Turkish Journal of Mathematics

Volume 40 | Number 2 Article 20

1-1-2016

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 453 – 469
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doi:10.3906/mat-1501-22

G-frames for operators in Hilbert C^*-modules

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Received: 12.01.2015 • **Accepted/Published Online:** 04.09.2015 • **Final Version:** 10.02.2016

Abstract: We present a generalization of g-frames related to an adjointable operator K on a Hilbert C^*-module, which we call K-g-frames. We obtain several characterizations of K-g-frames and we also give conditions under which the removal of an element from a K-g-frame leaves again a K-g-frame. In addition, we define a concept of dual, and using it we study the relation between a K-g-frame and a g-Bessel sequence with respect to different sequences of Hilbert C^*-modules.

Key words: Hilbert C^*-module, g-frame, operator, duality

1. Introduction

Frames for Hilbert spaces were formally defined by Duffin and Schaeffer [4] in their work on nonharmonic Fourier series, reintroduced and developed in 1986 by Daubechies et al. [3], and since then they have become the focus of active research, both in theory and in applications, such as the characterization of function spaces, digital signal processing, and scientific computations.

Sun [14] proposed the concept of g-frames as generalizations of Hilbert spaces frames and showed that this includes many other cases of generalizations of the frame concept. On the other hand, frames and g-frames were introduced in Hilbert C^*-modules that are generalizations of Hilbert spaces [7,11]. It should be remarked that, due to the complexity of the C^*-algebras involved in the Hilbert C^*-modules and the fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert C^*-modules, the generalizations of frame theory from Hilbert spaces to Hilbert C^*-modules are not trivial. The properties of frames and g-frames in Hilbert C^*-modules were further studied in [1,10,15].

Atomic systems for subspaces were first introduced by Feichtinger and Werther [6] based on examples arising in sampling theory. In [8], Găvruţa introduced atomic systems for operators in Hilbert spaces, and frames for operators allowing the reconstruction of elements from the range of a linear and bounded operator. Later, Asgari and Rahimi [2] applied the atomic systems theory to the situation of g-frames, thereby leading to the notion of g-frames for operators.

Recently, Najati et al. [13] generalized the notion of frames for operators from Hilbert spaces to Hilbert C^*-modules and studied some of their properties. In this paper we give a generalization of g-frames for operators in Hilbert C^*-modules and we extend some results in [2] to Hilbert C^*-modules.

The paper is organized in the following manner. In Section 2, we recall the definitions and basic properties.

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2010 *AMS Mathematics Subject Classification:* 46L99, 42C15.

Section 3 is devoted to the introduction of g-frames for operators in Hilbert C^*-modules. In this section we characterize g-frames for operators in several aspects and we also study the erasure of g-frames for operators. In Section 4, we investigate the duality of g-frames for operators in Hilbert C^*-modules.

2. Preliminaries

In the following we briefly recall some definitions and basic properties of operators and g-frames in Hilbert C^*-modules.

Throughout this paper, the symbols $\mathbb{J}$, $\mathbb{C}$, and $\mathcal{A}$ refer, respectively, to a finite or countable index set, the field of complex numbers, and a unital C^*-algebra with identity $1_{\mathcal{A}}$.

Definition 2.1 *(see [12]) A pre-Hilbert C^*-module over $\mathcal{A}$ or, simply, a pre-Hilbert $\mathcal{A}$-module, is a left $\mathcal{A}$-module $\mathscr{H}$ with a sesquilinear form $\langle\cdot,\cdot\rangle : \mathscr{H}\times\mathscr{H}\to\mathcal{A}$, called an $\mathcal{A}$-valued inner product, that possesses the following properties:*

(i) $\langle f, f\rangle \geq 0$ for all $f\in\mathscr{H}$ and $\langle f, f\rangle = 0$ if and only if $f=0$;

(ii) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g\in\mathscr{H}$;

(iii) $\langle af+g, h\rangle = a\langle f, h\rangle + \langle g, h\rangle$ for all $a\in\mathcal{A}$, $f, g, h\in\mathscr{H}$;

(iv) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$ whenever $\lambda\in\mathbb{C}$ and $f, g\in\mathscr{H}$.

For $f\in\mathscr{H}$, we define a norm on $\mathscr{H}$ by $\|f\|_{\mathscr{H}} = \|\langle f, f\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$. If $\mathscr{H}$ is complete with respect to this norm, it is called a Hilbert C^*-module over $\mathcal{A}$ or a Hilbert $\mathcal{A}$-module.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T:\mathscr{H}\to\mathscr{K}$ (not necessarily linear or bounded) is said to be adjointable if there exists a map $T^*:\mathscr{K}\to\mathscr{H}$ such that $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f\in\mathscr{H}$ and $g\in\mathscr{K}$.

From now on, we assume that $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ and $\{\mathscr{W}_j\}_{j\in\mathbb{J}}$ are two sequences of Hilbert $\mathcal{A}$-modules. We also reserve the notation $\mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and $\mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\mathscr{H})$ is abbreviated to $\mathrm{End}_{\mathcal{A}}^*(\mathscr{H})$.

For a unital C^*-algebra $\mathcal{A}$, let $\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ be the Hilbert $\mathcal{A}$-module defined by

$$\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}) = \left\{\{g_j\}_{j\in\mathbb{J}} : g_j\in\mathscr{V}_j, \sum_{j\in\mathbb{J}}\langle g_j, g_j\rangle \text{ converges in } \|\cdot\|\right\}.$$

Definition 2.2 *(see [16]) Let $T\in\mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\mathscr{K})$, then an adjointable operator $T^\dagger\in\mathrm{End}_{\mathcal{A}}^*(\mathscr{K},\mathscr{H})$ is called the Moore–Penrose inverse of T if*

$$TT^\dagger T = T,\ T^\dagger TT^\dagger = T^\dagger,\ (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T. \tag{2.1}$$

It has been proven that an adjointable operator between two Hilbert C^*-modules admits a Moore–Penrose inverse if and only if it has closed range (see [16, Theorem 2.2]).

Definition 2.3 *(see [7]) A sequence $\{f_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathcal{A}$-module $\mathscr{H}$ is said to be a frame for $\mathscr{H}$ if there exist two constants $A, B>0$ such that*

$$A\langle f, f\rangle \leq \sum_{j\in\mathbb{J}}\langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle \tag{2.2}$$

holds for every $f \in \mathscr{H}$. The numbers A and B are called frame bounds.

Definition 2.4 *(see [13]) Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$. A family $\{f_j\}_{j\in\mathbb{J}} \subseteq \mathscr{H}$ is called a K-frame for $\mathscr{H}$ if there exist constants $A_1,\, B_1 > 0$ such that*

$$A_1\langle K^*f,\, K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle f,\, f_j\rangle\langle f_j,\, f\rangle \leq B_1\langle f,\, f\rangle, \quad \forall\, f \in \mathscr{H}. \tag{2.3}$$

The numbers A_1 and B_1 are called K-frame bounds.

(Throughout the paper, the sums like those in the middles of (2.2) and (2.3) are assumed to be convergent in the norm sense.)

Definition 2.5 *(see [11]) One calls a sequence $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ if there exist two constants $C_1,\, D_1 > 0$ such that*

$$C_1\langle f,\, f\rangle \leq \sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle \leq D_1\langle f,\, f\rangle \tag{2.4}$$

holds for every $f \in \mathscr{H}$. We call C_1 and D_1 the g-frame bounds. The g-frame $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be λ-tight if $C_1 = D_1 = \lambda$. The sequence $\{\Lambda_j\}_{j\in\mathbb{J}}$ is called a g-Bessel sequence with g-Bessel bound D_1 if we only require the right-hand inequality of (2.4).

