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The M[–] and –[M] functors and five short lemma in H_v-modules

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Abstract: The largest class of multivalued systems satisfying the module-like axioms are the H_v-modules. The main tools concerning the class of H_v-modules with the ordinary modules are the fundamental relations. Based on the relation ε^*, exact sequences in H_v-modules are defined. In this paper, we introduce the H_v-module $M[A]$ and determine its heart and the connection between equivalence relations $\varepsilon^*_{M[A]}$ and ε^*_A. Moreover, we define the $M[-]$ and $-[M]$ functors and investigate the exactness and some concepts related to them. Finally, we prove the five short lemma in H_v-modules.

Key words: H_v-module, exact sequence, five short lemma, weak equality, fundamental relation ε^*

1. Introduction

A hyperstructure (or hypergroupoid) is a nonempty set H together with a hyperoperation defined on H, that is, a mapping of $H \times H$ into the family of nonempty subsets of H. In 1934, Marty introduced the concept of a hypergroup [12] as a nonempty set H equipped with a hyperoperation $* : H \times H \longrightarrow \mathcal{P}^*(H)$ that satisfies the associative law: $(x * y) * z = x * (y * z)$ for every $x, y, z \in H$ and the reproduction axiom is valid, i.e. $x * H = H * x = H$ for every $x \in H$; it means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x * u$ and $y \in v * x$. If A, B are nonempty subsets of H then $A * B$ is given by $A * B = \bigcup_{a \in A, b \in B} a * b$. Moreover, $a * A$ is used for $\{x\} * A$ and $A * x$ for $A * \{x\}$. Several books have been written to date on hyperstructures [2, 3, 9, 15]. The concept of H_v-structures as a larger class than the well-known hyperstructures was introduced by Vougiouklis at the Fourth Congress of AHA (Algebraic Hyperstructures and Applications) [16], where the axioms are replaced by the weak ones, that is, instead of the equality on sets one has nonempty intersections. The basic definitions and results of H_v-structures can be found in [6, 9, 15]. This concept has been further investigated by many researchers. The largest class of multivalued systems satisfying the module-like axioms is the class of H_v-modules (or H_v-vector spaces) [1, 4, 5, 7, 10, 11, 13, 14, 17].

In 2001, Davvaz and Ghadiri defined exact sequences in H_v-modules and proved some results in this respect [8]. In Section 2, we recall some basic concepts for the sake of completeness and we present some examples for the definitions. In Section 3, we introduce the concepts of $M[-]$ and $-[M]$ functors and investigate some related concepts. In Section 4, we determine the heart of $M[A]$ and the connection between equivalence relations $\varepsilon^*_{M[A]}$ and ε^*_A. Finally, we investigate the exactness of functors $M[-]$ and $-[M]$ and prove the five short lemma in H_v-modules.

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2. Basic concepts

In this section we recall some basic concepts. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the family of nonempty subsets of H. Every function $* : H \times H \to \mathcal{P}^*(H)$ is called a hyperoperation on H and $(H,*)$ is called a hyperstructure. The hyperstructure $(H,*)$ is called an H_v-group if

(1) The $*$ is weak associative, *i.e.* $x*(y*z)\cap(x*y)*z \neq \emptyset$,

(2) The reproduction axiom holds, *i.e.* $a*H = H*a = H$ for every $a \in H$.

We say H is weak commutative if for every $x,y \in H$, $x*y\cap y*x \neq \emptyset$.

A multivalued system $(R,+,\cdot)$ is called an H_v-ring if the following axioms hold

(1) $(R,+)$ is a weak commutative H_v-group,

(2) $(R,\cdot)$ is a weak associative, *i.e.* $x\cdot(y\cdot z)\cap(x\cdot y)\cdot z \neq \emptyset$ for every $x,y,z \in R$,

(3) The $\cdot$ hyperoperation is weak distributive with respect to $+$, *i.e.* for every $x,y,z \in R$, we have $x\cdot(y+z)\cap(x\cdot y+x\cdot z)\neq\emptyset$, $(x+y)\cdot z\cap(x\cdot z+y\cdot z)\neq\emptyset$.

For example, if $(H,+)$ is an H_v-group, then for every hyperoperation $\cdot$ such that $\{x,y\}\subseteq x\cdot y$ for every $x,y\in H$, the hyperstructure $(H,+,\cdot)$ is an H_v-ring. Therefore, we can construct some H_v-rings by a given H_v-group [15].

Let M be a nonempty set. Then M is called a left H_v-module over an H_v-ring R if $(M,+)$ is a weak commutative H_v-group and there exists a map $\cdot : R\times M \to \mathcal{P}^*(M)$ denoted by $(r,m)\mapsto rm$ such that for every $r_1,r_2\in R$ and every $m_1,m_2\in M$, we have

(1) $r_1(m_1+m_2)\cap(r_1m_1+r_1m_2)\neq\emptyset$,

(2) $(r_1+r_2)m_1\cap(r_1m_1+r_2m_1)\neq\emptyset$,

(3) $(r_1r_2)m_1\cap r_1(r_2m_1)\neq\emptyset$.

Let M_1 and M_2 be two H_v-modules over an H_v-ring R. A mapping $f : M_1 \longrightarrow M_2$ is called a strong H_v-homomorphism if for every $x,y\in M_1$ and every $r\in R$, we have $f(x+y)=f(x)+f(y)$ and $f(rx)=rf(x)$.

The H_v-modules M_1 and M_2 are called isomorphic if the H_v-homomorphism f is one to one and onto. It is denoted by $M_1\cong M_2$.

By using a certain type of equivalence relations, we can connect hyperstructures to usual structures. The smallest of these relations are called fundamental relations and denoted by $\beta^*,\gamma^*,\varepsilon^*$, so that if H is an H_v-group (H_v-ring, H_v-module over an H_v-ring R) then H/β^* is a group (H/γ^* is a ring, H/ε^* is an R/γ^*-module). The fundamental relation ε^* on an H_v-module M can be defined as follows:

Consider the left H_v-module M over an H_v-ring R. If ϑ denotes the set of all expressions consisting of finite hyperoperations of either on R and M or of the external hyperoperations applying on finite sets of elements of R and M, a relation ε can be defined on M whose transitive closure is the fundamental relation ε^*. The relation ε is defined as follows: for every $x,y\in M$, $x\ \varepsilon\ y$ if and only if $\{x,y\}\subseteq u$ for some $u\in\vartheta$; *i.e.*

$$x\varepsilon y \Leftrightarrow x,y\in\sum_{i=1}^{n} m_i^{'},\ m_i^{'}=m_i \text{ or } m_i^{'}=\sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} r_{ijk})m_i,$$

where $m_i\in M,\ r_{ijk}\in R$.

