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Primary and biprimary class sizes implying nilpotency of finite groups

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Abstract: Let G be a finite group. We prove that G is nilpotent if the set of conjugacy class sizes of primary and bipirimary elements is $\{1, m, n, mn\}$ with m and n coprime. Moreover, m and n are distinct primes power.

Key words: Finite groups, conjugacy class sizes, primary and biprimary elements

1. Introduction

Throughout this paper all groups considered are finite and G always denotes a group. For an element x of a group G we denote by x^G the conjugacy class containing x, and by $|x^G|$ the conjugacy class size of x. A primary element is an element of prime power order and a biprimary (triprimary) element is an element whose order is divisible by precisely two (three) primes. The rest of the notation and terminology is standard; readers may refer to [7].

In recent years, there has been tremendous interest in studying the structure of a group by some arithmetical conditions imposed on the conjugacy class sizes of G. A classical result due to Itô [8] is that a group G with two conjugacy class sizes is nilpotent and G is solvable if it has three conjugacy class sizes. Beltrán and Felipe [3, 2] studied groups with four conjugacy class sizes and proved that if the set of conjugacy class sizes of G is $\{1, m, n, mn\}$ with integers $m, n > 1$ coprime, then G is nilpotent with m and n distinct primes power.

To investigate the influence of partial conjugacy class sizes on the structure of groups is also an interesting topic. For instance, Li [11] proved that a group G is solvable if its conjugacy class size of every primary element is either 1 or m with m a fixed integer. In [9], Jiang and Shao showed that if the set of conjugacy class sizes of primary, biprimary, and triprimary elements is $\{1, m, n, mn\}$ with m and n coprime, then G is solvable.

In the present paper, we are concerned with the influence of conjugacy class sizes of primary and biprimary elements on the structure of groups. Our main result is the following:

Theorem A Let G be a group. Further let m, $n > 1$ be two coprime integers. If the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, n, mn\}$, then G is nilpotent. Furthermore, $m = p^a$ and $n = q^b$ for distinct primes p and q.

The authors proved in [14] that:

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Theorem B ([14, Main Theorem]) Let G be a solvable group and let m and n be two coprime integers. Suppose further that the conjugacy class size of every primary or biprimary element is one of $\{1, m, n, mn\}$ and all of these occur. Then G is nilpotent. In particular, $m = p^a$ and $n = q^b$ for distinct primes p and q.

As a result, our main task of this paper is to prove the solvability of G. That is:

Theorem C Let G be a group. Further let m, $n > 1$ be two coprime integers. If the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, n, mn\}$, then G is solvable.

In order to show Theorem C, first we prove a special case:

Theorem D Let G be a group and n be an integer coprime to p. If the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, p^a, n, p^a n\}$ with positive integer a, then G is solvable.

2. Preliminaries

Before taking up the problems, we first give some lemmas that will be used in the sequel.

Lemma 2.1 ([12, Theorem 5]) If for some prime p every primary p'-element of a group G has conjugacy class size prime to p, then the Sylow p-subgroup of G is a direct factor of G.

Lemma 2.2 ([10, Theorem 3.2]) Let G be a group such that p^a is the highest power of a prime p that divides the conjugacy class size of a biprimary element of G. Assume that there is a p-element in G whose conjugacy class size is precisely p^a. Then G has a normal p-complement.

Lemma 2.3 ([4, Corollary B]) Let G be a group and suppose that the conjugacy class size of every primary element is 1 or m. Then G is nilpotent. More precisely, $m = p^n$ for some prime p, and $G = P \times A$ with A abelian and P a p-group.

Lemma 2.4 Let G be a group and p a prime. Then every p-element has a p-power conjugacy class size if and only if $G = \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G)$.

Proof The sufficiency is obvious; we only prove the necessity. Since every p-element has a p-power conjugacy class size, we see that $\mathbf{O}_p(G) \in \mathrm{Syl}_p(G)$ by [1, Corollary 4]. By the Schur–Zassenhaus theorem, G has a Hall p'-subgroup, say H. On the other hand, for an arbitrary element $y \in G$, we may write $y = y_p \cdot y_{p'}$, where y_p and $y_{p'}$ are the p-part and the p'-part of y, respectively. Since $|y_p^G|$ is a p-power, there is some $g \in G$ such that $y_{p'} \leq H^g \leq \mathbf{C}_G(y_p)$, yielding $y_p \in \mathbf{C}_G(H)^g$. As a result, $y \in \mathbf{C}_G(H)^g H^g$, leading to $G \subseteq \bigcup_{g \in G} (\mathbf{C}_G(H)H)^g$. Consequently, $G = \mathbf{C}_G(H)H$, implying $H \trianglelefteq G$ and thus $G = \mathbf{O}_p(G) \times \mathbf{O}_{p'}(G)$. □

Lemma 2.5 ([14, Lemma 2.5]) Suppose that the three smallest nontrivial conjugacy class sizes of primary and biprimary elements are $a < b < c$ with $(a, b) = 1$ and $a^2 < c$. Then the set $W := \{g \in G | \ |g^G| = 1 \text{ or } a\}$ is a normal subgroup of G.