Definition 2.6 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. The operator $T_\Lambda : \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}) \to \mathscr{H}$ defined by*

$$T_\Lambda\{g_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}}\Lambda_j^* g_j \tag{2.5}$$

is called the synthesis operator.

The g-frame operator $S_\Lambda : \mathscr{H} \to \mathscr{H}$ is defined by

$$S_\Lambda f = \sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f, \tag{2.6}$$

which is a positive and self-adjoint operator. Moreover, if $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame, then S_Λ is invertible. This provides for all $f \in \mathscr{H}$ the reconstruction formula as follows:

$$\begin{aligned}\sum_{j\in\mathbb{J}} S_\Lambda^{-1}\Lambda_j^*\Lambda_j f &= S_\Lambda^{-1}S_\Lambda f = f\\ &= S_\Lambda S_\Lambda^{-1} f = \sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j S_\Lambda^{-1} f.\end{aligned} \tag{2.7}$$

The following lemmas will be used to prove our main results.

Lemma 2.7 *(see [15]) Let $\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)$ for all $j \in \mathbb{J}$, then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ if and only if there exist two constants $C_2, D_2 > 0$ such that*

$$C_2\|f\|^2 \leq \left\|\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle\right\| \leq D_2\|f\|^2, \quad \forall f \in \mathscr{H}. \tag{2.8}$$

Lemma 2.8 *(see [5]) Let $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ be Hilbert $\mathcal{A}$-modules. Also let $T' \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{G}, \mathscr{F})$ and $T \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{E}, \mathscr{F})$ with $\overline{Ran}(T^*)$ orthogonally complemented. The following statements are equivalent:*

(1) $T'T'^* \leq \lambda TT^*$ for some $\lambda > 0$.

(2) There exists $\mu > 0$ such that $\|T'^*z\| \leq \mu\|T^*z\|$ for all $z \in \mathscr{F}$.

(3) There exists $D \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{G}, \mathscr{E})$ such that $T' = TD$, i.e. $TX = T'$ has a solution.

(4) $Ran(T') \subseteq Ran(T)$.

3. G-frames for operators in Hilbert C^*-modules

In this section we introduce g-frames for operators in Hilbert C^*-modules and study conditions for a family of adjointable operators on a Hilbert C^*-module to be a g-frame for operator, as well as conditions for removing an element from a g-frame for operator to again obtain a g-frame for operator. Let us begin with

Definition 3.1 *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ and $\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)$ for all $j \in \mathbb{J}$, then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is said to be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ if there exist two constants $C, D > 0$ such that*

$$C\langle K^*f, K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle, \quad \forall f \in \mathscr{H}. \tag{3.1}$$

The numbers C and D are called K-g-frame bounds. Particularly, if

$$C\langle K^*f, K^*f\rangle = \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle, \quad \forall f \in \mathscr{H}, \tag{3.2}$$

then we call $\{\Lambda_j\}_{j\in\mathbb{J}}$ a C-tight K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

Remark 3.2 *(1) If $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ is a surjective operator, then every K-g-frame $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ is a g-frame. Indeed, if we let C, D be the K-g-frame bounds, then for any $f \in \mathscr{H}$ we have*

$$C\|(KK^*)^{-1}\|^{-1}\langle f, f\rangle \leq C\langle K^*f, K^*f\rangle \leq \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle.$$

(2) Every adjointable operator K on a finitely or countably generated Hilbert C^*-module $\mathscr{H}$ admits a K-g-frame. To see this, let $\{f_j\}_{j\in\mathbb{J}}$ be a frame of $\mathscr{H}$ with bounds A, B. For each $j \in \mathbb{J}$, let $\mathscr{V}_j = \mathcal{A}$ and define adjointable operator $\Lambda_j : \mathscr{H} \to \mathscr{V}_j$, $\Lambda_j f = \langle f, f_j\rangle$. It is easy to check that

$$A\langle f, f\rangle \leq \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle \leq B\langle f, f\rangle, \quad \forall f \in \mathscr{H}.$$

Now let $\Gamma_j = \Lambda_j K^*$ *for all* $j \in \mathbb{J}$*, then*

$$A\langle K^*f,\, K^*f\rangle \leq \sum_{j\in\mathbb{J}} \langle \Lambda_j K^*f,\, \Lambda_j K^*f\rangle = \sum_{j\in\mathbb{J}} \langle \Gamma_j f,\, \Gamma_j f\rangle \leq B\|K^*\|^2\langle f,\, f\rangle,$$

showing that $\{\Gamma_j\}_{j\in\mathbb{J}}$ *is a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

Example 3.3 *Let* l^∞ *be the set of all bounded complex-valued sequences. For any* $u = \{u_j\}_{j\in\mathbb{N}}$, $v = \{v_j\}_{j\in\mathbb{N}} \in l^\infty$*, we define*

$$uv = \{u_j v_j\}_{j\in\mathbb{N}},\ u^* = \{\overline{u}_j\}_{j\in\mathbb{N}},\ \|u\| = \max_{j\in\mathbb{N}} |u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|\cdot\|\}$ *is a* C^**-algebra.*

Let $\mathscr{H} = C_0$ *be the set of all sequences converging to zero. For any* $u, v \in \mathscr{H}$ *we define*

$$\langle u,\, v\rangle = uv^* = \{u_j \overline{v}_j\}_{j\in\mathbb{N}}.$$

Then $\mathscr{H}$ *is a Hilbert* $\mathcal{A}$*-module.*

Now let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ *and* $\mathbb{J} = \mathbb{N}$*. Let also* $\{e_j\}_{j\in\mathbb{J}}$ *be the standard orthogonal basis of* $\mathscr{H}$*. For each* $j \in \mathbb{J}$*, set* $\mathscr{V}_j = \overline{span}\{e_j\}$*, and define adjointable operator*

$$\Lambda_j : \mathscr{H} \to \mathscr{V}_j, \quad \Lambda_j f = \langle f,\, e_j\rangle e_j,$$

then for every $f \in \mathscr{H}$ *we have*

$$\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle = \sum_{j\in\mathbb{J}} \langle f,\, e_j\rangle\langle e_j,\, e_j\rangle\langle e_j,\, f\rangle = \{f_j\}_{j\in\mathbb{J}}\{\overline{f}_j\}_{j\in\mathbb{J}} = \langle f,\, f\rangle.$$

Fix $N \in \mathbb{N}$ *and define*

$$K : \mathscr{H} \to \mathscr{H}, \quad Ke_j = \begin{cases} je_j & \text{if } j \leq N, \\ 0 & \text{if } j > N. \end{cases}$$

It is easy to check that K *is adjointable and satisfies*

$$K^*e_j = \begin{cases} je_j & \text{if } j \leq N, \\ 0 & \text{if } j > N. \end{cases}$$

For any $f \in \mathscr{H}$*, let* $f = \sum_{j=1}^{\infty} c_j e_j$*, then*

$$\langle K^*f,\, K^*f\rangle = \left\langle \sum_{j=1}^{N} jc_j e_j,\, \sum_{j=1}^{N} jc_j e_j \right\rangle = \sum_{j=1}^{N} j^2 \langle c_j,\, c_j\rangle.$$

Hence

$$\frac{1}{N^2}\langle K^*f,\, K^*f\rangle = \sum_{j=1}^{N} (\frac{j}{N})^2 \langle c_j,\, c_j\rangle \leq \sum_{j=1}^{\infty} \langle c_j,\, c_j\rangle = \langle f,\, f\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle.$$

This shows that $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *with bounds* $\frac{1}{N^2}, 1$.

