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On the extended zero divisor graph of commutative rings

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Abstract: In this paper we present a new graph that is closely related to the classical zero-divisor graph. In our case two nonzero distinct zero divisors x and y of a commutative ring R are adjacent whenever there exist two nonnegative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. This yields an extension of the classical zero divisor graph $\Gamma(R)$ of R, which will be denoted by $\overline{\Gamma}(R)$. First we distinguish when $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide. Various examples in this context are given. We show that if $\overline{\Gamma}(R) \neq \Gamma(R)$, then $\overline{\Gamma}(R)$ must contain a cycle. We also show that if $\overline{\Gamma}(R) \neq \Gamma(R)$ and $\overline{\Gamma}(R)$ is complemented, then the total quotient ring of R is zero-dimensional. Among other things, the diameter and girth of $\overline{\Gamma}(R)$ are also studied.

Key words: Zero divisor graphs, extended zero divisor graphs

1. Introduction

Throughout this paper all rings are commutative with identity element with $1 \neq 0$.

Setup and notation. Let R be a ring. We use $Z(R)$ to denote the set of all zero divisors of R and $Z(R)^* := Z(R)\backslash\{0\}$. We denote by $\mathrm{Ann}(x)$ the annihilator of an element x of R. For an ideal I of R, $\sqrt{I}$ means the radical of I; in particular, $\mathrm{Nil}(R) := \sqrt{0}$ is the nilradical of R. For a nonzero nilpotent element x of R, n_x denotes the index of nilpotency of x. The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo a nonnegative integer $n \in \mathbb{N}^*$ will be noted by $\mathbb{Z}_n$. We use $\subset$ to mean "is a not necessarily proper subset of" and $\subsetneq$ to mean "is a proper subset of". Finally, $T(R) = S^{-1}R$, where S is the set of regular elements, the total quotient ring of R.

For general background information and terminology on commutative rings with zero divisors we refer the reader to [19].

The zero-divisor graph of a ring R, denoted by $\Gamma(R)$, is the simple graph associated to R such that its vertex set consists of all its nonzero zero divisors and that two distinct vertices are joined by an edge if and only if the product of these two vertices is zero. The idea of associating graphs with algebraic structures goes back to Beck in [17], where he was mainly interested in colorings. In his work all elements of the ring were vertices of the graph (see also [2]). It was Anderson and Livingston, in [11], who introduced the zero divisor graph of a commutative ring and initiated the study of the relation between ring-theoretic properties and graph theoretic ones. Since then, the zero divisor graphs of commutative rings have attracted the attention of several

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researchers (see, for instance, [4, 6, 7, 10, 11, 12, 14, 15, 20, 21]). It was proved, among other things, that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \leq 3$ and $\text{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. For a survey and recent results concerning zero divisor graphs, we refer the reader to [5].

Motivated then by the success of this new area of research several authors have recently introduced other graphs associated to some ring theoretic properties (see, for instance, [8, 9, 13, 16, 22, 23]). The main aim of studying these graphs is that one may find some results about the algebraic structures and vice versa.

In this paper, we introduce an extension of the classical zero divisor graph of a commutative ring R, denoted by $\overline{\Gamma}(R)$, which we call *the extended zero divisor graph* of R, such that its vertex set consists of all its nonzero zero divisors and that two distinct vertices x and y are joined by an edge if and only if there exist two nonnegative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Thus, obviously the classical graph $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$. Note also that $\overline{\Gamma}(R)$ is the empty graph if and only if R is an integral domain.