Lemma 2.6 ([6, Theorem 5.3.4]) Let $P \times Q$ be the direct product of a p-group P and a p'-group Q. Suppose that G is a p-group such that $\mathbf{C}_G(P) \leq \mathbf{C}_G(Q)$. Then Q acts trivially on G.

Lemma 2.7 ([14, Lemma 2.6]) Let $G = K \rtimes H$ and $g \in H$. Then $\mathbf{C}_G(g) = \mathbf{C}_K(g)\mathbf{C}_H(g)$.

Lemma 2.8 ([13, 9.1.10]) Let the group G possess a nilpotent Hall π-subgroup H. Then every π-subgroup of G is contained in a conjugate of H. In particular, all Hall π-subgroups of G are conjugate.

3. Proof of Theorem D

Proof If there exists a prime $r \in \pi(G) - (\{p\} \cup \pi(n))$, then Lemma 2.1 shows that the Sylow r-subgroup R of G is a direct factor of G, implying that the conjugacy class size of each r-element is an r-number. As a result, $R \leq \mathbf{Z}(G)$ and we may write $G = A \times B$, where $A \leq \mathbf{Z}(G)$ and B is a Hall $\{p\} \cup \pi(n)$-subgroup of G. As central factors are irrelevant in this context, we conclude that the set of conjugacy class sizes of primary and biprimary elements of B is $\{1, p^a, n, p^a n\}$. Without loss of generality, G can be assumed as a $\{p\} \cup \pi(n)$-group. Moreover, we may suppose that $|\pi(n)| \geq 2$ since, otherwise, G is a $\{p, q\}$-group for some prime q distinct from p, and the theorem follows immediately by [5, Theorem 2]. We divide the proof into several steps.

Step 1. There exists no p-element of conjugacy class size p^a.

Assume false. Then G has a normal p-complement K by Lemma 2.2. Further, for every primary element $x \in K$, we have

$$|G : K||K : \mathbf{C}_K(x)| = |G : \mathbf{C}_G(x)||\mathbf{C}_G(x) : \mathbf{C}_K(x)|,$$

yielding to $|x^K| = 1$ or n. As a result, K is nilpotent by Lemma 2.3. Moreover, G is solvable according to [7, Theorem 6.4.3], and we are done.

Step 2. There is no p'-element of conjugacy class size n.

Assume on the contrary that y is a p'-element of conjugacy class size n. By considering its primary decomposition, y can be assumed to be a q-element for some $q \in \pi(n)$. Further, for every primary q'-element $x \in \mathbf{C}_G(y)$, we obtain that $|\mathbf{C}_G(y) : \mathbf{C}_{\mathbf{C}_G(y)}(x)| = |\mathbf{C}_G(y) : \mathbf{C}_G(xy)| = 1$ or p^a, which follows by Lemma 2.1 that $\mathbf{C}_G(y) = \mathbf{C}_G(y)_q \times \mathbf{C}_G(y)_{q'}$, where $\mathbf{C}_G(y)_{q'}$ is the normal Hall q'-subgroup of $\mathbf{C}_G(y)$.

On the other hand, for every primary element $z \in \mathbf{C}_G(y)_{q'}$, we see that $\mathbf{C}_G(y)_q \leq \mathbf{C}_G(z)$, indicating that $\mathbf{C}_G(y) \cap \mathbf{C}_G(z) = \mathbf{C}_G(y)_q(\mathbf{C}_G(y)_{q'} \cap \mathbf{C}_G(z))$. This implies that $|\mathbf{C}_G(y)_{q'} : \mathbf{C}_{\mathbf{C}_G(y)_{q'}}(z)| = |\mathbf{C}_G(y)_{q'} : \mathbf{C}_G(y)_{q'} \cap \mathbf{C}_G(z)| = |\mathbf{C}_G(y) : \mathbf{C}_G(yz)| = 1$ or p^a. Then Lemma 2.3 gives that $\mathbf{C}_G(y)_{q'}$ is nilpotent and thus $\mathbf{C}_G(y)_{q'} = P \times B$, where $P \in \mathrm{Syl}_p(G)$ and B is a Hall $\{p, q\}'$-subgroup of $\mathbf{C}_G(y)$. Moreover, $\mathbf{C}_G(y) = \mathbf{C}_G(y)_{p'} \times P$. Let t be a primary p'-element of conjugacy class size p^a in G, which exists by Step 1. Without loss we may assume that $y \in \mathbf{C}_G(t)$, yielding $t \in \mathbf{C}_G(y)_{p'}$, against the fact that $\mathbf{C}_G(y)_{p'}$ is centralized by P.