The following proposition gives a necessary and sufficient condition for tight g-frames to be tight K-g-frames in Hilbert C^*-modules.

Proposition 3.4 *Let* $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be an* A*-tight g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and* $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$*; then* $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a tight* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *if and only if there exists a number* $M > 0$ *such that* $KK^* = M \cdot \mathrm{Id}_{\mathscr{H}}$*.*

Proof We assume first that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a B-tight K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$; then for any $f \in \mathscr{H}$ we have

$$B\langle K^*f, K^*f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle = A\langle f, f\rangle.$$

Thus, $\langle KK^*f, f\rangle = \frac{A}{B}\langle f, f\rangle$ and polarization formula shows that $\langle KK^*f, g\rangle = \frac{A}{B}\langle f, g\rangle$ for any $f, g \in \mathscr{H}$. Consequently, $KK^* = \frac{A}{B}\cdot \mathrm{Id}_{\mathscr{H}}$.

For the other implication, let $KK^* = M \cdot \mathrm{Id}_{\mathscr{H}}$ for some positive number M; then for any $f \in \mathscr{H}$ we have

$$\sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle = \frac{A}{M}\langle Mf, f\rangle = \frac{A}{M}\langle KK^*f, f\rangle = \frac{A}{M}\langle K^*f, K^*f\rangle.$$

This shows that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a $\frac{A}{M}$-tight K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. □

Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$; by $F_K(\mathscr{H})$ and $F^T_K(\mathscr{H})$ we denote the sets of all K-g-frames and all tight K-g-frames of $\mathscr{H}$ respectively. Our next result presents one relation between two sets of tight K-g-frames and the involved operators.

Proposition 3.5 *Let* $\mathscr{H}$ *be a finitely or countably generated Hilbert* $\mathcal{A}$*-module and* $K_1, K_2 \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$*, then* $F^T_{K_1}(\mathscr{H}) \subseteq F^T_{K_2}(\mathscr{H})$ *if and only if there exists a constant* $A > 0$ *such that* $AK_2K_2^* = K_1K_1^*$*.*

Proof Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with g-frame operator S_Λ; then for all $f \in \mathscr{H}$ we have $\langle f, f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j S_\Lambda^{-\frac{1}{2}} f, \Lambda_j S_\Lambda^{-\frac{1}{2}} f\rangle$. Therefore,

$$\langle K_1^*f, K_1^*f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*f, \Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*f\rangle, \tag{3.3}$$

showing that $\{\Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*\}_{j\in\mathbb{J}}$ is a tight K_1-g-frame of $\mathscr{H}$. Assume that $F^T_{K_1}(\mathscr{H}) \subseteq F^T_{K_2}(\mathscr{H})$; then $\{\Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*\}_{j\in\mathbb{J}}$ is also a tight K_2-g-frame of $\mathscr{H}$. Hence there exists $A > 0$ such that

$$A\langle K_2^*f, K_2^*f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*f, \Lambda_j S_\Lambda^{-\frac{1}{2}} K_1^*f\rangle. \tag{3.4}$$

Altogether we obtain $A\langle K_2^*f, K_2^*f\rangle = \langle K_1^*f, K_1^*f\rangle$, equivalently, $AK_2K_2^* = K_1K_1^*$.

Suppose now that $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ is a tight K_1-g-frame of $\mathscr{H}$; then there exists $A_1 > 0$ such that $A_1\langle K_1^*f, K_1^*f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle$ for all $f \in \mathscr{H}$. Thus

$$AA_1\langle K_2^*f, K_2^*f\rangle = A_1\langle K_1^*f, K_1^*f\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle.$$

Therefore, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a AA_1-tight K_2-g-frame of $\mathscr{H}$ and, consequently, $F^T_{K_1}(\mathscr{H}) \subseteq F^T_{K_2}(\mathscr{H})$. □

We also have the relation between two sets of K-g-frames and the ranges of the involved operators.

Proposition 3.6 *Let $K_1, K_2 \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$. If $Ran(K_2) \subseteq Ran(K_1)$ and $\overline{Ran}(K_1^*)$ is orthogonally complemented, then $F_{K_1}(\mathscr{H}) \subseteq F_{K_2}(\mathscr{H})$.*

Proof By Lemma 2.8 we know that there exists $\mu > 0$ such that $K_2K_2^* \leq \mu K_1K_1^*$. Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a K_1-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with bounds C, D; then

$$\frac{C}{\mu}\langle K_2^*f,\, K_2^*\rangle \leq C\langle K_1^*f,\, K_1^*\rangle \leq \sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle \leq D\langle f,\, f\rangle, \quad \forall\, f \in \mathscr{H}.$$

This shows that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K_2-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with bounds $\frac{C}{\mu}$, D. Therefore, we have $F_{K_1}(\mathscr{H}) \subseteq F_{K_2}(\mathscr{H})$. □

The converse of the above proposition remains true if we replace $F_{K_1}(\mathscr{H})$ by $F^T_{K_1}(\mathscr{H})$.

Proposition 3.7 *Let $K_1, K_2 \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$. If $F^T_{K_1}(\mathscr{H}) \subseteq F_{K_2}(\mathscr{H})$ and $\overline{Ran}(K_1^*)$ is orthogonally complemented, then $Ran(K_2) \subseteq Ran(K_1)$.*

Proof Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a C-tight K_1-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Since $F^T_{K_1}(\mathscr{H}) \subseteq F_{K_2}(\mathscr{H})$, it follows that there exists $D > 0$ such that $\|K_2^*f\|^2 \leq \frac{C}{D}\|K_1^*f\|^2$ for all $f \in \mathscr{H}$. By Lemma 2.8 we get $Ran(K_2) \subseteq Ran(K_1)$. □

The following is a simple but very useful representation for K-g-frames.

Theorem 3.8 *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ and let $\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{K}_j)$ for each $j \in \mathbb{J}$. Suppose that the operator $T : \mathscr{H} \to \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ is given by $Tf = \{\Lambda_j f\}_{j\in\mathbb{J}}$ and $\overline{Ran(T)}$ is orthogonally complemented. Then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ if and only if there exist constants C, $D > 0$ such that*

$$C\|K^*f\|^2 \leq \left\|\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle\right\| \leq D\|f\|^2, \quad \forall\, f \in \mathscr{H}. \tag{3.5}$$

Proof Evidently, every K-g-frame of $\mathscr{H}$ satisfies (3.5).

