}$ be a bilinear form on M. For any $x,y,z\in hg(M)$, B is called super-symmetric if $B(x,y)=(-1)^{|x||y|}B(y,x)$. B is called invariant if $B(x\cdot y,z)=B(x,y\cdot z)$. B is called even if $B(L_{\bar{0}},L_{\bar{1}})=B(L_{\bar{1}},L_{\bar{0}})=0$.*

Definition 7.5 *A Hom-associative Rota–Baxter superalgebra $(M,\cdot,\tau,R)$ with a Rota–Baxter operator of weight 0 is called a quadratic Hom-associative Rota–Baxter superalgebra if there exists a nondegenerate, super-symmetric, and even invariant bilinear form B on $(M,\cdot,\tau,R)$ such that τ satisfies $B(\tau(x),y)=B(x,\tau(y))$. It is denoted by $(M,\cdot,\tau,R,B)$. A quadratic Hom-associative Rota–Baxter superalgebra is called involutive if $\tau^2=I_M$.*

Theorem 7.6 *Let $(M,\cdot,\tau,R,B)$ be an involutive multiplicative quadratic Hom-associative Rota–Baxter superalgebra with a Rota–Baxter operator of weight 0. Assume that even linear maps $d=0: R=M_1\to M_0=M$, $\phi_0=\tau$, $\phi_1=\tau$. Define an even bilinear map $l_2: M_i\times M_j\to M_{i+j}\quad(0\le i+j\le 1)$ by*

$$\begin{cases} l_2(x,y) = R(x)\cdot y + x\cdot R(y) - (-1)^{|x||y|}(y\cdot R(x)+R(y)\cdot x) & x,y\in hg(M_0),\\ l_2(x,a) = -(-1)^{|x||a|}l_2(a,x) = 0 & x\in hg(M_0), a\in hg(M_1),\\ l_2(a,b) = 0 & a,b\in hg(M_1),\end{cases}$$

and an even trilinear map $l_3: M_0\times M_0\times M_0\to M_1$ by

$$l_3(x,y,z)=B(l_2(x,y),z).$$

If $R\circ\tau=\tau\circ R$ and $R(x)\cdot y=x\cdot R(y)$, then $(\mathbb{M}: M_1\xrightarrow{d=0}M_0,l_2,l_3,\phi_0,\phi_1)$ is a skeletal Hom-Lie 2-superalgebra.

Proof It is obvious that even linear maps l_2 and l_3 are skew-supersymmetric. By the Rota–Baxter identity (24), we deduce that equations (8) and (10) in Definition 2.1 hold. □

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