Suppose that $\gamma^*(r)$ is the equivalence class containing $r \in R$ and $\varepsilon^*(x)$ is the equivalence class containing $x \in M$. On M/ε^* the $\oplus$ and the external product $\odot$ using the γ^* classes in R are defined as follows:

For every $x, y \in M$, and for every $r \in R$,

$$\varepsilon^*(x) \oplus \varepsilon^*(y) = \varepsilon^*(c), \text{ for every } c \in \varepsilon^*(x) + \varepsilon^*(y),$$

$$\gamma^*(r) \odot \varepsilon^*(x) = \varepsilon^*(d), \text{ for every } d \in \gamma^*(r) \cdot \varepsilon^*(x).$$

The kernel of canonical map $\phi : M \longrightarrow M/\varepsilon^*_M$ is called the heart of M and it is denoted by ω_M, *i.e.* $\omega_M = \{x \in M \mid \phi(x) = 0\}$, where 0 is the unit element of the group $(M/\varepsilon^*, \oplus)$. One can prove that the unit element of the group $(M/\varepsilon^*, \oplus)$ is equal to ω_M. By the definition of ω_M, we have

$$\omega_{\omega_M} = Ker(\phi : \omega_M \longrightarrow \omega_M/\varepsilon^*_{\omega_M} = 0) = \omega_M.$$

The kernel of a strong H_v-homomorphism $f : A \longrightarrow B$ is defined as follows:

$$Ker(f) = \{a \in A \mid f(a) \in \omega_B\}.$$

Let M_1 and M_2 be two H_v-modules over an H_v-ring R and let $\varepsilon^*_{M_1}$, $\varepsilon^*_{M_2}$, and $\varepsilon^*_{M_1 \times M_2}$ be the fundamental relations on M_1, M_2, and $M_1 \times M_2$ respectively; then

$$(x_1, x_2)\varepsilon^*_{M_1 \times M_2}(y_1, y_2) \Leftrightarrow x_1\varepsilon^*_{M_1}y_1 \text{ and } x_2\varepsilon^*_{M_2}y_2; \text{ for all } (x_1, x_2), (y_1, y_2) \in M_1 \times M_2$$

and it is easy to see that $(M_1 \times M_2)/\varepsilon^*_{M_1 \times M_2} \cong M_1/\varepsilon^*_{M_1} \times M_2/\varepsilon^*_{M_2}$ [14, 15].

Definition 2.1 *[8] Let M be an H_v-module and X, Y be nonempty subsets of M. We say X is weak equal to Y and write $X \overset{w}{=} Y$ if and only if for every $x \in X$ there exists $y \in Y$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$ and for every $y \in Y$ there exists $x \in X$ such that $\varepsilon^*_M(x) = \varepsilon^*_M(y)$.*

Definition 2.2 *[8] Let $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{n-1} \xrightarrow{f_n} M_n$ be a sequence of H_v-modules and strong H_v-homomorphisms. We say this sequence is exact if for every $2 \leq i \leq n$, $Im(f_{i-1}) \overset{w}{=} Ker(f_i)$.*

Definition 2.3 *[8] A function $f : M_1 \longrightarrow M_2$ is called weak-monic if for every $m_1, m_1' \in M_1$, $f(m_1) = f(m_1')$ implies $\varepsilon^*_{M_1}(m_1) = \varepsilon^*_{M_1}(m_1')$ and f is called weak-epic if for every $m_2 \in M_2$ there exists $m_1 \in M_1$ such that $\varepsilon^*_{M_2}(m_2) = \varepsilon^*_{M_1}(f(m_1))$. Finally f is called weak-isomorphism if f is weak-monic and weak-epic.*

We present the following example for the above definitions.

Example 1 *Let R be an H_v-ring. Consider the following H_v-modules on R.*

(1) $M = \{a, b\}$ together with the following hyperoperations:

$*_M$	a	b
a	a	b
b	b	a

and $\underset{(r,m) \mapsto \{a\}}{\cdot_M : R \times M \to \mathcal{P}^*(M)}$

(2) $M_1 = \{0,1,2\}$ *together with the following hyperoperations:*

$*_{M_1}$	0	1	2
0	0	1	2
1	1	0,2	1
2	2	1	0

and $\underset{(r,m_1)\mapsto\{0\}}{\cdot_{M_1} : R \times M_1 \to \mathcal{P}^*(M_1)}$

(3) $M_2 = \{\bar{0},\bar{1},\bar{2}\}$ *together with the following hyperoperations:*

$*_{M_2}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

and $\underset{(r,m_2)\mapsto M_2}{\cdot_{M_2} : R \times M_2 \to \mathcal{P}^*(M_2)}$

Since $\{0,2\} \subseteq 1 *_{M_1} 1$, $r \cdot m_1 = 0$ *for every* $r \in R$ *and every* $m_1 \in M_1$ *and* $0 *_{M_1} 0 = 0$, *we obtain* $M_1/\varepsilon^*_{M_1} = \{\varepsilon^*_{M_1}(0) = \varepsilon^*_{M_1}(2) = \{0,2\}, \varepsilon^*_{M_1}(1) = \{1\}\}$. *Moreover, since* $\varepsilon^*_{M_1}(0) + \varepsilon^*_{M_1}(1) = \varepsilon^*_{M_1}(1)$, *it follows that* $\omega_{M_1} = \varepsilon^*_{M_1}(0) = \{0,2\}$. *Since* $r \cdot_{M_2} m_2 = M_2$ *for every* $r \in R$ *and every* $m_2 \in M_2$, *we obtain* $M_2/\varepsilon^*_{M_2} = \{\{\bar{0},\bar{1},\bar{2}\}\}$ *and* $\omega_{M_2} = \varepsilon^*_{M_2}(\bar{0}) = \varepsilon^*_{M_2}(\bar{1}) = \varepsilon^*_{M_2}(\bar{2}) = M_2$.