Step 3. If x is a p-element of conjugacy class size $p^a n$, then $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'} \nleq \mathbf{Z}(G)$ is the abelian Hall p'-subgroup of $\mathbf{C}_G(x)$, respectively. On the other hand, if y is an r-element of conjugacy class size $p^a n$ with prime $r \neq p$, then $\mathbf{C}_G(y) = \mathbf{C}_G(y)_p \times \mathbf{C}_G(y)_{p'}$, where $\mathbf{C}_G(y)_p \nleq \mathbf{Z}(G)$ is the abelian Sylow p-subgroup of $\mathbf{C}_G(y)$ and $\mathbf{C}_G(y)_{p'}$ is the Hall π'-subgroup of $\mathbf{C}_G(y)$.

Let x be a p-element of conjugacy class size $p^a n$. Then for every primary p'-element $z \in \mathbf{C}_G(x)$, we obtain that $\mathbf{C}_G(xz) = \mathbf{C}_G(x) \cap \mathbf{C}_G(z) \leq \mathbf{C}_G(x)$. By the maximality of $p^a n$, we see that $\mathbf{C}_G(xz) = \mathbf{C}_G(x) \leq \mathbf{C}_G(z)$ and thus $z \in \mathbf{Z}(\mathbf{C}_G(x))$. This shows that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_p$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and $\mathbf{C}_G(x)_{p'}$ is the abelian Hall p'-subgroup of $\mathbf{C}_G(x)$. Assume that $\mathbf{C}_G(x)_{p'} \leq \mathbf{Z}(G)$.

Then $\mathbf{C}_G(x)_{p'} = \mathbf{Z}(G)_{p'}$ and thus $|G:\mathbf{Z}(G)|_{p'} = n$. We prove that $\mathbf{C}_G(t)_p = \mathbf{Z}(G)_p$ for every noncentral primary p'-element t. If not, we may select some element $w \in \mathbf{C}_G(t)_p - \mathbf{Z}(G)$ satisfying $|w^G| = n$ or $p^a n$ by Step 2. Both cases indicate that $\mathbf{Z}(G)_{p'}$ is a Hall p'-subgroup of $\mathbf{C}_G(w)$, yielding that $t \in \mathbf{Z}(G)$, contrary to the choice of t. Hence, $|G:\mathbf{Z}(G)|_p = p^a$ and thus $|G:\mathbf{Z}(G)| = |G:\mathbf{Z}(G)|_p|G:\mathbf{Z}(G)|_{p'} = p^a n$, against the existence of a primary or biprimary element of conjugacy class size $p^a n$.

Let y be an r-element of conjugacy class sizes $p^a n$ with $r \neq p$. The same argument above implies that $\mathbf{C}_G(y) = \mathbf{C}_G(y)_r \times \mathbf{C}_G(y)_{r'} = \mathbf{C}_G(y)_p \times \mathbf{C}_G(y)_{p'}$, where $\mathbf{C}_G(y)_p$ is the abelian Sylow p-subgroup of $\mathbf{C}_G(y)$ and $\mathbf{C}_G(y)_{p'}$ is the Hall p'-subgroup of $\mathbf{C}_G(y)$. Suppose that $\mathbf{C}_G(y)_p \leq \mathbf{Z}(G)$. Then $\mathbf{C}_G(y)_p = \mathbf{Z}(G)_p$, yielding that $|G:\mathbf{Z}(G)|_p = p^a$. We prove that $\mathbf{C}_G(w)_{p'} = \mathbf{Z}(G)_{p'}$ for every noncentral p-element w. Otherwise, we select some element $t \in \mathbf{C}_G(w)_{p'} - \mathbf{Z}(G)$, a primary element. Then $|t^G| = p^a$ or $p^a n$ by Step 2. Further, both cases imply that $\mathbf{Z}(G)_p$ is a Sylow p-subgroup of $\mathbf{C}_G(t)$, yielding that $w \in \mathbf{Z}(G)$, contrary to the choice of w. This shows that $|G:\mathbf{Z}(G)|_{p'} = n$ and thus $|G:\mathbf{Z}(G)| = |G:\mathbf{Z}(G)|_p|G:\mathbf{Z}(G)|_{p'} = p^a n$, against our assumption.

In the following, we divide the proof into two cases: $p^a > n$ and $p^a < n$.

Case 1. $p^a > n$.

Step 4. $L_p := \{x \in G | x \text{ is a } p\text{-element such that } |x^G| = 1 \text{ or } n\}$ is an abelian normal Sylow p-subgroup of G.

Since $p^a > n$, by Lemma 2.5 we see that $W := \{x \in G|\ |x^G| = 1 \text{ or } n\}$ is a normal subgroup of G. Moreover, $W = L_p \times \mathbf{Z}(G)_{p'}$ since there is no p'-element of conjugacy class size n by Step 2. As a result, $L_p \trianglelefteq G$. Moreover, L_p is abelian since $|u^{L_p}|$ divides $(|L_p|, n) = 1$ for each element $u \in L_p$.