For the converse, we suppose that a sequence $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ fulfills (3.5). For all $f \in \mathscr{H}$, the left-hand inequality of (3.5) gives $\|K^*f\|^2 \leq \frac{1}{C}\|Tf\|^2$. Then Lemma 2.8 implies that there exists a constant $\mu > 0$ such that $KK^* \leq \mu T^*T$ and hence

$$\frac{1}{\mu}\langle K^*f,\, K^*f\rangle \leq \langle Tf,\, Tf\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle, \quad \forall\, f \in \mathscr{H}.$$

To complete the proof, it remains to show that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence. For any $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ and any $\mathbb{I} \subset \mathbb{J}$, the right-hand inequality of (3.5) leads to

$$\begin{aligned}\left\|\sum_{j\in\mathbb{I}}\Lambda_j^*(g_j)\right\|^2 &= \sup_{f\in\mathscr{H},\,\|f\|=1}\left\|\left\langle\sum_{j\in\mathbb{I}}\Lambda_j^*(g_j),\, f\right\rangle\right\|^2 = \sup_{f\in\mathscr{H},\,\|f\|=1}\left\|\sum_{j\in\mathbb{I}}\langle\Lambda_j^*(g_j),\, f\rangle\right\|^2\\ &\leq \sup_{f\in\mathscr{H},\,\|f\|=1}\left\|\sum_{j\in\mathbb{I}}\langle g_j,\, g_j\rangle\right\|\left\|\sum_{j\in\mathbb{I}}\langle\Lambda_j f,\, \Lambda_j f\rangle\right\| \leq D\left\|\sum_{j\in\mathbb{I}}\langle g_j,\, g_j\rangle\right\|.\end{aligned}$$

Thus the series $\sum_{j\in\mathbb{J}}\Lambda_j^*(g_j)$ converges in $\mathscr{H}$ unconditionally. Since

$$\langle Tf,\, \{g_j\}_{j\in\mathbb{J}}\rangle = \sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, g_j\rangle = \Big\langle f,\, \sum_{j\in\mathbb{J}}\Lambda_j^*(g_j)\Big\rangle,$$

T is adjointable. Now for any $f\in\mathscr{H}$ we have

$$\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle = \langle Tf,\, Tf\rangle \le \|T\|^2\langle f,\, f\rangle,$$

as desired. □

Using Theorem 3.8 we can easily prove the following result, which offers a condition for getting a K-g-frame from a g-frame.

Proposition 3.9 *Let $\{\Lambda_j\in \mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with bounds C,D and the synthesis operator T_Λ. Let $K\in\mathrm{End}_{\mathcal{A}}^*(\mathscr{H})$ and $\overline{Ran}(T_\Lambda^*)$ be orthogonally complemented. Then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.*

Proof Let S_Λ be the g-frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$; then for every $f\in\mathscr{H}$ the reconstruction formula gives $Kf=\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j S_\Lambda^{-1}Kf$ and so

$$\begin{aligned}
\|K^*f\|^2 &= \sup_{h\in\mathscr{H},\,\|h\|=1}\|\langle K^*f,\, h\rangle\|^2 = \sup_{h\in\mathscr{H},\,\|h\|=1}\Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j S_\Lambda^{-1}Kh,\, \Lambda_j f\rangle\Big\|^2\\
&\le \sup_{h\in\mathscr{H},\,\|h\|=1}\Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j S_\Lambda^{-1}Kh,\, \Lambda_j S_\Lambda^{-1}Kh\rangle\Big\|\,\Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle\Big\|\\
&\le \sup_{h\in\mathscr{H},\,\|h\|=1} D\|S_\Lambda^{-1}Kh\|^2\Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle\Big\|\\
&\le DC^{-2}\|K\|^2\Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle\Big\|.
\end{aligned}$$

It follows that

$$D^{-1}C^2\|K\|^{-2}\|K^*f\|^2 \le \Big\|\sum_{j\in\mathbb{J}}\langle \Lambda_j f,\, \Lambda_j f\rangle\Big\|.$$

□

In the two next theorems, we generalized the results in [2] for K-g-frames in Hilbert spaces to Hilbert C^*-modules.

Theorem 3.10 *Let $K\in\mathrm{End}_{\mathcal{A}}^*(\mathscr{H})$ and let $\Lambda_j\in\mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\mathscr{K}_j)$ for each $j\in\mathbb{J}$. Suppose that the operator $T:\mathscr{H}\to\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ is given by $Tf=\{\Lambda_j f\}_{j\in\mathbb{J}}$ and $\overline{Ran(T)}$ is orthogonally complemented. Then the following statements are equivalent:*

(i) $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

(ii) $\{\Lambda_j\}_{j\in\mathbb{J}}$ *is a g-Bessel sequence for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and there exists a g-Bessel sequence* $\{\Gamma_j\}_{j\in\mathbb{J}}$ *for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *such that*

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^*\Gamma_j f, \quad \forall\, f \in \mathscr{H}. \tag{3.6}$$

(iii) The series $\sum_{j\in\mathbb{J}} \Lambda_j^* g_j$ *converges in* $\mathscr{H}$ *for all* $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ *and there exists* $M > 0$ *such that for every* $f \in \mathscr{H}$*, there is* $\{g_{j,f}\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ *satisfying*

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^* g_{j,f} \textit{ and } \sum_{j\in\mathbb{J}} \langle g_{f,j},\, g_{f,j}\rangle \le M\langle f,\, f\rangle.$$

Proof (i)$\Rightarrow$(ii). Let C and D be the bounds of $\{\Lambda_j\}_{j\in\mathbb{J}}$; then for every $f \in \mathscr{H}$,

$$C\langle K^*f,\, K^*f\rangle \le \sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle \le D\langle f,\, f\rangle.$$

The left-hand inequality is equivalent to $CKK^* \le T^*T$. By Lemma 2.8 we know that there exists $\Gamma \in \mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\, \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}))$ such that $K = T^*\Gamma$. Let P_n be the projection on $\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ that maps each element to its n-th component, i.e. $P_n g = \{u_j\}_{j\in\mathbb{J}}$, where

$$u_j = \begin{cases} g_n & \text{if } j = n, \\ 0 & \text{if } j \ne n, \end{cases}$$

for each $g = \{g_j\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$. If we define $\Gamma_j = P_j\Gamma$, then for each $f \in \mathscr{H}$ we have

$$\sum_{j\in\mathbb{J}} \langle \Gamma_j f,\, \Gamma_j f\rangle = \sum_{j\in\mathbb{J}} \langle P_j\Gamma f,\, P_j\Gamma f\rangle = \sum_{j\in\mathbb{J}} \langle (\Gamma f)_j,\, (\Gamma f)_j\rangle = \langle \Gamma f,\, \Gamma f\rangle \le \|\Gamma\|^2\langle f,\, f\rangle.$$

Hence $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Now

$$Kf = T^*\Gamma f = \sum_{j\in\mathbb{J}} \Lambda_j^*(\Gamma f)_j = \sum_{j\in\mathbb{J}} \Lambda_j^*(P_j\Gamma f) = \sum_{j\in\mathbb{J}} \Lambda_j^*\Gamma_j f.$$

(ii)$\Rightarrow$(iii). Since $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence, $\sum_{j\in\mathbb{J}} \Lambda_j^* g_j$ converges with respect to the norm topology for all $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$. Let D be the g-Bessel bound of $\{\Gamma_j\}_{j\in\mathbb{J}}$ and taking $g_{j,f} = \Gamma_j f$ for all $j \in \mathbb{J}$ and all $f \in \mathscr{H}$ we get

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^* g_{j,f} \text{ and } \sum_{j\in\mathbb{J}} \langle g_{f,j},\, g_{f,j}\rangle = \sum_{j\in\mathbb{J}} \langle \Gamma_j f,\, \Gamma_j f\rangle \le D\langle f,\, f\rangle.$$

(iii)$\Rightarrow$(i). Since $\sum_{j\in\mathbb{J}} \Lambda_j^* g_j$ converges with respect to the norm topology for all $\{g_j\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ and $\{\Lambda_j f\}_{j\in\mathbb{J}} \in \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ for all $f \in \mathscr{H}$, we conclude, as in the proof of Theorem 3.8, that T is adjointable. Therefore,

$$\left\| \sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle \right\| = \|\langle Tf,\, Tf\rangle\| \le \|T\|^2\|f\|^2.$$