Since $(M_1 \times M_2)/\varepsilon^*_{M_1 \times M_2} \cong M_1/\varepsilon^*_{M_1} \times M_2/\varepsilon^*_{M_2}$, *it follows that*

$$M_1 \times M_2/\varepsilon^*_{M_1 \times M_2} = \{\{(0,\bar{0}),(0,\bar{1}),(0,\bar{2}),(2,\bar{0}),(2,\bar{1}),(2,\bar{2})\},\{(1,\bar{0}),(1,\bar{1}),(1,\bar{2})\}\}.$$

Note that $\omega_{M_1 \times M_2} = \omega_{M_1} \times \omega_{M_2}$. *The subsets* $X = \{(2,\bar{1}),(2,\bar{2}),(1,\bar{1}),(1,\bar{2})\}$ *and* $Y = \{(0,\bar{2}),((1,\bar{0})\}$ *of* $M_1 \times M_2$ *are weakly equal. Now consider* $f \in M[M_1 \times M_2]$, *where* $f(a) = (2,\bar{2})$, $f(b) = (1,\bar{0})$ *and* $g \in M_1[M_1 \times M_2]$, *where* $g(0) = (1,\bar{1})$, $g(1) = (2,\bar{2})$, $g(2) = (1,\bar{1})$. *Then* f *is weak-epic and* g *is weak-monic.*

3. M[–] and –[M] functors

Let $f : A \longrightarrow B$ be a strong H_v-homomorphism of H_v-modules over an H_v-ring R. Then $F : A/\varepsilon^*_A \longrightarrow B/\varepsilon^*_B$, where $F(\varepsilon^*_A(a)) = \varepsilon^*_B(f(a))$ is an R/γ^*-homomorphism of R/γ^*-modules. Let R be a weak-commutative H_v-ring and **H** be the set of all H_v-modules and all strong R-homomorphisms. One can show that **H** is a category. Furthermore, set **H*** the category of R/γ^*-modules and R/γ^*-homomorphisms. Then $T :$ **H** $\longrightarrow$ **H***, defined by $T(A) = A/\varepsilon^*_A$ and $T(f : A \longrightarrow B) = F : A/\varepsilon^* \longrightarrow B/\varepsilon^*_B$, where $F(\varepsilon^*_A(a)) = \varepsilon^*_B(f(a))$ is a covariant functor [8]. Now we want to introduce $M[-]$ and $-[M]$ functors and investigate some related concepts.

Suppose that M and N are two H_v-modules and $M[N]$ is the set of all functions on M with values in N. First we equip $M[N]$ to appropriate hyperoperations to be an H_v-module. Then we introduce the functors $M[-]$ and $-[M]$ and investigate some related concepts. Throughout this paper, the hyperoperations in M, N and $M[N]$ will be shown with the same symbols.

Theorem 3.1 *The* $M[N]$ *with the following hyperoperations is an* H_v*-module.*

$$f + g = \{h \in M[N] \mid h(x) \in f(x) + g(x),\ \forall x \in M\},$$
$$r \cdot f = \{k \in M[N] \mid k(x) \in r \cdot f(x),\ \forall x \in M\}.$$

Proof The hyperoperations $+$ and $\cdot$ in $M[N]$ are well defined and for $+$ and $\cdot$ in N are well defined. Let $f,g,h \in M[N]$. We have

$$\begin{aligned}(f+g)+h &= \{l \in M[N] \mid l(x) \in f(x)+g(x),\ \forall x \in M\} + h = \textstyle\bigcup_{l \in f+g} l+h \\ &= \{L \in M[N] \mid L(x) \in l(x)+h(x),\ \forall x \in M,\ l(x) \in f(x)+g(x)\}\end{aligned}$$

and

$$\begin{aligned}f+(g+h) &= f + \{k \in M[N] \mid k(x) \in g(x)+h(x),\ \forall x \in M\} = \textstyle\bigcup_{k \in g+h} f+k \\ &= \{K \in M[N] \mid K(x) \in f(x)+k(x),\ \forall x \in M,\ k(x) \in g(x)+h(x)\}.\end{aligned}$$

Since N is an H_v-group, for all $x \in M$ there exists $n_x \in [(f(x)+g(x))+h(x)] \cap [f(x)+(g(x)+h(x))]$. We define $u \in M[N]$ by $u(x) = n_x$, according to the choice axiom. Then $u \in [(f+g)+h] \cap [f+(g+h)]$ and associativity is satisfied.

For the reproduction axiom let $f,g \in M[A]$. Then for all $x \in M$, $f(x), g(x) \in N$ and so there exists $y_x \in N$ such that $f(x) \in g(x)+y_x$. We define $h \in M[N]$ by $h(x) = y_x$; then $f \in g+h$. Similarly, there exists $h^{'} \in M[N]$ such that $f \in h^{'}+g$. Since N is an H_v-module, the conditions of H_v-modules are satisfied in $M[N]$. We check only one of the H_v-module conditions. Let $r_1, r_2 \in R$ and $f \in M[N]$. Since N is an H_v-module, it follows that for every $x \in M$ there exists $n_x \in [(r_1+r_2)f(x)] \cap [r_1 f(x) + r_2 f(x)]$. We define $h \in M[N]$ by $h(x) = n_x$. Obviously, $h \in [(r_1+r_2)f] \cap [(r_1 f + r_2 f)] \neq \emptyset$. $\square$

Lemma 3.2 *Let $f : A \longrightarrow B$ be a strong H_v-homomorphism and M be an H_v-module. Then*

(1) The map $\bar{f}: M[A] \longrightarrow M[B]$ defined by $\bar{f}(\phi) = f \circ \phi$ is a strong H_v-homomorphism.

(2) The map $\bar{f}: B[M] \longrightarrow A[M]$ defined by $\bar{f}(\phi) = \phi \circ f$ is a strong H_v-homomorphism.

Proof (1) Let $\phi_1, \phi_2 \in M[A]$. Then

$$\begin{aligned}&\bar{f}(\phi_1+\phi_2) = \{f \circ h \mid h \in M[A],\ h(m) \in \phi_1(m)+\phi_2(m),\ \forall m \in M\}, \\ &\bar{f}(\phi_1) + \bar{f}(\phi_2) = f \circ \phi_1 + f \circ \phi_2 = \{h^{'} \in M[B] \mid h^{'}(m) \in f \circ \phi_1(m) + f \circ \phi_2(m)\}.\end{aligned}$$

Suppose that $f \circ h \in \bar{f}(\phi_1+\phi_2)$, where $h \in M[A]$ and $h(m) \in \phi_1(m)+\phi_2(m)$ for every $m \in M$. Then $f(h(m)) \in f(\phi_1(m)+\phi_2(m)) = f(\phi_1(m)) + f(\phi_2(m))$. Therefore, $\bar{f}(\phi_1+\phi_2) \subseteq \bar{f}(\phi_1) + \bar{f}(\phi_2)$.

Conversely, suppose that $h^{'} \in \bar{f}(\phi_1) + \bar{f}(\phi_2)$. We need to find an $h \in M[A]$ such that $h^{'} = foh$ and $h(m) \in \phi_1(m)+\phi_2(m)$. By hypothesis for $m \in M,$ we have

$$h^{'}(m) = b_m \in f \circ \phi_1(m) + f \circ \phi_2(m) = f(\phi_1(m)+\phi_2(m)) \subseteq Im(f).$$

Therefore, $b_m \in f(\phi_1(m)+\phi_2(m))$. Now, according to the choice axiom, we can select $a \in f^{-1}(b_m)$ such that $a \in \phi_1(m)+\phi_2(m)$ and define $h(m) = a$.