If L_p is not a Sylow p-subgroup of G, then there exists a p-element y such that $|y^G| = p^a n$ by Step 1, which leads to $\mathbf{C}_G(y) = \mathbf{C}_G(y)_p \times \mathbf{C}_G(y)_{p'}$ with the abelian Hall p'-subgroup $\mathbf{C}_G(y)_{p'}$ such that $\mathbf{C}_G(y)_{p'} \nleq \mathbf{Z}(G)$ by Step 3. Taking an arbitrary primary element $z \in \mathbf{C}_G(y)_{p'} - \mathbf{Z}(G)$, we see that $\mathbf{C}_G(y) \leq \mathbf{C}_G(z)$ and thus $\mathbf{C}_{L_p}(y) \leq \mathbf{C}_{L_p}(z)$, which follows by Lemma 2.6 that $z \in \mathbf{C}_G(L_p) =: M$. Consequently, $\mathbf{C}_G(y)_{p'} \leq M$. On the other hand, because z has conjugacy class size p^a or $p^a n$, we see that $|\mathbf{C}_G(z):\mathbf{C}_G(y)| = 1$ or n. Note that $L_p \leq \mathbf{C}_G(z)$. This indicates $L_p \leq \mathbf{C}_G(y)$ and thus $y \in M$. We conclude that M contains all p-elements of G as L_p is abelian. As a result, $\mathbf{C}_G(y) \leq M$ and $|G:M|$ is a p'-number. Note that $M \leq \mathbf{C}_G(k)$ for every $k \in L_p - \mathbf{Z}(G)$ because L_p is abelian. This yields that n divides $|G:M|$.

Along with the equality

$$|G:M||M:\mathbf{C}_G(y)| = |G:\mathbf{C}_G(y)| = p^a n,$$

we see clearly that $|G:M| = n$ and $|M:\mathbf{C}_G(y)| = p^a$, indicating that $\mathbf{C}_G(y)_{p'}$ is a Hall p'-subgroup of M. Further, every p-element of M has conjugacy class size 1 or p^a in M. By Lemma 2.4, we see that $M = M_p \times \mathbf{C}_G(y)_{p'}$, where $M_p \in \mathrm{Syl}_p(G)$ and $\mathbf{C}_G(y)_{p'} \nleq \mathbf{Z}(G)$ by Step 3. If we choose a noncentral primary element $w \in \mathbf{C}_G(y)_{p'}$, we get $|w^G| = n$, against Step 2.

Step 5. Conclusion in Case 1.

Since G has an abelian normal Sylow p-subgroup L_p, we obtain that G has a p-complement H by the Schur–Zassenhaus theorem. If H is abelian, then G is solvable by [7, Theorem 6.4.3], which follows by Theorem B that G is nilpotent. Write $G = P \times H$, where $P \in \mathrm{Syl}_p(G)$. Consequently, $H \leq \mathbf{Z}(G)$, a contradiction to our assumption. As a result, H is nonabelian. Lemma 2.7 implies that the set of conjugacy class sizes of primary elements of H is $\{1,n\}$. Hence, H is nilpotent by Lemma 2.3, yielding that G is solvable, and the theorem is proved.

Case 2. $p^a < n$.

Step 6. Let q be a prime dividing n. Denote that $L_q := \{x \in G | \, x$ is a q-element such that $|x^G| = 1$ or $p^a\}$. If L_q is not central, then L_q is the normal Sylow q-subgroup of G.

Since $p^a < n$, we obtain that $L_{p'} := \{x \in G| \ x$ is a p'-element such that $|x^G| = 1$ or $p^a\}$ is an abelian normal p'-subgroup of G if we apply a similar argument as in Step 4. Further, L_q is abelian.

Assume that $L_q \nleq \mathbf{Z}(G)$. If L_q is not a Sylow q-subgroup of G, then there exists a q-element w satisfying $|w^G| = p^a n$ according to Step 2. Step 3 implies that $\mathbf{C}_G(w) = \mathbf{C}_G(w)_p \times \mathbf{C}_G(w)_{p'}$, where $\mathbf{C}_G(w)_p \nleq \mathbf{Z}(G)$ is the abelian Sylow p-subgroup of $\mathbf{C}_G(w)$ and $\mathbf{C}_G(w)_{p'}$ is the Hall p'-subgroup of $\mathbf{C}_G(w)$. For each element $u \in \mathbf{C}_G(w)_p - \mathbf{Z}(G)$, we have $\mathbf{C}_G(w) \leq \mathbf{C}_G(u)$, yielding $\mathbf{C}_{L_q}(w) \leq \mathbf{C}_{L_q}(u)$. By Lemma 2.6, we obtain that $u \in \mathbf{C}_G(L_q) =: N$ and thus $\mathbf{C}_G(x)_p \leq N$. On the other hand, we see that $|\mathbf{C}_G(u) : \mathbf{C}_G(w)| = 1$ or p^a. Since $L_q \leq \mathbf{C}_G(u)$, it follows that $L_q \leq \mathbf{C}_G(w)$, leading to $w \in N$. Consequently, every q-element of G lies in N. Fix $y \in L_q$ a noncentral q-element. We see that $\mathbf{C}_G(w)_p \leq N \leq \mathbf{C}_G(y)$ as L_q is abelian. Moreover, $|\mathbf{C}_G(y) : N||N : \mathbf{C}_G(w)_p| = |\mathbf{C}_G(y) : \mathbf{C}_G(w)_p|$ is a p'-number, which implies that $\mathbf{C}_G(w)_p \in \mathrm{Syl}_p(N)$ and $\mathbf{C}_G(w)_p \in \mathrm{Syl}_p(\mathbf{C}_G(y))$.