In Section 2, we are interested in studying when $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide. The main result in this section is Theorem 2.1. It gives some conditions on R that characterize when $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide. In this context several examples are given (see Example 2.3, Proposition 2.4, and Example 2.5). Then we study this property for finite direct products of rings (Proposition 2.8). Section 3 is devoted to the study of the diameter of extended graphs of commutative rings. Obviously, as an extension of the classical graph of $\Gamma(R)$, $\overline{\Gamma}(R)$ is also connected and has diameter of at most 3. In Theorem 3.5 we give the analog result of [7, Theorem 2.2] where the diameter of $\Gamma(R)$ is studied in the case where $Z(R) = \text{Nil}(R) \neq \{0\}$ (see Theorem 3.5). In Theorem 3.2 we give the analog of [11, Theorem 2.5] where we characterize the case where $\overline{\Gamma}(R)$ has a vertex adjacent to every other vertex. This allows us to characterize when the graph $\overline{\Gamma}(R)$ is complete (see Theorem 3.3). We also study the diameter of the graph of finite direct products of rings (see Proposition 3.6). Finally, in Section 4, we study the girth of $\overline{\Gamma}(R)$. Also, since $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$ and by [10, Theorem 2.4], we deduce that $\text{gr}(\overline{\Gamma}(R)) \in \{3, 4, \infty\}$. In Theorem 4.1, we show that $\overline{\Gamma}(R)$ contains a cycle when $\overline{\Gamma}(R) \neq \Gamma(R)$. In Theorem 4.4, we give the analog of [7, Theorem 2.11] in which the girth of $\Gamma(R)$ is studied when $Z(R) = \text{Nil}(R) \neq \{0\}$. In Theorems 4.5 and 4.6, situations where $\text{gr}(\overline{\Gamma}(R)) = 4$ are given. In Proposition 4.8, we show that if $\overline{\Gamma}(R) \neq \Gamma(R)$ and $\overline{\Gamma}(R)$ is complemented, then the total quotient ring of R is zero-dimensional. At the end of the paper, the girth of the graph of finite direct products of rings and $\mathbb{Z}_n$ is studied (see Propositions 4.9 and 4.10).

For completeness, it is convenient to recall some notions on graph theory used in this paper. Here we use the terminology given in [5]. For a general background on graph theory, we refer the reader to [18].

Let G be a (undirected) graph. We say that G is connected if there is a path between any two distinct vertices. For distinct vertices x and y in G, the distance between x and y, denoted by $d(x, y)$, is the length of a shortest path connecting x and y, and if no such path exists, we set $d(x, y) = \infty$ (by convention $d(x, x) = 0$). The diameter of the graph G is the quantity $\text{diam}(\Gamma) := sup\{d(x, y)|x \text{ and } y \text{ are vertices of } G\}$. A cycle of length $n \in \mathbb{N}^*$ in G is a path of the form $x_1 - x_2 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the girth of G, denoted by $\text{gr}(G)$, as the length of a shortest cycle in G, provided that G contains a cycle; otherwise, $\text{gr}(G) = \infty$. A graph G is said to be complete if any two distinct vertices are adjacent. A complete bipartite graph is a graph G, which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$. When $G = K^{1,n}$, G is called a star graph. Finally,

$\overline{K}^{m,3}$ is the graph formed by joining a graph $G_1 = K^{m,3}$ ($= A \cup B$ with $|A| = m$ and $|B| = 3$) to the star graph $G_2 = K^{1,3}$ by identifying the center of G_2 and a point of B.

2. When do $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide?

We begin with the main result in this section, which studies when $\overline{\Gamma}(R)$ and $\Gamma(R)$ coincide.

Theorem 2.1 *Let R be a ring. The following statements are equivalent:*

1. $\overline{\Gamma}(R) = \Gamma(R)$.
2. *R satisfies the two following conditions:*
   (i) If $\mathrm{Nil}(R) \neq \{0\}$, *then every nonzero nilpotent element has index* 2, *and*
   (ii) For every $x \in Z(R) \backslash \mathrm{Nil}(R)$, $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$.
3. *R satisfies the two following conditions:*
   (i) If $\mathrm{Nil}(R) \neq \{0\}$, *then every nonzero nilpotent element has index* 2, *and*
   (ii) For every $x \in Z(R)$, $\sqrt{\mathrm{Ann}(x)} \backslash \mathrm{Nil}(R) \subset \mathrm{Ann}(x)$.

To prove this theorem, we need the following lemma.

Recall that, for a nonzero nilpotent element x of R, we use n_x to denote the index of nilpotency of x.