Similarly, one can show that $\overline{f}\,(r\phi) = r\,\overline{f}\,(\phi)$.

(2) Let $\phi_1, \phi_2 \in B[M]$. Then

$$\overline{f}\,(\phi_1 + \phi_2) = \{h \circ f \mid h \in B[M], h(b) \in \phi_1(b) + \phi_2(b)\},$$
$$\overline{f}\,(\phi_1) + \overline{f}\,(\phi_2) = \phi_1 \circ f + \phi_2 \circ f = \{h^{'} \in A[M] \mid h^{'}(a) \in \phi_1 \circ f(a) + \phi_2 \circ f(a)\}.$$

Suppose that $h \circ f \in \overline{f}\,(\phi_1 + \phi_2)$, where $h \in B[M]$ and $h(b) \in \phi_1(b) + \phi_2(b)$ for every $b \in B$. Since $Im(f) \subseteq B$, we have $h(f(a)) \in \phi_1(f(a)) + \phi_2(f(a))$ for every $a \in A$. Therefore, $\overline{f}\,(\phi_1 + \phi_2) \subseteq \overline{f}\,(\phi_1) + \overline{f}\,(\phi_2)$.

Conversely, suppose that $h^{'} \in \overline{f}\,(\phi_1) + \overline{f}\,(\phi_2)$. We need to find an $h \in B[M]$ such that $h^{'} = h \circ f$ and $h(b) \in \phi_1(b) + \phi_2(b)$. For every $b \in Im(f) \subseteq B$ we define $h(b) = h^{'}(a)$, where $f(a) = b$ and for every $b \in B \backslash Im(f)$ according to the choice axiom we select an m_b in $\phi_1(b) + \phi_2(b) \subseteq M$ and define $h(b) = m_b$. Then h satisfies the requirement conditions.

Similarly, one can show that $\overline{f}\,(r\phi) = r\,\overline{f}\,(\phi)$. □

Lemma 3.3 *Let M be an H_v-module and $f : A \longrightarrow B$ be a morphism in the category **H**. Then*

(1) $M[-] : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ defined by $M[-](A) = M[A]$ and $M[-](f) = \overline{f} : M[A] \longrightarrow M[B]$, where $\overline{f}\,(\phi) = f \circ \phi$ is a covariant functor.

(2) $-[M] : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ defined by $-[M](A) = A[M]$ and $-[M](f) = \overline{f} : B[M] \longrightarrow A[M]$, where $\overline{f}\,(\phi) = \phi \circ f$ is a contravariant functor.

Proof (1) By Theorem 3.1 if A is an H_v-module, then $M[-](A) = M[A]$ is an H_v-module. By Lemma 3.2 if $f : A \longrightarrow B$ is a strong H_v-homomorphism, then $M[-](f) = \overline{f}$ is a strong H_v-homomorphism. Now let $A \xrightarrow{f} B \xrightarrow{g} C$ be a strong H_v-homomorphism in **H**. Then

$$M[-](g \circ f)(\phi) = g \circ f \circ \phi = g(f \circ \phi) = M[-](g)(f \circ \phi) = M[-](g) \circ M[-](f)(\phi)$$

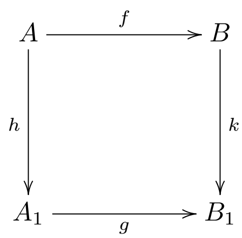
and for every $A \in obj\,\mathbf{H}$ we have $M[-](1_A)(\phi) = 1_A \circ \phi = \phi$. Then $M[-](1_A) = 1_{M[-](A)}$ and so $M[-]$ is a covariant functor.

(2) By Theorem 3.1 if A is an H_v-module, then $-[M](A) = A[M]$ is an H_v-module. By Lemma 3.2 if $f : A \longrightarrow B$ is a strong H_v-homomorphism, then $-[M](f) = \overline{f}$ is a strong H_v-homomorphism. Now let $A \xrightarrow{f} B \xrightarrow{g} C$ be a strong H_v-homomorphism in **H**. Then

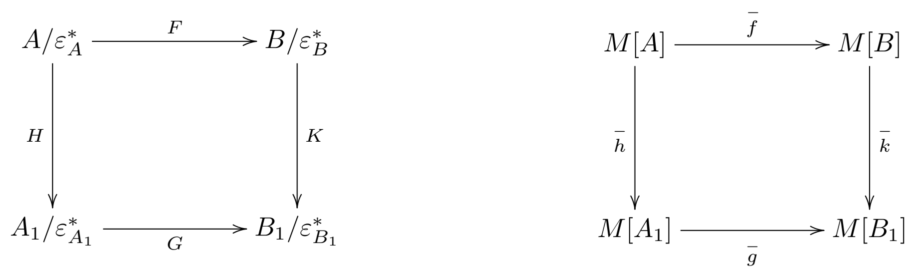
$$-[M](g \circ f)(\phi) = \phi \circ g \circ f = (\phi \circ g)f = -[M](f)(\phi \circ g) = -[M](f) \circ -[M](g)(\phi),$$

and for every $A \in obj\,\mathbf{H}$ we have $-[M](1_A)(\phi) = \phi \circ 1_A = \phi$. Then, $-[M](1_A) = 1_{-[M](A)}$ and so $-[M]$ is a contravariant functor. □

Lemma 3.4 *Let*



be a commutative diagram of H_v *-modules and strong* H_v *-homomorphisms. Then the following diagrams are commutative.*



Proof We have $T(A) = A/\varepsilon^*_A$ and $T(f: A \longrightarrow B) = F: A/\varepsilon^* \longrightarrow B/\varepsilon^*_B$, where $F(\varepsilon^*_A(a)) = \varepsilon^*_B(f(a))$. Therefore,

$$K \circ F = T(k) \circ T(f) = T(k \circ f) = T(g \circ h) = T(g) \circ T(h) = G \circ H.$$

We have $M[-](A) = M[A]$, $M[-](f: A \longrightarrow B) = \bar{f}: M[A] \longrightarrow M[B]$, where $\bar{f}\,(\phi) = f \circ \phi$. Therefore,

$$\begin{aligned} \bar{k} \circ \bar{f} &= M[-](k) \circ M[-](f) = M[-](k \circ f) = M[-](g \circ h) \\ &= M[-](g) \circ M[-](h) = \bar{g} \circ \bar{h}\,. \end{aligned}$$

□

We know that the combination of two covariant functors is a covariant functor. Therefore, the map $S = T \circ M[-]: \mathbf{H} \longrightarrow \mathbf{H^*}$ is a covariant functor, where

$$S(A) = M[A]/\varepsilon^*_{M[A]} \text{ and } S(f: A \longrightarrow B) = \bar{F}: M[A]/\varepsilon^*_{M[A]} \longrightarrow M[B]/\varepsilon^*_{M[B]},$$

where $\bar{F}\,(\varepsilon^*_{M[A]}(\phi)) = \varepsilon^*_{M[B]}(f \circ \phi)$.