Now for every $f \in \mathscr{H}$ we compute

$$\begin{aligned}\|K^*f\|^2 &= \sup_{h\in\mathscr{H},\,\|h\|=1} \|\langle K^*f,\, h\rangle\|^2 = \sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, g_{j,f}\rangle\Big\|^2 \\ &\le \sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle\Big\| \, \Big\|\sum_{j\in\mathbb{J}} \langle g_{j,f},\, g_{j,f}\rangle\Big\| \\ &\le M \Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle\Big\|.\end{aligned}$$

Thus, $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$, by Theorem 3.8. □

Corollary 3.11 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be an A-tight K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with synthesis operator T_Λ. Suppose that $\overline{Ran}(T^*_\Lambda)$ is orthogonally complemented. Then there exists a g-Bessel sequence $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ such that $Kf = \sum_{j\in\mathbb{J}} \Lambda^*_j \Gamma_j f$ for all $f \in \mathscr{H}$ and*

$$\sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Gamma_j h,\, \Gamma_j h\rangle\Big\| \ge \frac{1}{A}. \tag{3.7}$$

Proof By Theorem 3.10, there exists a g-Bessel sequence $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ such that $Kf = \sum_{j\in\mathbb{J}} \Lambda^*_j \Gamma_j f$ for all $f \in \mathscr{H}$. A simple calculation shows that $K^*f = \sum_{j\in\mathbb{J}} \Gamma^*_j \Lambda_j f$. Now for all $f \in \mathscr{H}$ we obtain

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle\Big\| &= A\cdot \|K^*f\|^2 = A\cdot \Big\|\sum_{j\in\mathbb{J}} \Gamma^*_j \Lambda_j f\Big\|^2 \\ &= A\cdot \sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Gamma_j h\rangle\Big\|^2 \\ &\le A\cdot \sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Lambda_j f,\, \Lambda_j f\rangle\Big\| \, \Big\|\sum_{j\in\mathbb{J}} \langle \Gamma_j h,\, \Gamma_j h\rangle\Big\|.\end{aligned}$$

It follows that

$$\sup_{h\in\mathscr{H},\,\|h\|=1} \Big\|\sum_{j\in\mathbb{J}} \langle \Gamma_j h,\, \Gamma_j h\rangle\Big\| \ge \frac{1}{A}.$$

□

Theorem 3.12 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with synthesis operator T_Λ and $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$. Suppose that $\overline{Ran}(T^*_\Lambda)$ is orthogonally complemented. Then $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame if and only if there exists $\Gamma \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}),\, \mathscr{H})$ with $\overline{Ran}(\Gamma^*)$ orthogonally complemented such that $Ran(K) \subseteq Ran(\Gamma)$ and $\Lambda_j = P_j\Gamma^*$ for each $j \in \mathbb{J}$, where P_j is the projection on $\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})$ that maps each element to its j-th component.*

Proof Suppose first that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Taking $\Gamma = T_\Lambda$, then the definition of K-g-frames along with Lemma 2.8 gives $Ran(K) \subseteq Ran(\Gamma)$ and $\Lambda_j = P_j\Gamma^*$ for all $j \in \mathbb{J}$.

Conversely, assume that $\Lambda_j = P_j\Gamma^*$ for all $j \in \mathbb{J}$, where $\Gamma \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}), \mathscr{H})$ with $\overline{Ran}(\Gamma^*)$ orthogonally complemented and $Ran(K) \subseteq Ran(\Gamma)$. Then by Lemma 2.8, there exists $A > 0$ such that $AKK^* \leq \Gamma\Gamma^*$. Thus for all $f \in \mathscr{H}$ we have

$$A\|K^*f\|^2 \leq \|\Gamma^* f\|^2 = \left\|\sum_{j\in\mathbb{J}}\langle(\Gamma^*f)_j,\, (\Gamma^*f)_j\rangle\right\| = \left\|\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\, \Lambda_j f\rangle\right\|.$$

Now the conclusion follows from Theorem 3.8. $\square$

We conclude this section with the results showing that the removal of an element from a K-g-frame can leave again a K-g-frame.

Theorem 3.13 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with bounds A, B. Let $j_0 \in \mathbb{J}$ be given. Suppose that K has closed range and $Ran(K)$ is orthogonally complemented. We have the following results:*

(i) If $\overline{span}\{\Lambda_j^*(\mathscr{K}_j)\}_{j\in\mathbb{J}\setminus\{j_0\}} \subset Ran(K)$ and $Ran(\Lambda_{j_0}^*\Lambda_{j_0}) \perp Ran(K)$, then $\{\Lambda_j\}_{j\in\mathbb{J}\setminus\{j_0\}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

(ii) If $\overline{span}\{\Lambda_j^*(\mathscr{K}_j)\}_{j\in\mathbb{J}\setminus\{j_0\}} \subset Ran(K)$ and $A - \|K^\dagger\|^2\|\Lambda_{j_0}\|^2 > 0$, then $\{\Lambda_j\}_{j\in\mathbb{J}\setminus\{j_0\}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$, where $K^\dagger$ is the Moore–Penrose inverse of K.

Proof (i) For any $h \in Ran(K)$, since $h \in Ker(\Lambda_{j_0}^*\Lambda_{j_0})$, we have

$$A\langle K^*h,\, K^*h\rangle \leq \sum_{j\in\mathbb{J}}\langle\Lambda_j h,\, \Lambda_j h\rangle = \sum_{j\in\mathbb{J}\setminus\{j_0\}}\langle\Lambda_j h,\, \Lambda_j h\rangle.$$

Since $Ran(K)$ is orthogonally complemented, every $f \in \mathscr{H}$ has a decomposition as $f = f_1 + f_2$, where $f_1 \in Ran(K)$ and $f_2 \in (Ran(K))^\perp$. A direct calculation shows $K^*f_2 = 0$. We also have $f_2 \perp \overline{span}\{\Lambda_j^*(\mathscr{K}_j)\}_{j\in\mathbb{J}\setminus\{j_0\}}$. Therefore,

$$A\langle K^*f,\, K^*f\rangle = A\langle K^*f_1,\, K^*f_1\rangle \leq \sum_{j\in\mathbb{J}\setminus\{j_0\}}\langle\Lambda_j f_1,\, \Lambda_j f_1\rangle = \sum_{j\in\mathbb{J}\setminus\{j_0\}}\langle\Lambda_j f,\, \Lambda_j f\rangle.$$

(ii) Since K has closed range, every $h \in Ran(K)$ can be expressed as $h = KK^\dagger h = (KK^\dagger)^*h = (K^\dagger)^*K^*h$. Thus

$$\langle h,\, h\rangle = \langle(K^\dagger)^*K^*h,\, (K^\dagger)^*K^*h\rangle \leq \|(K^\dagger)^*\|^2\langle K^*h,\, K^*h\rangle = \|K^\dagger\|^2\langle K^*h,\, K^*h\rangle.$$