Lemma 2.2 *Let R be a ring and let $x \in R \backslash \{0\}$. Then:*

1. *If x is nilpotent, then* $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(x^n)$ *for every integer* $n \geq 2$.
2. *If x is not nilpotent, then we have the equivalence:*
   $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$ *if and only if* $\mathrm{Ann}(x^n) = \mathrm{Ann}(x)$ *for every integer* $n \geq 2$.

Proof 1. Let x be a nonzero nilpotent element of R. If $n_x = 2$, then for every integer $n \geq 2$, $\mathrm{Ann}(x^n) = \mathrm{Ann}(0) = R \supsetneq \mathrm{Ann}(x)$. Now consider $n_x \geq 3$ and suppose by contradiction that there is $n \geq 2$ such that $\mathrm{Ann}(x^n) = \mathrm{Ann}(x)$. Since for $n \geq n_x$ we have $\mathrm{Ann}(x^n) = \mathrm{Ann}(0) = R$, n must be between 2 and $n_x - 1$. Then we have $x^{n_x - n} \in \mathrm{Ann}(x^n) = \mathrm{Ann}(x)$. Thus, $x^{n_x-n}x = x^{n_x-n+1} = 0$, which is absurd since $2 \leq n_x - n + 1 \leq n_x - 1$.

2. Let x be a nonnilpotent element such that $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$. Let $y \in \mathrm{Ann}(x^n)$ for some integer $n \geq 2$; then $yx^n = 0$, which implies that $yx \in \mathrm{Ann}(x^{n-1})$. By induction $\mathrm{Ann}(x^{n-1}) = \mathrm{Ann}(x)$, and hence $y \in \mathrm{Ann}(x^2) = \mathrm{Ann}(x)$, as desired. □

Proof of Theorem 2.1. (1) $\Rightarrow$ (2). Suppose that there exists a nilpotent element x such that $n_x \geq 3$. By Lemma 2.2, $\mathrm{Ann}(x) \subsetneq \mathrm{Ann}(x^n)$ for every integer $n \geq 2$. We may assume that $2 \leq n < n_x$. Consider an element $y \in \mathrm{Ann}(x^n) \backslash \mathrm{Ann}(x)$; then $x^n y = 0$ and $xy \neq 0$, which contradicts the fact that $\overline{\Gamma}(R) = \Gamma(R)$. Now let $x \in Z(R) \backslash \mathrm{Nil}(R)$. Since $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2)$ it remains to show the other inclusion. Let $y \neq x$ be an element of $\mathrm{Ann}(x^2)$. Then x and y are adjacent in $\overline{\Gamma}(R)$, which equal to $\Gamma(R)$. Hence $xy = 0$, and therefore $y \in \mathrm{Ann}(x)$.

(2) $\Rightarrow$ (3). Let $y \in \sqrt{\mathrm{Ann}(x)}\backslash\mathrm{Nil}(R)$. Then there exists $n \in \mathbb{N}^*$ such that $y^n x = 0$. Then $x \in \mathrm{Ann}(y^n) = \mathrm{Ann}(y)$ by Lemma 2.2 and therefore $xy = 0$.

(3) $\Rightarrow$ (1). Let x and y be two adjacent vertices in $\overline{\Gamma}(R)$. Then there exist two positive integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Three cases occur:

Case 1: If $x, y \in \mathrm{Nil}(R)$ such that $n_x = n_y = 2$, then $n = m = 1$ (by $2(i)$). This means that x and y are adjacent vertices in $\Gamma(R)$.

Case 2: If $x \notin \mathrm{Nil}(R)$ and $y \in \mathrm{Nil}(R)$, then $m = 1$ (by $2(i)$). Hence, $x \in \sqrt{\mathrm{Ann}(y)}\backslash\mathrm{Nil}(R)$ and by hypothesis $xy = 0$. Thus, x and y are adjacent vertices in $\Gamma(R)$.