Lemma 3.5 *For every* $A \in obj\,\boldsymbol{H}$, $\tau_A : T(A) \longrightarrow S(A)$ *defined by* $\tau_A(\varepsilon^*_A(a)) = \varepsilon^*_{M[A]}(\phi_a)$ *is a* R/γ^* *-homomorphism, where* $\phi_a : M \longrightarrow A$ *defined by* $\phi_a(m) = a$ *for every* $m \in M$. *Then the family* $\tau = (\tau_A : T(A) \longrightarrow S(A))_{A \in obj\boldsymbol{H}}$ *is a natural transformation from* T *to* S.

Proof We have

$$\tau_A(\varepsilon^*_A(a) \oplus \varepsilon^*_A(b)) = \tau_A(\varepsilon^*_A(a+b)) = \varepsilon^*_{M[A]}(\phi_t),$$

where $t \in a+b$. On the other hand, we obtain

$$\begin{aligned}
\tau_A(\varepsilon_A^*(a)) \oplus \tau_A(\varepsilon_A^*(b)) &= \varepsilon_{M[A]}^*(\phi_a) \oplus \varepsilon_{M[A]}^*(\phi_b) = \varepsilon_{M[A]}^*(\phi_a + \phi_b) \\
&= \varepsilon_{M[A]}^*(\{\phi \in M[A] \mid \phi(m) \in \phi_a(m) + \phi_b(m),\ \forall m \in M\}) \\
&= \varepsilon_{M[A]}^*(\{\phi \in M[A] \mid \phi(m) \in a+b,\ \forall m \in M\}) \\
&= \varepsilon_{M[A]}^*(\phi_t),
\end{aligned}$$

where $t \in a+b$. Therefore, $\tau_A(\varepsilon_A^*(a) \oplus \varepsilon_A^*(b)) = \tau_A(\varepsilon_A^*(a)) \oplus \tau_A(\varepsilon_A^*(b))$. Similarly, we have

$$\begin{aligned}
\tau_A(\gamma^*(r) \odot \varepsilon_A^*(a)) &= \tau_A(\varepsilon_A^*(d)),\ \text{for some } d \in \gamma^*(r) \cdot \varepsilon_A^*(a) \\
&= \varepsilon_{M[A]}^*(\phi_d),\ \text{for some } d \in r \cdot a
\end{aligned}$$

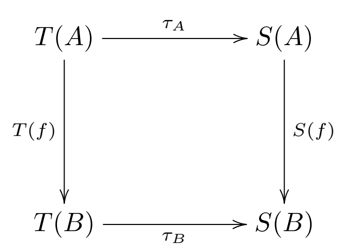
and

$$\begin{aligned}
\gamma^*(r) \odot \tau_A(\varepsilon_A^*(a)) &= \gamma^*(r) \odot \varepsilon_{M[A]}^*(\phi_a) \\
&= \varepsilon_{M[A]}^*(h) \text{ for some } h \in r \cdot \phi_a \\
&= \varepsilon_{M[A]}^*(h),
\end{aligned}$$

where for every $m \in M$, $h(m) \in r \cdot \phi_a(m) = r \cdot a$. Therefore,

$$\tau_A(\gamma^*(r) \odot \varepsilon_A^*(a)) = \gamma^*(r) \odot \tau_A(\varepsilon_A^*(a)).$$

Now let $f : A \longrightarrow B$ be a morphism in H and consider the following diagram.



We have

$$\begin{aligned}
&S(f) \circ \tau_A(\varepsilon_A^*(a)) = S(f)(\varepsilon_{M[A]}^*(\phi_a)) = \varepsilon_{M[B]}^*(f \circ \phi_a), \\
&\tau_B \circ T(f)(\varepsilon_A^*(a)) = \tau_B(\varepsilon_B^*(f(a))) = \varepsilon_{M[B]}^*(\phi_{f(a)}).
\end{aligned}$$

Obviously, $f \circ \phi_a = \phi_{f(a)}$ and so $S(f) \circ \tau_A = \tau_B \circ T(f)$ and $\tau : T \longrightarrow S$ is a natural transformation. □

Lemma 3.6 *Let H_1 and H_2 be two H_v-modules. Then $H_1 \times H_2$ is a product object in **H** category.*

Proof The proof is straightforward. □

Note that Lemma 3.6 can be generalized to the cartesian product of n arbitrary H_v-modules.

Theorem 3.7 *Let M be an H_v-module. Then $M[H_1 \times H_2] \cong M[H_1] \times M[H_2]$.*

Proof It is easy to see that the map $\phi : M[H_1] \times M[H_2] \longrightarrow M[H_1 \times H_2]$ defined by $\phi(f_1, f_2) = f : M \longrightarrow H_1 \times H_2$, where $f(m) = (f_1(m), f_2(m))$ is well defined. Now we have

$$\begin{aligned}
&\phi((f_1, g_1) + (f_2, g_2)) = \phi(\{(f, g) \mid f \in f_1 + f_2,\ g \in g_1 + g_2\}) \\
&= \{h \mid h(m) = (f(m), g(m)),\ f(m) \in f_1(m) + f_2(m),\ g(m) \in g_1(m) + g_2(m)\}.
\end{aligned}$$

On the other hand, we have

$$\phi((f_1,g_1)) = h \in M[H_1 \times H_2] \text{ such that } h(m) = (f_1(m), g_1(m)),$$
$$\phi((f_2,g_2)) = k \in M[H_1 \times H_2] \text{ such that } k(m) = (f_2(m), g_2(m)).$$

And

$$\begin{aligned} h+k &= \{l \mid l(m) \in h(m)+k(m) = (f_1(m), g_1(m)) + (f_2(m), g_2(m))\} \\ &= \{l \mid l(m) = (f(m), g(m)),\ f(m) \in f_1(m)+f_2(m),\ g(m) \in g_1(m)+g_2(m)\}. \end{aligned}$$

Therefore, $\phi((f_1,g_1)+(f_2,g_2)) = \phi((f_1,g_1)) + \phi((f_2,g_2))$.

Similarly, one can show that $\phi(r(f,g)) = r\phi((f,g))$.