We claim that there exists some $g \in \mathbf{C}_G(y)$ such that $v \in \mathbf{C}_G(\mathbf{C}_G(w)_p^g)$ for an arbitrary element $v \in \mathbf{C}_G(y) - \mathbf{Z}(G)$. If there exists some component v_i of v with conjugacy class size $p^a n$, say v_1, then we see easily that $\mathbf{C}_G(v) = \mathbf{C}_G(v_1)$. Moreover, $|\mathbf{C}_G(y) : \mathbf{C}_G(yv_1)| = n$. By Sylow's theorem, we see that there exists some $g \in \mathbf{C}_G(y)$ such that $\mathbf{C}_G(w)_p^g \leq \mathbf{C}_G(yv_1) \leq \mathbf{C}_G(v_1) = \mathbf{C}_G(v)$, leading to $v \in \mathbf{C}_G(\mathbf{C}_G(w)_p^g)$. Hence, we assume that every component has no conjugacy class size $p^a n$. Let v_1 be the p-component of v and $v_2, \ldots, v_t$ be all the p'-components of v. We show that $\mathbf{C}_G(v_2 \cdots v_t) = \mathbf{C}_G(v_2)$. If $t = 2$, there is nothing to prove. Assume then that $t > 2$ and $j \in \{3, \ldots, t\}$. Then $|v_2^G| = p^a$ and $|v_j^G| = p^a$ by Step 2. Moreover, it follows that $p^a = |v_2^G| \mid |(v_2 v_j)^G| \leq |v_2^G||v_j^G| = p^{2a} < p^a n$ by [13, 1.3.11], yielding $\mathbf{C}_G(v_2 v_j) = \mathbf{C}_G(v_2)$. Further, $\mathbf{C}_G(v_2 \cdots v_t) = \mathbf{C}_G(v_2)$, as required. This gives that $\mathbf{C}_G(v) = \mathbf{C}_G(v_1 v_2)$. In particular, $\mathbf{C}_G(v) = \mathbf{C}_G(v_1) = \mathbf{C}_G(v_2)$ if we apply a similar argument above. Recall that $|\mathbf{C}_G(y) : \mathbf{C}_G(yv_1)| = n$. By Sylow's theorem, there exists some $g \in \mathbf{C}_G(y)$ such that $\mathbf{C}_G(w)_p^g \leq \mathbf{C}_G(yv_1) \leq \mathbf{C}_G(v_1)$, leading to $v_1 \in \mathbf{C}_G(\mathbf{C}_G(w)_p^g)$. If v_2 is a q-element, then $\mathbf{C}_G(w)_p \leq N \leq \mathbf{C}_G(v_2)$ is also a Sylow p-subgroup of $\mathbf{C}_G(v_2)$ by the second argument of this step, leading to $v_2 \in \mathbf{C}_G(\mathbf{C}_G(w)_p)$; if v_2 is a q'-component, then $\mathbf{C}_G(yv_2) = \mathbf{C}_G(v_2) = \mathbf{C}_G(y)$, which also implies that $v_2 \in \mathbf{C}_G(\mathbf{C}_G(w)_p^g)$. Consequently, $\mathbf{C}_G(w)_p^g \leq \mathbf{C}_G(v_1 v_2) = \mathbf{C}_G(v)$, yielding $v \in \mathbf{C}_G(\mathbf{C}_G(w)_p^g)$, as claimed.

Therefore, $\mathbf{C}_G(y) = \bigcup_{g \in \mathbf{C}_G(y)} \mathbf{C}_G(\mathbf{C}_G(w)_p)^g$, which forces that $\mathbf{C}_G(w)_p$ must be central in $\mathbf{C}_G(y)$. However, $\mathbf{C}_G(w)_p$ is not central in G by Step 3. Thus, if we choose some noncentral element $u_1 \in \mathbf{C}_G(y)_p$, we have $\mathbf{C}_G(y) \leq \mathbf{C}_G(u_1)$, leading to $|u_1^G| = p^a$, against Step 1.

Step 7. Conclusion in Case 2.