Case 3: If $x \notin \mathrm{Nil}(R)$ and $y \notin \mathrm{Nil}(R)$, then $x \in \sqrt{\mathrm{Ann}(y^m)}\backslash\mathrm{Nil}(R) \subset \mathrm{Ann}(y^m)$ and so $xy^m = 0$. Thus, $y \in \sqrt{\mathrm{Ann}(x)}\backslash\mathrm{Nil}(R) \subset \mathrm{Ann}(x)$, as desired. □

One can consider, for example, $\mathbb{Z}_{24}$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, and $\mathbb{Z}_2[X,Y]/(XY^2, X^3)$ to get an example of a ring R that contains a nilpotent element with index of at least three and such that there is an element $x \in Z(R)\backslash\mathrm{Nil}(R)$ with $\mathrm{Ann}(x^2) \neq \mathrm{Ann}(x)$.

To show that conditions (i) and (ii) of both (2) and (3) of Theorem 2.1 are independent, we give the following examples. First note that one can show easily that $2(ii)$ is equivalent to $3(ii)$.

Example 2.3 *1. To give an example of a ring R that contains a nilpotent element with index two and contains an element $x \in Z(R)\backslash\mathrm{Nil}(R)$ such that $\mathrm{Ann}(x^2) \neq \mathrm{Ann}(x)$, we can consider the following rings: $\mathbb{Z}_{12}$, $\mathbb{Z}_{18}$, $\mathbb{Z}_{36}$, $\mathbb{Z}_2[X,Y]/(XY^2, X^2)$, and $\mathbb{Z}_2 \times \mathbb{Z}_4$.*

2. The following rings can be used as an example of a ring R that contains a nilpotent element with index at least three and such that $\mathrm{Nil}(R) = Z(R)$: $\mathbb{Z}_{2^m}$ (with $m \geq 3$) and $\mathbb{Z}_2[X]/(X^3)$.

However, it does not seem easy to get an example of a ring R that contains a nilpotent element with index of at least three and such that $Z(R) \neq \mathrm{Nil}(R)$ with $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$ for every element $x \in Z(R)\backslash\mathrm{Nil}(R)$. Then, in order to construct such an example, one should establish at first some of its properties. For that, we set the following result.

Proposition 2.4 *Let R be a ring that satisfies the following properties: $\mathrm{Nil}(R) \neq \{0\}$, $\mathrm{Nil}(R) \subsetneq Z(R)$, and $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$ for every $x \in Z(R)\backslash\mathrm{Nil}(R)$.*

Then, for every $y \in \mathrm{Nil}(R)$ and every $x \in Z(R)\backslash\mathrm{Nil}(R)$, $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

Consequently, $\mathrm{Ann}(x)\mathrm{Nil}(R) = \{0\}$ for every $x \in Z(R)\backslash\mathrm{Nil}(R)$.

If furthermore there exists an element $t \in \mathrm{Nil}(R)$ such that $t^2 \neq 0$, then, for every $x \in Z(R)\backslash\mathrm{Nil}(R)$, $t \notin \mathrm{Ann}(x)$.

Consequently, for every $x \in Z(R)\backslash\mathrm{Nil}(R)$, $\mathrm{Ann}(x) \subset \mathrm{Ann}(t) \subset \mathrm{Nil}(R)$ such that $z^2 = 0$ for every $z \in \mathrm{Ann}(x)$.

Proof First, consider $y \in \mathrm{Nil}(R)$ and $x \in Z(R)\backslash\mathrm{Nil}(R)$ and suppose that $\mathrm{Ann}(x) \not\subseteq \mathrm{Ann}(y)$. Then there is an element $a \in R$ such that $ax = 0$ and $ay \neq 0$. Then $a(x+y) = ay \neq 0$. However, for $n \in \mathbb{N}$ with $y^n = 0$, we have $a(x+y)^n = 0$. This means that $\mathrm{Ann}(x+y) \neq \mathrm{Ann}(x+y)^n$, which is absurd (by hypothesis and Lemma 2.2).