For all $f \in \mathscr{H}$, again applying the fact that $f = f_1 + f_2$ with $f_1 \in Ran(K)$ and $f_2 \in (Ran(K))^\perp$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{J}\setminus\{j_0\}}\langle\Lambda_j f,\, \Lambda_j f\rangle &= \sum_{j\in\mathbb{J}\setminus\{j_0\}}\langle\Lambda_j f_1,\, \Lambda_j f_1\rangle = \sum_{j\in\mathbb{J}}\langle\Lambda_j f_1,\, \Lambda_j f_1\rangle - \langle\Lambda_{j_0} f_1,\, \Lambda_{j_0} f_1\rangle \\
&\geq A\langle K^*f_1,\, K^*f_1\rangle - \|\Lambda_{j_0}\|^2\langle f_1,\, f_1\rangle \\
&\geq A\langle K^*f_1,\, K^*f_1\rangle - \|K^\dagger\|^2\|\Lambda_{j_0}\|^2\langle K^*f_1,\, K^*f_1\rangle \\
&= (A - \|K^\dagger\|^2\|\Lambda_{j_0}\|^2)\langle K^*f_1,\, K^*f_1\rangle \\
&= (A - \|K^\dagger\|^2\|\Lambda_{j_0}\|^2)\langle K^*f,\, K^*f\rangle.
\end{aligned}$$

$\square$

Theorem 3.14 *Let* $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and* $j_0 \in \mathbb{J}$ *be given. Let* $T_{\Lambda'}$ *be the synthesis operator of* $\{\Lambda_j\}_{j\in\mathbb{J}\backslash\{j_0\}}$ *and suppose that* $\overline{Ran}(T^*_{\Lambda'})$ *is orthogonally complemented. Then* $\{\Lambda_j\}_{j\in\mathbb{J}\backslash\{j_0\}}$ *is a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *if and only if there exists a g-Bessel sequence* $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and* $0 \leq \lambda < 1$ *such that*

$$\left\|K^*f - \sum_{j\in\mathbb{J}\backslash\{j_0\}} \Gamma_j^*\Lambda_j f\right\| \leq \lambda\|K^*f\|, \quad \forall f \in \mathscr{H}. \tag{3.8}$$

Proof We assume first that $\{\Lambda_j\}_{j\in\mathbb{J}\backslash\{j_0\}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Then by Theorem 3.10, there exists a g-Bessel sequence $\{\Theta_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}\backslash\{j_0\}}$ for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ such that $K^*f = \sum_{j\in\mathbb{J}\backslash\{j_0\}} \Theta_j^*\Lambda_j f$ for all $f \in \mathscr{H}$. For every $j \in \mathbb{J}$, let $\Gamma_j = \Theta_j$ if $j \in \mathbb{J}\backslash\{j_0\}$, and $\Gamma_j = 0$ if $j = j_0$. Then it is clear that $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Hence (3.8) holds for any given $\lambda \in [0, 1)$.

For the other implication, let $L_{\mathbb{J}\backslash\{j_0\}}f = \sum_{j\in\mathbb{J}\backslash\{j_0\}} \Gamma_j^*\Lambda_j f$ for any $f \in \mathscr{H}$. Then

$$\begin{aligned}\|L_{\mathbb{J}\backslash\{j_0\}}f\|^2 &= \sup_{g\in\mathscr{H},\, \|h\|=1} \left\|\sum_{j\in\mathbb{J}\backslash\{j_0\}} \langle\Lambda_j f, \Gamma_j h\rangle\right\|^2 \\ &\leq \sup_{h\in\mathscr{H},\, \|h\|=1} \left\|\sum_{j\in\mathbb{J}\backslash\{j_0\}} \langle\Lambda_j f, \Lambda_j f\rangle\right\| \left\|\sum_{j\in\mathbb{J}\backslash\{j_0\}} \langle\Gamma_j h, \Gamma_j h\rangle\right\| \leq D\left\|\sum_{j\in\mathbb{J}\backslash\{j_0\}} \langle\Lambda_j f, \Lambda_j f\rangle\right\|,\end{aligned} \tag{3.9}$$

where D is the g-Bessel bound of $\{\Gamma_j\}_{j\in\mathbb{J}}$. We also have

$$\|K^*f\| - \|L_{\mathbb{J}\backslash\{j_0\}}f\| \leq \|K^*f - L_{\mathbb{J}\backslash\{j_0\}}f\| \leq \lambda\|K^*f\|,$$

and hence

$$\|L_{\mathbb{J}\backslash\{j_0\}}f\| \geq (1-\lambda)\|K^*f\|. \tag{3.10}$$

This along with the inequality (3.9) yields

$$\frac{(1-\lambda)^2}{D}\|K^*f\|^2 \leq \left\|\sum_{j\in\mathbb{J}\backslash\{j_0\}} \langle\Lambda_j f, \Lambda_j f\rangle\right\|.$$

By Theorem 3.8, $\{\Lambda_j\}_{j\in\mathbb{J}\backslash\{j_0\}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. □

4. Duality of g-frames for operators in Hilbert C^*-modules

In this section we define a concept of dual by means of a bounded operator and the associated synthesis operators to investigate the relation between a g-frame for operator and a g-Bessel sequence with respect to different sequences of Hilbert C^*-modules.

Definition 4.1 *Let* $\{f_j\}_{j\in\mathbb{J}}$ *be a* K*-frame and* $\{g_j\}_{j\in\mathbb{J}}$ *be a Bessel sequence of* $\mathscr{H}$*. Then* $\{g_j\}_{j\in\mathbb{J}}$ *is called a dual* K*-frame of* $\{f_j\}_{j\in\mathbb{J}}$ *if*

$$Kf = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle g_j, \quad \forall f \in \mathscr{H}. \tag{4.1}$$

Inspired by the concept of Q-dual fusion frames in [9], we next introduce what we call V-dual K-g-frames in Hilbert C^*-modules.

Definition 4.2 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{W}_j)\}_{j\in\mathbb{J}}$ be a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{W}_j\}_{j\in\mathbb{J}}$ with synthesis operators T_Λ and T_Γ, respectively. Then $\{\Gamma_j\}_{j\in\mathbb{J}}$ is called a V-dual K-g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$ if there exists a bounded operator $V : \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}) \to \mathscr{L}^2(\{\mathscr{W}_j\}_{j\in\mathbb{J}})$ such that $T_\Gamma V T^*_\Lambda = K$.*

To derive our main result, we need the following lemma.

Lemma 4.3 *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H})$ and $\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)$ for all $j \in \mathbb{J}$. Suppose that $\{f_{jk}\}_{k\in I_j}$ is a frame for $\mathscr{V}_j$ with frame bounds A_j and B_j and let $0 < A = \inf_{j\in\mathbb{J}} A_j \le \sup_{j\in\mathbb{J}} B_j = B < \infty$. Then the following conditions are equivalent:*

(i) $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

(ii) $\{\Lambda^*_j f_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$ is a K-frame for $\mathscr{H}$.