Let $t \in \mathbf{C}_G(y)$ be an arbitrary element. Write $t = t_q \cdot t_{q'}$ as before. If we apply a similar argument in

Step 6, we obtain that

$$\mathbf{C}_G(y)=\bigcup_{g\in\mathbf{C}_G(y)}\mathbf{C}_{\mathbf{C}_G(y)}(\mathbf{C}_G(y)_p)^gL_q=\bigcup_{g\in\mathbf{C}_G(y)}(\mathbf{C}_{\mathbf{C}_G(y)}(\mathbf{C}_G(y)_pL_q))^g,$$

which yields $\mathbf{C}_G(y)=\mathbf{C}_{\mathbf{C}_G(y)}(\mathbf{C}_G(y)_p)L_q$. Hence, $|G:\mathbf{C}_{\mathbf{C}_G(y)}(\mathbf{C}_G(y)_p)|$ is a $\{p,q\}$-number. Now, if there exists some noncentral element $u\in\mathbf{C}_G(y)_p$ that has conjugacy class size n or p^an, we see that n is a q-power, against $|\pi(n)|\geq 2$. $\square$

4. Proof of Theorem C

Proof If we reason similarly as to the proof of Theorem D, we may assume that G is a $(\pi(m)\cup\pi(n))$-group with $|\pi(m)|\geq 2$ and $|\pi(n)|\geq 2$. Write $\pi:=\pi(m)$. The proof will be completed in several following steps.

Step 1. If x is a primary π-element of conjugacy class size mn, then $\mathbf{C}_G(x)=\mathbf{C}_G(x)_\pi\times\mathbf{C}_G(x)_{\pi'}$, where $\mathbf{C}_G(x)_{\pi'}\nleq\mathbf{Z}(G)$ is an abelian Hall π'-subgroup of $\mathbf{C}_G(x)$. Analogously, if y is a primary π'-element of conjugacy class size mn, then $\mathbf{C}_G(y)=\mathbf{C}_G(y)_\pi\times\mathbf{C}_G(y)_{\pi'}$, where $\mathbf{C}_G(y)_\pi\nleq\mathbf{Z}(G)$ is an abelian Hall π-subgroup of $\mathbf{C}_G(y)$.

This follows exactly by a similar argument as in Step 3 of Theorem D.

Step 2. G has no primary π-element of conjugacy class size m. Analogously, there exists no π'-element of conjugacy class size n.

By the symmetry of m and n, we only prove the first statement. Let x be a primary π-element of conjugacy class size m. We may consider x as a p-element with $p\in\pi$. Then for every primary p'-element $y\in\mathbf{C}_G(x)$, we have $|\mathbf{C}_G(x):\mathbf{C}_{\mathbf{C}_G(x)}(y)|=|\mathbf{C}_G(x):\mathbf{C}_G(xy)|=1$ or n, which follows by Lemma 2.1 that $\mathbf{C}_G(x)=\mathbf{C}_G(x)_p\times\mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_{p'}$ is the Hall p'-subgroup of $\mathbf{C}_G(x)$.

For each primary element $y\in\mathbf{C}_G(x)_{p'}$, we have $\mathbf{C}_G(x)_p\leq\mathbf{C}_G(y)$, implying $\mathbf{C}_G(x)\cap\mathbf{C}_G(y)=\mathbf{C}_G(x)_p(\mathbf{C}_G(x)_{p'}\cap\mathbf{C}_G(y))$. As a result, $|\mathbf{C}_G(x)_{p'}:\mathbf{C}_{\mathbf{C}_G(x)_{p'}}(y)|=|\mathbf{C}_G(x)_{p'}:\mathbf{C}_G(x)_{p'}\cap\mathbf{C}_G(y)|=|\mathbf{C}_G(x):\mathbf{C}_G(xy)|=1$ or n. If n occurs, then n is a prime power according to Lemma 2.3, against our assumption. Hence, $\mathbf{C}_G(x)_{p'}$ is abelian, implying that G has an abelian Hall π'-subgroup H. Let $y\in G$ be a primary or biprimary element of conjugacy class sizes n. We may assume without loss that y is a q-element with prime $q\in\pi$ if we consider the primary decomposition of y. As a result, there is some $g\in G$ such that $x^g\in\mathbf{C}_G(y)$, yielding that $y\in\mathbf{C}_G(x^g)=\mathbf{C}_G(x)_\pi^g\times H^g$. Moreover, $H^g\leq\mathbf{C}_G(y)$, against $|y^G|=n$.

Without loss of generality, we will assume that $n<m$ in the following.

Step 3. Write $L_\pi:=\{x\in G|x$ is a π-element with $|x^G|=1$ or $n\}$. Then L_π is a nontrivial abelian normal π-subgroup of G.

By Lemma 2.5, the set $W:=\{x\in G||x^G|=1$ or $n\}$ is a normal subgroup of G. Moreover, it follows by Step 2 that $W=L_\pi\times\mathbf{Z}(G)_{\pi'}$ and, consequently, L_π is a nontrivial normal π-subgroup of G. Further, for each primary element $y\in L_\pi$, we have that $|y^{L_\pi}|$ divides $(|L_\pi|,n)=1$, indicating that L_π is abelian.