Now let $f \in M[H_1 \times H_2]$, where $f(m) = (h_{1m}, h_{2m})$. We define $f_1 \in M[H_1]$ by $f(m) = h_{1m}$ and $f_2 \in M[H_2]$ by $f(m) = h_{2m}$. Obviously, $\phi((f_1,f_2)) = f$.

Finally, suppose that $\phi((f_1,f_2)) = \phi((g_1,g_2))$. Then, for every $m \in M$, we obtain $(f_1(m), f_2(m)) = (g_1(m), g_2(m))$ and so $(f_1,f_2) = (g_1,g_2)$. □

Note that in finite mode in Theorem 3.7 we have

$$\begin{aligned} |M[H_1] \times M[H_2]| &= |M[H_1]| \times |M[H_2]| = |H_1|^{|M|} \times |H_2|^{|M|} \\ &= |H_1 \times H_2|^{|M|} = |M[H_1 \times H_2]|. \end{aligned}$$

Therefore, it is sufficient to show that ϕ is one to one or onto.

Corollary 3.8 *Let M, H_1, $H_2, \cdots, H_n$ be H_v-modules. Then*

$$M[H_1 \times H_2 \times H_3 \times \cdots \times H_n] \cong M[H_1] \times M[H_2] \times M[H_3] \times \cdots \times M[H_n].$$

4. Five short lemma in H_v-modules

Let $f : A \longrightarrow B$ be a strong H_v-homomorphism of H_v-modules over an H_v-ring R. Then we have $f(\omega_A) \subseteq \omega_B$ and so $\omega_A \subseteq Ker(f)$. Furthermore, $Ker(f) = \omega_A$ if and only if f is weak-monic [8]. In this section, we determine the heart of $M[A]$ and the connection between equivalence relations $\varepsilon^*_{M[A]}$ and ε^*_A. Moreover, we check the exactness of $M[-]$ and $-[M]$ functors. Finally, we investigate the five short lemma in H_v-modules.

Lemma 4.1 *If $\varepsilon^*_{M[A]}(f) = \varepsilon^*_{M[A]}(g)$, then $\varepsilon^*_A(f(m)) = \varepsilon^*_A(g(m))$, for every $m \in M$; i.e. if for some $m \in M$, $\varepsilon^*_A(f(m)) \neq \varepsilon^*_A(g(m))$ then $\varepsilon^*_{M[A]}(f) \neq \varepsilon^*_{M[A]}(g)$.*

Proof Suppose that $f\ \varepsilon^*_{M[A]}\ g$. Then there exist $f_0 = f, f_1, \cdots, f_n = g$ in $M[A]$ such that $f_i\ \varepsilon_{M[A]}\ f_{i+1}$ for $i = 0,1,\ldots,n-1$. Therefore, $\{f_i, f_{i+1}\} \subseteq \sum_{j=1}^{n_i} g'_{ij}$, for $i = 0,1,\ldots,n-1$, where $g'_{ij} = g_{ij}$ or $g'_{ij} = \sum_{k=1}^{n_{ij}} (\prod_{l=1}^{l_{ijk}} r_{ijkl}) g_{ij}$ for $g_{ij} \in M[A]$ and $r_{ijkl} \in R$. Now, since

$$\sum_{j=1}^{n_i} g'_{ij} = \{h \in M[N] \mid h(m) \in g'_{i1}(m) + g'_{i2}(m) + \cdots + g'_{in_i}(m),\ \forall m \in M\},$$

we have $\{f_i(m), f_{i+1}(m)\} \subseteq \sum_{j=1}^{n_i} g'_{ij}(m)$ for every $m \in M$ and so there exist $a_0 = f_0(m) = f(m), a_1 = f_1(m), \ldots, a_n = f_n(m) = g(m) \in A$ such that $a_i \ \varepsilon_A \ a_{i+1}$, for $i = 0, 1, \ldots, n-1$. Therefore, for every $m \in M$, we have $f(m) \ \varepsilon_A^* \ g(m)$. □

In the following example we show that the converse of Lemma 4.1 is not true in general.

Example 2 *Consider $f, g \in M[M_1 \times M_2]$ as in Example 1 and define $f(a) = (2, \bar{2})$, $f(b) = (1, \bar{0})$ and $g(a) = (0, \bar{1})$, $g(b) = (1, \bar{2})$. By Example 1 we have $\varepsilon^*_{M_1 \times M_2}(f(a)) = \varepsilon^*_{M_1 \times M_2}(g(a))$ and $\varepsilon^*_{M_1 \times M_2}(f(b)) = \varepsilon^*_{M_1 \times M_2}(g(b))$. Since for every $r \in R$ and every $m_1 \in M_1$, $rm_1 = \{0\}$ and, on the other hand, for every two elements m_2 and m_2' of M_2, $m_2 *_{M_2} m_2'$ is a singleton, it follows that $\varepsilon^*_{M[M_1 \times M_2]}(f) \neq \varepsilon^*_{M[M_1 \times M_2]}(g)$.*

In the following lemma, we determine the heart of $M[A]$.

Lemma 4.2 *Let M and A be two H_v-modules. Then $\omega_{M[A]} = M[\omega_A]$.*

Proof Suppose that $f \in \omega_{M[A]}$. Then for every $g \in M[A]$ we have

$$\varepsilon^*_{M[A]}(g) = \varepsilon^*_{M[A]}(f) \oplus \varepsilon^*_{M[A]}(g)\Big(= \varepsilon^*_{M[A]}(f+g)\Big).$$

Now by Lemma 4.1 for every $m \in M$ we obtain

$$\varepsilon^*_A((f+g)(m)) = \varepsilon^*_A(g(m)).$$

However, for every $m \in M$ we have $(f+g)(m) = \{l(m) \mid l \in f+g\} = f(m) + g(m)$. Hence,

$$\varepsilon^*_A((f+g)(m)) = \varepsilon^*_A(f(m)+g(m)) = \varepsilon^*_A(f(m)) \oplus \varepsilon^*_A(g(m)) = \varepsilon^*_A(g(m)).$$

Therefore, for every $m \in M$, we obtain $\varepsilon^*_A(f(m)) \in \omega_A$ and so $f \in M[\omega_A]$.

Conversely, suppose that $f \in M[\omega_A]$. Then for every $g \in M[A]$ and all $m \in M$ we have

$$\varepsilon^*_A(f(m)+g(m)) = \varepsilon^*_A(f(m)) \oplus \varepsilon^*_A(g(m)) = \varepsilon^*_A(g(m)).$$

Therefore, for every $g \in M[A]$ and all $m \in M$, we have $f(m) + g(m) \in \varepsilon^*_A(g(m))$ and we obtain

$$\varepsilon^*_{M[A]}(f) \oplus \varepsilon^*_{M[A]}(g) = \varepsilon^*_{M[A]}(f+g) = \varepsilon^*_{M[A]}(\{l \mid l(m) \in f(m)+g(m)\}) = \varepsilon^*_{M[A]}(g)$$

and consequently $f \in \omega_{M[A]}$. □

In the following, we want to investigate the exactness of $-[M]$ and $M[-]$ functors.