Proof The claim (i)$\Leftrightarrow$(ii) follows from the fact that for any $f \in \mathscr{H}$ we have

$$\begin{aligned} A\sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle &\le \sum_{j\in\mathbb{J}} A_j\langle \Lambda_j f, \Lambda_j f\rangle \le \sum_{j\in\mathbb{J}}\sum_{k\in I_j}\langle \Lambda_j f, f_{jk}\rangle\langle f_{jk}, \Lambda_j f\rangle \\ &= \sum_{j\in\mathbb{J}}\sum_{k\in I_j}\langle f, \Lambda^*_j f_{jk}\rangle\langle \Lambda^*_j f_{jk}, f\rangle \le \sum_{j\in\mathbb{J}} B_j\langle \Lambda_j f, \Lambda_j f\rangle \le B\sum_{j\in\mathbb{J}}\langle \Lambda_j f, \Lambda_j f\rangle. \end{aligned}$$

□

Theorem 4.4 *Let $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ and $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{W}_j)\}_{j\in\mathbb{J}}$ be two sequences of adjointable operators on $\mathscr{H}$. For each $j \in \mathbb{J}$, let $\{f_{jk}\}_{k\in I_j}$ be a frame for $\mathscr{V}_j$ and $\{\widetilde{f}_{jk}\}_{k\in I_j}$ be a frame for $\mathscr{W}_j$ with frame bounds A_j, B_j and $\widetilde{A}_j, \widetilde{B}_j$, respectively. Suppose that $0 < A = \inf_{j\in\mathbb{J}} A_j \le B = \sup_{j\in\mathbb{J}} B_j$, $0 < \widetilde{A} = \inf_{j\in\mathbb{J}} \widetilde{A}_j \le \widetilde{B} = \sup_{j\in\mathbb{J}} \widetilde{B}_j$. Then the following conditions are equivalent:*

(i) $\{\Gamma^*_j \widetilde{f}_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$ is a dual K-frame of $\{\Lambda^*_j f_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$.

(ii) $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a V-dual K-g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$, where $V : \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}) \to \mathscr{L}^2(\{\mathscr{W}_j\}_{j\in\mathbb{J}})$ is defined by $V\{h_j\}_{j\in\mathbb{J}} = \big\{\sum_{k\in I_j}\langle h_j, f_{jk}\rangle \widetilde{f}_{jk}\big\}_{j\in\mathbb{J}}$.

Proof By Lemma 4.3, we have that $\{\Lambda^*_j f_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$ being a K-frame of $\mathscr{H}$ is equivalent to $\{\Lambda_j\}_{j\in\mathbb{J}}$ being a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. The same occurs with the sequences $\{\Gamma^*_j \widetilde{f}_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$. Thus it remains only to prove the duality.

Let us denote by T_j and $\widetilde{T}_j$ the synthesis operators associated to the frames $\{f_{jk}\}_{k\in I_j}$ and $\{\widetilde{f}_{jk}\}_{k\in I_j}$ respectively. Recall that T_j is an operator from $\ell^2(\mathcal{A})$ onto $\mathscr{V}_j$ defined by

$$T_j\{c_k\}_{k\in I_j} = \sum_{k\in I_j} c_k f_{jk},\ \forall\{c_k\}_{k\in I_j} \in \ell^2(\mathcal{A}) = \left\{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{J}} a_j a^*_j \text{ converges in } \|\cdot\|\right\}.$$

Now define the operator $V : \mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}}) \to \mathscr{L}^2(\{\mathscr{W}_j\}_{j\in\mathbb{J}})$ by

$$V\{h_j\}_{j\in\mathbb{J}} = \left\{\sum_{k\in I_j}\langle h_j,\, f_{jk}\rangle \widetilde{f}_{jk}\right\}_{j\in\mathbb{J}}.$$

Since

$$\sum_{j\in\mathbb{J}}\langle \widetilde{T}_j T_j^*(h_j),\, \widetilde{T}_j T_j^*(h_j)\rangle \le \sum_{j\in\mathbb{J}} \|\widetilde{T}_j\|^2\|T_j^*\|^2\langle h_j,\, h_j\rangle,$$

it follows that

$$\|V\{h_j\}_{j\in\mathbb{J}}\|^2 = \left\|\sum_{j\in\mathbb{J}}\langle \widetilde{T}_j T_j^*(h_j),\, \widetilde{T}_j T_j^*(h_j)\rangle\right\| \le \widetilde{B}B\left\|\sum_{j\in\mathbb{J}}\langle h_j,\, h_j\rangle\right\| = \widetilde{B}B\|\{h_j\}_{j\in\mathbb{J}}\|^2.$$

Therefore, V is well defined and also bounded.

Let T_Λ and T_Γ be the synthesis operators of $\{\Lambda_j\}_{j\in\mathbb{J}}$ and $\{\Gamma_j\}_{j\in\mathbb{J}}$ respectively; then

$$\begin{aligned} T_\Gamma V T_\Lambda^* f = T_\Gamma V\{\Lambda_j f\}_{j\in\mathbb{J}} &= T_\Gamma\left\{\sum_{k\in I_j}\langle \Lambda_j f,\, f_{jk}\rangle \widetilde{f}_{jk}\right\}_{j\in\mathbb{J}} \\ &= \sum_{j\in\mathbb{J}}\sum_{k\in I_j}\langle \Lambda_j f,\, f_{jk}\rangle \Gamma_j^*\widetilde{f}_{jk} = \sum_{j\in\mathbb{J}}\sum_{k\in I_j}\langle f,\, \Lambda_j^* f_{jk}\rangle \Gamma_j^*\widetilde{f}_{jk}. \end{aligned}$$

Hence the last term is equal to Kf for all $f \in \mathscr{H}$ if and only if $\{\Gamma_j^*\widetilde{f}_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$ is a dual K-frame of $\{\Lambda_j^* f_{jk}\}_{j\in\mathbb{J},\, k\in I_j}$. □

If we let $\mathscr{V}_j = \mathscr{W}_j$ for each $j \in \mathbb{J}$ and $V = \mathrm{Id}_{\mathscr{L}^2(\{\mathscr{V}_j\}_{j\in\mathbb{J}})}$ in definition 4.2, then

$$Kf = \sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f \tag{4.2}$$

for all $f \in \mathscr{H}$. In this case we call $\{\Gamma_j\}_{j\in\mathbb{J}}$ a dual K-g-frame of $\{\Lambda_j\}_{j\in\mathbb{J}}$.

Example 4.5 *Let $K \in \mathrm{End}_{\mathcal{A}}^*(\mathscr{H})$ and $T \in \mathrm{End}_{\mathcal{A}}^*(\mathscr{H})$ an invertible operator such that T^*K is surjective. Let also $T_j \in \mathrm{End}_{\mathcal{A}}^*(\mathscr{V}_j)$ be invertible for each $j \in \mathbb{J}$ and suppose that $0 < m = \inf_{j\in\mathbb{J}}\|T_j^{-1}\|^{-1} \le \sup_{j\in\mathbb{J}}\|T_j\| = M < \infty$. Let $\{\Lambda_j \in \mathrm{End}_{\mathcal{A}}^*(\mathscr{H},\, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ be a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ with bounds $A,\, B$. For all $j \in \mathbb{J}$, let $\Gamma_j = T_j\Lambda_j T$; then for any $f \in \mathscr{H}$ we have*

$$\begin{aligned} \sum_{j\in\mathbb{J}}\langle \Gamma_j f,\, \Gamma_j f\rangle = \sum_{j\in\mathbb{J}}\langle T_j\Lambda_j T f,\, T_j\Lambda_j T_j f\rangle &\ge \sum_{j\in\mathbb{J}}\|T_j^{-1}\|^{-2}\langle \Lambda_j T f,\, \Lambda_j T_j f\rangle \\ &\ge m^2\sum_{j\in\mathbb{J}}\langle \Lambda_j T f,\, \Lambda_j T_j f\rangle \ge m^2 A\langle K^*Tf,\, K^*Tf\rangle \\ &\ge m^2 A\|[(K^*T)^*(K^*T)]^{-1}\|^{-1}\langle f,\, f\rangle. \end{aligned}$$

Similarly we have $\sum_{j\in\mathbb{J}}\langle \Gamma_j f,\, \Gamma_j f\rangle \le M^2B\|T\|^2\langle f,\, f\rangle$. Therefore, $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. Denote the g-frame operator of $\{\Gamma_j\}_{j\in\mathbb{J}}$ by S_Γ and set $\Phi_j = T_j^*T_j\Lambda_j T S_\Gamma^{-1}T^*K^*$ for each $j \in \mathbb{J}$.