Write $L_q:=\{x\in G|x$ to be a q-element such that $|x^G|=1$ or $n\}$ with $q\in\pi$. Then L_π is the direct product of the subgroups L_q for all primes $q\in\pi$. As a sequence, L_q is an abelian normal subgroup of G.

Step 4. If L_q is not central in G, then L_q is a Sylow q-subgroup of G.

Assume that $L_q \nleq \mathbf{Z}(G)$. If L_q is not a Sylow q-subgroup of G, then there exists some q-element w of conjugacy class size mn by Step 2. Moreover, Step 1 gives that $\mathbf{C}_G(w) = \mathbf{C}_G(w)_\pi \times \mathbf{C}_G(w)_{\pi'}$ with $\mathbf{C}_G(w)_{\pi'} \nleq \mathbf{Z}(G)$ abelian. For every $u \in \mathbf{C}_G(w)_{\pi'}$, we have $\mathbf{C}_G(w) \leq \mathbf{C}_G(u)$ and, in particular, $\mathbf{C}_{L_q}(w) \leq \mathbf{C}_{L_q}(u)$. By applying Lemma 2.6, we get $u \in \mathbf{C}_G(L_q) =: N$ and, therefore, $\mathbf{C}_G(w)_{\pi'} \leq N$. On the other hand, $|\mathbf{C}_G(u) : \mathbf{C}_G(w)| = 1$ or n since u has conjugacy class size m or mn. Note that $L_q \leq \mathbf{C}_G(u)$. This implies that $L_q \leq \mathbf{C}_G(w)$ and thus $w \in N$. We conclude that N contains all q-elements of G.

Fix $y \in L_q - \mathbf{Z}(G)$. Then $\mathbf{C}_G(w)_{\pi'} \leq N \leq \mathbf{C}_G(y)$. Moreover, $|\mathbf{C}_G(y) : N||N : \mathbf{C}_G(w)_{\pi'}| = |\mathbf{C}_G(y) : \mathbf{C}_G(w)_{\pi'}|$ is a π-number, indicating that both $|\mathbf{C}_G(y) : N|$ and $|N : \mathbf{C}_G(w)_{\pi'}|$ are π-numbers. Therefore, $\mathbf{C}_G(w)_{\pi'} \nleq \mathbf{Z}(G)$ is an abelian Hall π'-subgroup of N and $\mathbf{C}_G(y)$. Let $R \leq \mathbf{C}_G(w)_{\pi'} \leq N$ be a noncentral Sylow r-subgroup of $\mathbf{C}_G(y)$ with $r \in \pi'$. We prove that for every noncentral element $v \in \mathbf{C}_G(y) - \mathbf{Z}(G)$, there exists some $g \in \mathbf{C}_G(y)$ such that $v \in \mathbf{C}_G(R^g)$.

If there is some component v_i of conjugacy class size mn, say v_1, then $\mathbf{C}_G(v) = \mathbf{C}_G(v_1)$. Moreover, if v_1 is a q-component, then $R \leq N \leq \mathbf{C}_G(v_1)$, yielding $v \in \mathbf{C}_G(R)$; if v_1 is a q'-component, then $\mathbf{C}_G(yv_1) = \mathbf{C}_G(v_1) \leq \mathbf{C}_G(y)$ and $|\mathbf{C}_G(y) : \mathbf{C}_G(yv_1)| = n$. By Sylow's theorem, there exists some $g \in \mathbf{C}_G(y)$ such that $R^g \leq \mathbf{C}_G(yv_1) = \mathbf{C}_G(v_1) = \mathbf{C}_G(v)$, leading to $v \in \mathbf{C}_G(R^g)$. As a consequence, we assume that v has no component of conjugacy class size mn. Write $v = (v_1 \cdots v_r) \cdot (v_{r+1} \cdots v_t)$, where $v_1, ..., v_r$ are all the π-components of v and $v_{r+1}, ..., v_t$ are all the π'-components of v, respectively. Note that $\mathbf{C}_G(w)_{\pi'}$ is an abelian Hall π'-subgroup of $\mathbf{C}_G(y)$. Then every π'-element of $\mathbf{C}_G(y)$ is contained in a conjugate of $\mathbf{C}_G(w)_{\pi'}$ by applying Lemma 2.8. As a result, there exists some $g \in \mathbf{C}_G(y)$ such that $v_{r+1} \cdots v_t \in \mathbf{C}_G(w)^g_{\pi'}$, leading to $v_{r+1} \cdots v_t \in \mathbf{C}_G(R^g)$. Hence, $R^g \leq \mathbf{C}_G(v_{r+1} \cdots v_t)$. On the other hand, by Step 2, we see that each v_i has conjugacy class size n with $i \in \{1, \dots, r\}$. For every $j \in \{2, \dots, r\}$, we see that $n = |v_1^G| \mid |(v_1 v_j)^G| \leq |v_1^G||v_j^G| = n^2$ by [13, 1.3.11], and this implies that $\mathbf{C}_G(v_1) = \mathbf{C}_G(v_1 v_j)$, implying $\mathbf{C}_G(v_1 \cdots v_r) = \mathbf{C}_G(v_1)$. Analogously, $\mathbf{C}_G(y) = \mathbf{C}_G(yv_1) = \mathbf{C}_G(v_1)$. If v_1 is a q'-element, then by Sylow's theorem, there exists some $g \in \mathbf{C}_G(y)$ such that $R^g \leq \mathbf{C}_G(v_1)$ and thus $v_1 \in \mathbf{C}_G(R^g)$; if v_1 is a q-element, then by a similar argument above we obtain that $R \leq N \leq \mathbf{C}_G(v_1)$. This shows that $R^g \leq \mathbf{C}_G(v_1 \cdot v_{r+1} \cdots v_t) = \mathbf{C}_G(v)$, yielding $v \in \mathbf{C}_G(R^g)$.