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence. Then for every $a \in A$ we have $f(a) \in Im(f) \overset{w}{=} Ker(g)$ and so $\varepsilon^*_B(f(a)) = \varepsilon^*_B(b)$ for some $b \in Ker(g)$. Now we obtain

$$\varepsilon^*_C(g(f(a))) = G(\varepsilon^*_B(f(a))) = G(\varepsilon^*_B(b)) = \varepsilon^*_C(g(b)) = \omega_C.$$

Therefore, for every $a \in A$ we have $g(f(a)) \in \omega_C$.

Now, by considering $-[M]$ functor on the exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, we obtain

$$C[M] \xrightarrow{\bar{g}} B[M] \xrightarrow{\bar{f}} A[M].$$

We want to check the exactness of this sequence. We have

$$Im(\bar{g}) = \{\bar{g}\ (\phi) \mid \phi \in C[M]\} = \{\phi \circ g \mid \phi \in C[M]\},$$
$$Ker(\bar{f}) = \{\psi \in B[M] \mid \ \bar{f}\ (\psi) = \psi \circ f \in \omega_{A[M]} = A[\omega_M]\}.$$

Let ϕ be a function in $C[M]$ such that $\phi(\omega_C) \cap \omega_M = \emptyset$ (note that it is necessary for $\omega_M \neq M$). Then for every $a \in A$, since $g \circ f(a) \in \omega_C$ and $\phi(\omega_C) \cap \omega_M = \emptyset$, we obtain $\varepsilon^*_M(\phi(g(f(a)))) \neq \omega_M$. On the other hand, for every $\psi \in Ker(\bar{f})$ and every $a \in A$, $\varepsilon^*_M(\psi(f(a))) = \omega_M$. Thus, by Lemma 4.1 for $\phi \circ g \in Im(\bar{g})$ there is no member of $Ker(\bar{g})$ such that its class is equal to the class of $\phi \circ g$. Therefore, in general the $-[M]$ functor is not exact. The same discussion is established for the $M[-]$ functor.

Example 3 *Consider the* H_v*-modules* M, M_1, *and* M_2 *as Example 1 and the sequence* $M \xrightarrow{f} M_1 \xrightarrow{i} M_1$, *where* $f(a) = 0$, $f(b) = 2$, *and* i *is identity. It is easy to see that the sequence* $M \xrightarrow{f} M_1 \xrightarrow{i} M_1$ *is exact. However, the sequence*

$$M_1[M_1 \times M_2] \xrightarrow{\bar{i}} M_1[M_1 \times M_2] \xrightarrow{\bar{f}} M[M_1 \times M_2]$$

is not exact, because for $\phi \in M_1[M_1 \times M_2]$ *defined by* $\phi(0) = (1, \bar{1})$, $\phi(1) = (2, \bar{1})$, *and* $\phi(2) = (1, \bar{2})$ *there is no member of* $Ker(\bar{f})$ *such that its class is equal to the class of* ϕ.

In the following theorem we show that if the converse of Lemma 4.1 is established, then the functors $M[-]$ and $-[M]$ are exact.

Theorem 4.3 *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be an exact sequence of* H_v*-modules and strong* H_v*-homomorphisms. If the converse of Lemma 4.1 is established, then the sequences*

$$C[M] \xrightarrow{\bar{g}} B[M] \xrightarrow{\bar{f}} A[M] \tag{1}$$

$$M[A] \xrightarrow{\bar{f}} M[B] \xrightarrow{\bar{g}} M[C] \tag{2}$$

are exact sequences.

Proof We prove (2). The proof of (1) is similar. Suppose that $h \in Im(\bar{f})$. Then there exists $\phi \in M[A]$ such that $h = \bar{f}\ (\phi) = f \circ \phi \in M[B]$. For every $m \in M$, $f \circ \phi(m) \in Im(f)$ and so there exists $b_m \in Ker(g)$ such that $\varepsilon^*_B(f \circ \phi(m)) = \varepsilon^*_B(b_m)$. Now we define $k \in M[B]$ by $k(m) = b_m$. Since $\bar{g}\ (k) = gok \in M[\omega_C] = \omega_{M[C]}$, we obtain $k \in Ker\ \bar{g}$. Finally, by the converse of Lemma 4.1 we have $\varepsilon^*_{M[B]}(h) = \varepsilon^*_{M[B]}(k)$.

Conversely, let $k \in Ker(\overline{g})$; then $\overline{g}\ (k) = g \circ k \in \omega_{M[C]} = M[\omega_C]$. Therefore, for all $m \in M$, $g \circ k(m) \in \omega_C$ and $k(m) \in Ker(g)$. Then there exists $b_m = f(a) \in Im(f)$ for some $a \in A$ such that $\varepsilon^*_B(b_m) = \varepsilon^*_B(k(m))$. We define $\psi \in M[A]$ by $\psi(m) = a$ and set $\phi = f \circ \psi = \overline{f}\ (\psi) \in Im(\overline{f})$. Now by the converse of Lemma 4.1 we obtain $\varepsilon^*_{M[B]}(k) = \varepsilon^*_{M[B]}(\phi)$. □

Lemma 4.4 *Let A, B, and C be H_v-modules. Then*

(1) $\omega_A \xrightarrow{i} A \xrightarrow{f} B$ *is exact if and only if f is weak-monic.*

(2) $B \xrightarrow{g} C \xrightarrow{j} \omega_C$ *is exact if and only if g is weak-epic.*

(3) $\omega_A \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{j} \omega_C$ *is exact if and only if f is weak-monic, g is weak-epic, and* $Im(f) \overset{w}{=} Ker(g)$.

Proof (1) Suppose that the given sequence is exact. It is sufficient to show that $Ker(f) = \omega_A$. We always have $\omega_A \subseteq Ker(f)$. On the other hand, if $a \in Ker(f)$, then there exists $a_1 \in Im(i) = \omega_A$ such that $\varepsilon^*_A(a) = \varepsilon^*_A(a_1) = \omega_A$ and so $a \in \omega_A$. Therefore, $Ker(f) = \omega_A$ and f is weak-monic.

Conversely, suppose that f is weak-monic. Then, $Ker(f) = \omega_A = Im(i)$ and consequently $Ker(f) \overset{w}{=} Im(i)$.