Then it is easy to check that $\{\Phi_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *is a g-Bessel sequence for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. *Now for any* $f \in \mathscr{H}$ *we obtain*

$$\begin{aligned}\sum_{j\in\mathbb{J}} \Phi_j^*\Lambda_j f &= K\sum_{j\in\mathbb{J}} TS_\Gamma^{-1}T^*\Lambda_j^*T_j^*T_j\Lambda_j f \\ &= KTS_\Gamma^{-1}\sum_{j\in\mathbb{J}} T^*\Lambda_j^*T_j^*T_j\Lambda_j f \\ &= KTS_\Gamma^{-1}S_\Gamma T^{-1}f = Kf.\end{aligned}$$

This implies that $\{\Phi_j\}_{j\in\mathbb{J}}$ *is a dual* K*-g-frame of* $\{\Lambda_j\}_{j\in\mathbb{J}}$.

Proposition 4.6 *Let* $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and* $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a dual* K*-g-frame of* $\{\Lambda_j\}_{j\in\mathbb{J}}$ *with synthesis operator* T_Γ. *Suppose that* $\overline{Ran}(T_\Gamma^*)$ *is orthogonally complemented. Then* $\{\Gamma_j\}_{j\in\mathbb{J}}$ *is also a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. *Moreover,* $\{\Lambda_j\}_{j\in\mathbb{J}}$ *and* $\{\Gamma_j\}_{j\in\mathbb{J}}$ *in (4.2) can be interchanged if and only if* $K = K^*$.

Proof Denote by D the g-Bessel bound of $\{\Lambda_j\}_{j\in\mathbb{J}}$; then for every $f \in \mathscr{H}$ we have

$$\begin{aligned}\|K^*f\|^2 &= \sup_{h\in\mathscr{H},\,\|h\|=1}\|\langle K^*f,\, h\rangle\|^2 = \sup_{h\in\mathscr{H},\,\|h\|=1}\left\|\sum_{j\in\mathbb{J}}\langle\Gamma_j f,\, \Lambda_j h\rangle\right\|^2 \\ &\le \sup_{h\in\mathscr{H},\,\|h\|=1}\left\|\sum_{j\in\mathbb{J}}\langle\Gamma_j f,\, \Gamma_j f\rangle\right\|\left\|\sum_{j\in\mathbb{J}}\langle\Lambda_j h,\, \Lambda_j h\rangle\right\| \le D\left\|\sum_{j\in\mathbb{J}}\langle\Gamma_j f,\, \Gamma_j f\rangle\right\|.\end{aligned}$$

By Theorem 3.8, $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. The "Moreover" part follows immediately from the observation that for all $f, g \in \mathscr{H}$ we have

$$\langle K^*f,\, g\rangle = \langle f,\, Kg\rangle = \left\langle f,\, \sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j g\right\rangle = \left\langle\sum_{j\in\mathbb{J}}\Lambda_j^*\Gamma_j f,\, g\right\rangle.$$

□

Proposition 4.7 *Let* $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *with bounds* C, D *and* $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a dual* K*-g-frame of* $\{\Lambda_j\}_{j\in\mathbb{J}}$. *If* K *is right-invertible, then* $\{\Gamma_j\}_{j\in\mathbb{J}}$ *is a g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$.

Proof Assume that $KT = \mathrm{Id}_{\mathscr{H}}$. Then $f = \sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j Tf$ for all $f \in \mathscr{H}$. Now

$$\begin{aligned}\|f\|^2 &= \left\|\left\langle\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j Tf,\, f\right\rangle\right\|^2 = \left\|\sum_{j\in\mathbb{J}}\langle\Lambda_j Tf,\, \Gamma_j f\rangle\right\|^2 \\ &\le \left\|\sum_{j\in\mathbb{J}}\langle\Lambda_j Tf,\, \Lambda_j Tf\rangle\right\|\left\|\sum_{j\in\mathbb{J}}\langle\Gamma_j f,\, \Gamma_j f\rangle\right\| \le D\|T\|^2\left\|\sum_{j\in\mathbb{J}}\langle\Gamma_j f,\, \Gamma_j f\rangle\right\|.\end{aligned}$$

By Lemma 2.7, $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$. □

Proposition 4.8 *Let* $\{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a* K*-g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{V}_j\}_{j\in\mathbb{J}}$ *and* $\{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathscr{H}, \mathscr{V}_j)\}_{j\in\mathbb{J}}$ *be a dual* K*-g-frame of* $\{\Lambda_j\}_{j\in\mathbb{J}}$*. For each* $j \in \mathbb{J}$*, let* $(\{f_{jk}\}_{k\in I_j}, \{g_{jk}\}_{k\in I_j})$ *be a pair of dual frames of* $\mathscr{V}_j$ *with frame bounds* A_j, B_j *and* $\widetilde{A}_j$, $\widetilde{B}_j$*, respectively. Suppose that* $0 < A = \inf_{j\in\mathbb{J}} A_j \le B = \sup_{j\in\mathbb{J}} B_j$, $0 < \widetilde{A} = \inf_{j\in\mathbb{J}} \widetilde{A}_j \le \widetilde{B} = \sup_{j\in\mathbb{J}} \widetilde{B}_j$*. Then* $\{\Gamma_j^* g_{jk}\}_{j\in\mathbb{J}, k\in I_j}$ *is a dual* K*-frame of* $\{\Lambda_j^* f_{jk}\}_{j\in\mathbb{J}, k\in I_j}$*.*

Proof By Lemma 4.3, $\{\Lambda_j^* f_{jk}\}_{j\in\mathbb{J}, k\in I_j}$ is a K-frame of $\mathscr{H}$. For all $f \in \mathscr{H}$, since

$$\sum_{j\in\mathbb{J}} \sum_{k\in I_j} \langle f, \Gamma_j^* g_{jk}\rangle \langle \Gamma_j^* g_{jk}, f\rangle \le \sum_{j\in\mathbb{J}} \widetilde{B}_j \langle \Gamma_j f, \Gamma_j f\rangle \le \widetilde{B} \sum_{j\in\mathbb{J}} \langle \Gamma_j f, \Gamma_j f\rangle,$$

we conclude that $\{\Gamma_j^* g_{jk}\}_{j\in\mathbb{J}, k\in I_j}$ is a Bessel sequence of $\mathscr{H}$. Now the result follows from the following equality:

$$Kf = \sum_{j\in\mathbb{J}} \Gamma_j^* \Lambda_j f = \sum_{j\in\mathbb{J}} \Gamma_j^* \Bigg(\sum_{k\in I_j} \langle \Lambda_j f, f_{jk}\rangle g_{jk} \Bigg) = \sum_{j\in\mathbb{J}} \sum_{k\in I_j} \langle f, \Lambda_j^* f_{jk}\rangle \Gamma_j^* g_{jk}$$

for all $f \in \mathscr{H}$. □

Acknowledgments

We would like to thank the referees for their many very helpful comments and suggestions that helped us improve the presentation of this paper. The first author was partially supported by the Natural Science Foundation of Jiangxi, China (Grant no. 20151BAB201007) and the National Natural Science Foundation of China (Grant no. 11271148).

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