This shows that $z^2 = 0$ for every nilpotent element $z \in \mathrm{Ann}(x)$. Thus, it remains to prove that $\mathrm{Ann}(t) \subset \mathrm{Nil}(R)$. If not, there is then an element $a \in \mathrm{Ann}(y)$ such that $a \notin \mathrm{Nil}(R)$. Thus, $\mathrm{Ann}(a) \subset \mathrm{Ann}(y)$, but $y \in \mathrm{Ann}(a)$ implies that $y^2 = 0$, which is absurd. □

Now we are in a position to give the desired example. For that we use a new ring construction recently introduced in [1].

Let R be a commutative ring with $1 \neq 0$ and let M_1 and M_2 be R-submodules of a commutative R-algebra L such that $(M_1)^2 := \{xy | x, y \in M_1\} \subset M_2$. Then we call 2-trivial extension of R by (M_1, M_2) the ring denoted by $R \ltimes_2 M_1 \ltimes M_2$ whose underlying group is $A \times M_1 \times M_2$ with multiplication given by

$$(a, m_1, m_2)(b, n_1, n_2) = (ab, an_1 + bm_1, an_2 + bm_2 + m_1 n_1).$$

Note that this construction is an extension of the well-known trivial extension of a ring by a module (see, for instance, [3]). In fact, $R \ltimes_2 0 \ltimes M_2$ can be seen as the trivial extension of a ring R by M_2.

Note that $\mathrm{Nil}(R \ltimes_2 M_1 \ltimes M_2) = \mathrm{Nil}(R) \ltimes_2 M_1 \ltimes M_2$ and $Z(R \ltimes_2 M_1 \ltimes M_2) = \{(r, m_1, m_2) \in R \ltimes M_1 \ltimes M_2 | r \in Z(R) \cup Z(M_1) \cup Z(M_2)\}$. Also note that if M_1 contains an element m such that $m^2 \neq 0$, then $(0, m, 0)$ is a nilpotent element of index 3.

Example 2.5 *Let $R = \mathbb{Z} \ltimes \mathbb{Z}_2$. Then $\mathbb{Z} \cong (\mathbb{Z} \ltimes \mathbb{Z}_2)/(\{0\} \ltimes \mathbb{Z}_2)$ is an R-module with the scalar multiplication defined as follows: $(a, \overline{n})x := ax$ for every $(a, n, x) \in \mathbb{Z}^3$.*

The ring $S = R \ltimes_2 \mathbb{Z} \ltimes \mathbb{Z}$ satisfies the following properties:

- *S contains a nilpotent element with index three.*
- $\mathrm{Nil}(R) \subsetneq Z(R)$.
- *For every $x \in Z(R) \backslash \mathrm{Nil}(R)$, $\mathrm{Ann}(x^2) = \mathrm{Ann}(x)$.*

Proof To get the result it suffices to show that, for every $x \in Z(S) \backslash \mathrm{Nil}(S)$,

$$\mathrm{Ann}(x) = \{((0, \overline{0}), 0, 0); (0, \overline{1}), 0, 0)\}.$$

This equality is a simple consequence of the fact that $\mathrm{Nil}(S) = \{((0, \overline{n}), s, t) | (n, s, t) \in \mathbb{Z}^3\}$ and $Z(S) \backslash \mathrm{Nil}(S) = \{((2k, \overline{n}), s, t) | k \in \mathbb{Z}^* \textit{ and } (n, s, t) \in \mathbb{Z}^3\}$. □

The following particular cases are simple consequences of Theorem 2.1.

Corollary 2.6 *Let R be a ring. If R contains a nilpotent element of index 3, then $\overline{\Gamma}(R) \neq \Gamma(R)$.*

Corollary 2.7 *Let R be a reduced ring. Then $\overline{\Gamma}(R) = \Gamma(R)$.*

Proof Assume that there is an element $x \in Z(R)^*$ such that $\mathrm{Ann}(x) \neq \mathrm{Ann}(x^2)$. Then there is $z \in Z(R)^*$ such that $zx^2 = 0$ and $zx \neq 0$, and hence $zx \in \mathrm{Nil}(R) \backslash \{0\}$, a contradiction since R is reduced. □

Now we show when $\overline{\Gamma}(R) = \Gamma(R)$ for the finite direct product of rings.