(2) Suppose that the given sequence is exact. Then $Im(g) \overset{w}{=} Ker(j)$ and so for every $c \in Ker(j)\Big(= C$ since $\omega_{\omega_C} = \omega_C\Big)$ there exists $b \in B$ such that $\varepsilon^*_C(g(b)) = \varepsilon^*_C(c)$. Therefore, g is weak-epic.

Conversely, suppose that g is weak-epic. Then for every $c \in C\Big(= Ker(j)\Big)$ there exists $b \in B$ such that $\varepsilon^*_C(g(b)) = \varepsilon^*_C(c)$. On the other hand, for all $g(b) \in Im(g) \subseteq C$ there exist some $t \in B$ such that $\varepsilon^*_C(g(b)) = \varepsilon^*_C(g(t))$, where $g(t) \in C = Ker(j)$ and consequently $Im(g) \overset{w}{=} Ker(j)$.

(3) It follows from (1), (2), and the definition of exactness. □

Lemma 4.5 *Let $f : A \longrightarrow B$ be a strong H_v-homomorphism of H_v-modules. Then f is weak-epic if and only if F is onto. Moreover, f is weak-monic if and only if F is one to one. Finally, f is a weak isomorphism if and only if F is an isomorphism.*

Proof Suppose that f is weak-epic and $\varepsilon^*_B(b) \in B/\varepsilon^*_B$. Since f is weak-epic, there exists $a \in A$ such that $\varepsilon^*_B(f(a)) = \varepsilon^*_B(b)$. However, $\varepsilon^*_B(f(a)) = F(\varepsilon^*_A(a))$. Therefore, $F(\varepsilon^*_A(a)) = \varepsilon^*_B(b)$ and consequently F is onto.

Conversely, let F be onto. Then for every $b \in B$ there exists $\varepsilon^*_A(a) \in A/\varepsilon^*_A$ such that $F(\varepsilon^*_A(a)) = \varepsilon^*_B(b)$. However, $F(\varepsilon^*_A(a)) = \varepsilon^*_B(f(a))$. Therefore, there exists $a \in A$ such that $\varepsilon^*_B(f(a)) = \varepsilon^*_B(b)$ and consequently f is weak-epic. The second part is proved in [8]. The third part is an obvious result. □

Theorem 4.6 *Let* $\omega_A \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{j} \omega_C$ *be an exact sequence of H_v-modules and strong H_v-homomorphisms over an H_v-ring R. Then*

$$0 = \omega_A/\varepsilon^*_{\omega_A} \xrightarrow{I} A/\varepsilon^*_A \xrightarrow{F} B/\varepsilon^*_B \xrightarrow{G} C/\varepsilon^*_C \xrightarrow{J} \omega_c/\varepsilon^*_{\omega_c} = 0$$

is an exact sequence of R/γ^*-homomorphisms and R/γ^*-modules.

Proof It follows from Lemma 4.4, Lemma 4.5, and Theorem 4.8 of [8] that say if $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence, then $A/\varepsilon^*_A \xrightarrow{F} B/\varepsilon^*_B \xrightarrow{G} C/\varepsilon^*_C$ is an exact sequence. □

Theorem 4.7 ***(Five short lemma in H_v-modules)*** *Let*

$$\begin{array}{ccccccccc}
\omega_A & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \omega_C \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \\
\omega_{A_1} & \longrightarrow & A_1 & \xrightarrow[f_1]{} & B_1 & \xrightarrow[g_1]{} & C_1 & \longrightarrow & \omega_{C_1}
\end{array}$$

be a commutative diagram of H_v-modules and H_v-homomorphisms over an H_v-ring R with both rows exact. Then

(1) If h and l are weak-monic, then k is weak-monic.

(2) If h and l are weak-epic, then k is weak-epic.

(3) If h and l are weak isomorphisms, then k is a weak isomorphism.

Proof (1) By Lemma 3.4 and Theorem 4.6 the following diagram of R/γ^*-modules and R/γ^*-homomorphisms is commutative with both rows exact:

$$\begin{array}{ccccccccc}
0 = \omega_A/\varepsilon^*_{\omega_A} & \longrightarrow & A/\varepsilon^*_A & \xrightarrow{F} & B/\varepsilon^*_B & \xrightarrow{G} & C/\varepsilon^*_C & \longrightarrow & C/\varepsilon^*_{\omega_C} = 0 \\
 & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle L} & & \\
0 = \omega_{A_1}/\varepsilon^*_{\omega_{A_1}} & \longrightarrow & A_1/\varepsilon^*_{A_1} & \xrightarrow[F_1]{} & B_1/\varepsilon^*_{B_1} & \xrightarrow[G_1]{} & C_1/\varepsilon^*_{C_1} & \longrightarrow & \omega_{C_1}/\varepsilon^*_{\omega_{C_1}} = 0.
\end{array}$$

By Lemma 4.5, H and L are one to one R/γ^*-homomorphisms. Then by the five short lemma in modules K is a one to one R/γ^*-homomorphism. Therefore, by Lemma 4.5, k is a weak-monic R-homomorphism.

Alternative Proof. It is sufficient to show that $Ker(k) = \omega_B$. We always have $\omega_B \subseteq Ker(k)$. On the other hand, suppose that $b \in Ker(k)$. Then $k(b) \in \omega_{B_1}$ and so $g_1(k(b)) \in g_1(\omega_{B_1})$. Since $g_1(\omega_{B_1}) \subseteq \omega_{C_1}$, we have $g_1(k(b)) \in \omega_{C_1}$. Since $g_1 \circ k = l \circ g$ and l is weak monic, we obtain $g(b) \in Ker(l) = \omega_C$. Then $b \in Ker(g) \overset{w}{=} Im(f)$ and consequently

$$\varepsilon^*_B(b) = \varepsilon^*_B(f(a)) \text{ for some } a \in A. \tag{3}$$

Since k is a strong H_v-homomorphism, we have $\varepsilon^*_{B_1}(k(b)) = \varepsilon^*_{B_1}(k(f(a)))$. Since $k \circ f = f_1 \circ h$ and $b \in Ker(k)$, we obtain $\varepsilon^*_{B_1}(k(b)) = \varepsilon^*_{B_1}(f_1(h(a))) = \omega_{B_1}$. Therefore, $f_1(h(a)) \in \omega_{B_1}$. Since f_1 is weak-monic we obtain $h(a) \in \omega_{A_1}$ and since h is weak-monic it follows that $a \in \omega_A$. Thus, $f(a) \in f(\omega_A) \subseteq \omega_B$ and by Eq. (3) we obtain $\varepsilon^*_B(b) = \varepsilon^*_B(f(a)) = \omega_B$. Therefore, $b \in \omega_B$ and the proof is complete.

(2) It is similar to (1).

(3) It follows from (1) and (2). □

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