Therefore, $\mathbf{C}_G(y) = \bigcup_{g \in \mathbf{C}_G(y)} \mathbf{C}_{\mathbf{C}_G(y)}(R)^g$, which implies that R must be central in $\mathbf{C}_G(y)$. However, we know that R is not central in G, and so if we take some noncentral $u_1 \in R$, we have $\mathbf{C}_G(y) \leq \mathbf{C}_G(R) \leq \mathbf{C}_G(u_1)$. This provides an r-element u_1 of conjugacy class size n, against Step 2.

Step 5. Final contradiction.

We will complete this theorem in the following two cases:

Case 1. L_π is a Hall π-subgroup of G.

By the Schur–Zassenhaus theorem, G has a π-complement H. If H is abelian, then G is solvable, and we are done. Assume then that H is nonabelian. Then it follows by Lemma 2.7 that the conjugacy class sizes of primary elements of H are $\{1, n\}$. Then Lemma 2.3 implies that H is nilpotent, yielding that G is also solvable, and the theorem is proved.

Case 2. L_π is not a Hall π-subgroup of G.

In this case, there must be some prime $p \in \pi$ such that $L_p \leq \mathbf{Z}(G)$. Further, by Step 2 there exists some $q \in \pi$ such that L_q is not central in G, and thus L_q is a Sylow q-subgroup of G by Step 4. Fix y a q-element of conjugacy class size n. Let t be a p-element of conjugacy class size mn in G. Without loss, we assume that $t \in \mathbf{C}_G(y)$. It follows by Step 1 that $\mathbf{C}_G(t) = \mathbf{C}_G(t)_\pi \times \mathbf{C}_G(t)_{\pi'}$ with $\mathbf{C}_G(t)_{\pi'} \nleq \mathbf{Z}(G)$ abelian. Notice that $|\mathbf{C}_G(y) : \mathbf{C}_G(y) \cap \mathbf{C}_G(t)| = m$. Then $\mathbf{C}_G(t)_{\pi'}$ is also a Hall π'-subgroup of $\mathbf{C}_G(y)$. According to Lemma 2.8, all the Hall π'-subgroups of $\mathbf{C}_G(y)$ are conjugate.

Since $\mathbf{C}_G(t)_{\pi'} \nleq \mathbf{Z}(G)$, there exists a noncentral Sylow r-subgroup R of $\mathbf{C}_G(y)$ for some prime $r \in \pi'$. The same arguments in Step 4 give that $v_{p'} \leq \mathbf{C}_G(R^g)$ for every element $v \in \mathbf{C}_G(y)$. Thus, if we take into account that L_q is a normal Sylow q-subgroup of G, we have

$$\mathbf{C}_G(y) = \bigcup_{g \in \mathbf{C}_G(y)} \mathbf{C}_{\mathbf{C}_G(y)}(R)^g L_q = \bigcup_{g \in \mathbf{C}_G(y)} (\mathbf{C}_{\mathbf{C}_G(y)}(R) L_q)^g.$$

This implies that $\mathbf{C}_G(y) = \mathbf{C}_{\mathbf{C}_G(y)}(R) L_q$, and accordingly, $|\mathbf{C}_G(y) : \mathbf{C}_{\mathbf{C}_G(y)}(R)|$ is a q-number. Now we take some noncentral $u_1 \in R$, which has conjugacy class size m or mn. Observe that $\mathbf{C}_{\mathbf{C}_G(y)}(R) \leq \mathbf{C}_G(u_1) \cap \mathbf{C}_G(y) = \mathbf{C}_G(u_1 y) \leq \mathbf{C}_G(y)$, so that $u_1 y$ has conjugacy class size n or mn. The first case leads to $\mathbf{C}_G(y) \leq \mathbf{C}_G(u)$, which is a contradiction, and so $u_1 y$ has conjugacy class size mn and it follows that m is a q-power. By Theorem D, we obtain that G is solvable and the theorem is established. $\square$

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