Proposition 2.8 *Let $(R_i)_{1 \leq i \leq n}$ be a finite family of rings with $n \in \mathbb{N}^* \backslash \{1\}$. Then $\overline{\Gamma}(\prod_{i=1}^{n} R_i) = \Gamma(\prod_{i=1}^{n} R_i)$ if and only if R_i is reduced for every $1 \leq i \leq n$.*

Proof It suffices to prove the case where $n = 2$.

($\Rightarrow$) Suppose that R_1 is not reduced. Then there is an element $x_1 \neq 0$ such that $x_1^2 = 0$. We have $(1,0)(x_1,1) = (x_1,0) \neq (0,0)$ but $(1,0)(x_1,1)^2 = (0,0)$. Then, by Theorem 2.1, $\overline{\Gamma}(R_1 \times R_2) \neq \Gamma(R_1 \times R_2)$, a contradiction.

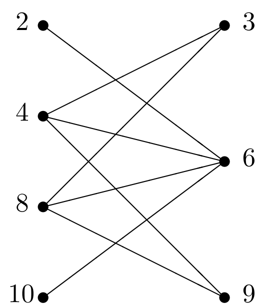
($\Leftarrow$) Use Corollary 2.7. □

As a simple consequence of Proposition 2.8, we determine when the graph $\overline{\Gamma}(\mathbb{Z}_n)$ coincides with $\Gamma(\mathbb{Z}_n)$.

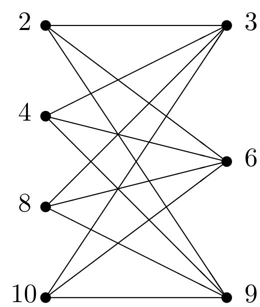
Corollary 2.9 *Let* $n = \prod_{i=1}^{k} P_i^{\alpha_i}$ *be the prime factorization of an integer* n *with* $k \in \mathbb{N}^*$. *Consider* $m := \operatorname{Sup}\{\alpha_i \mid 1 \leq i \leq k\}$. *Then* $\overline{\Gamma}(\mathbb{Z}_n) \neq \Gamma(\mathbb{Z}_n)$ *if and only if either* $m \geq 3$ *or* ($m = 2$ *and* $k \geq 2$).

Consequently, $\overline{\Gamma}(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$ *if and only if either* $n = p^2$ *for some prime* p *or* n *is square-free. In particular, if* $\mathbb{Z}_n$ *has nonzero nilpotent elements, then* $\overline{\Gamma}(\mathbb{Z}_n) = \Gamma(\mathbb{Z}_n)$ *if and only if* $\Gamma(\mathbb{Z}_n)$ *is complete.*

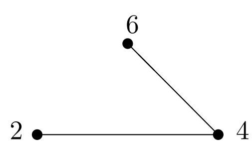
We end this section with the following simple examples.



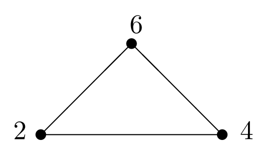
$\Gamma(\mathbb{Z}_{12})$



$\overline{\Gamma}(\mathbb{Z}_{12})$



$\Gamma(\mathbb{Z}_8)$



$\overline{\Gamma}(\mathbb{Z}_8)$

3. Diameter of extended graphs of rings

In this section, we study the diameter of extended graphs of rings.

Certainly, as an extension of the classical graph of [11, Theorem 2.3], $\overline{\Gamma}(R)$ has diameter of at most 3.

Theorem 3.1 Let R be a ring. Then $\overline{\Gamma}(R)$ is connected with $\operatorname{diam}(\overline{\Gamma}(R)) \leq 3$.

Now we determine some situations where $\operatorname{diam}(\overline{\Gamma}(R)) \leq 2$.

In the following result, as an analog of [11, Theorem 2.5], we characterize when $\overline{\Gamma}(R)$ has a vertex adjacent to every other vertex (i.e. when $\overline{\Gamma}(R)$ has a spanning tree that is a star graph).

Theorem 3.2 *Let* R *be a ring. Then there is a vertex* x *of* $\overline{\Gamma}(R)$ *that is adjacent to every other vertex if and only if either* $R \cong \mathbb{Z}_2 \times D$, *where* D *is an integral domain, or* $Z(R) = \sqrt{\operatorname{Ann}(x^{n_x - 1})}$.

Proof ($\Rightarrow$) Suppose that x is adjacent to every other vertex of $\overline{\Gamma}(R)$. If x is a nilpotent element, then for every nonzero zero divisor element $y \neq x$ there are two positive integers α, β such that $y^{\alpha}x^{\beta} = 0$ with $y^{\alpha} \neq 0$ and $x^{\beta} \neq 0$ (since x and y are adjacent in $\overline{\Gamma}(R)$); thus, $\beta < n_x$ and $y^{\alpha}x^{n_x-1} = 0$ and hence $y \in \sqrt{\mathrm{Ann}(x^{n_x-1})}$. Finally, since x is nilpotent, $x \in \sqrt{\mathrm{Ann}(x^{n_x-1})}$, and therefore $Z(R) = \sqrt{\mathrm{Ann}(x^{n_x-1})}$.

If $x \notin \mathrm{Nil}(R)^*$ then $x^2 = x$, if not there are two positive integers α and β such that $(x^2)^{\alpha}x^{\beta} = x^{2\alpha+\beta} = 0$, a contradiction since $x \notin \mathrm{Nil}(R)^*$, so $R = Rx \oplus R(1-x)$. Hence, we may assume that $R = R_1 \times R_2$ with $(1,0)$ adjacent to every other vertex. For any $1 \neq z \in R_1$, $(z,0)$ is a zero divisor, so there are $n, m \in \mathbb{N}^*$ such that $(z,0)^n(1,0)^m = (0,0)$ and $(z,0)^n \neq (0,0)$, a contradiction. Hence, $R_1 \cong \mathbb{Z}_2$. If R_2 is not an integral domain, then there is a nonzero $t \in Z(R_2)$. Then $(1,t)$ is a zero divisor of R, which is not adjacent to $(1,0)$, a contradiction. Thus, R_2 must be an integral domain.

($\Leftarrow$). If $R \cong \mathbb{Z}_2 \times D$ for D an integral domain, then $(\overline{1},0)$ is adjacent to every other vertex. If $Z(R) = \sqrt{\mathrm{Ann}(x^{n_x-1})}$ for some nonzero $x \in R$, then x is adjacent to every other vertex. □

We next determine when $\overline{\Gamma}(R)$ is a complete graph (i.e. where the diameter of $\overline{\Gamma}(R)$ is one). In [11, Theorem 2.8], it was proved that the graph $\Gamma(R)$ is complete if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for every x, $y \in Z(R)^*$ (i.e. $Z(R)^2 = 0$). For our case, we have the following result.

Theorem 3.3 *Let R be a ring. Then $\overline{\Gamma}(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R) = \mathrm{Nil}(R)$ and for every x, $y \in Z(R)^*$ $x^{n_x-1}y^{n_y-1} = 0$.*

Proof ($\Leftarrow$) By definition.

($\Rightarrow$) Suppose that $\overline{\Gamma}(R)$ is complete.

If $Z(R) = \mathrm{Nil}(R)$ then, by definition, $x^{n_x-1}x^{n_x-1} = 0$ for every element $x \in \mathrm{Nil}(R)$. Since $\overline{\Gamma}(R)$ is complete, for all distinct elements x, $y \in Z(R)^*$ there are two positive integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. Necessarily $n < n_x$ and $m < m_y$, and therefore $x^{n_x-1}y^{n_y-1} = 0$.

Now suppose that $Z(R) \neq \mathrm{Nil}(R)$. Since $\overline{\Gamma}(R)$ is complete and by Theorem 3.2, we have $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain. Hence, for distinct a, $b \in D\backslash\{0\}$, $(0,a)$ and $(0,b)$ are adjacent in $\overline{\Gamma}(R)$, and then there are two positive integers n and m such that $(0,a)^n(0,b)^m = (0,0)$, so $a = 0$ or $b = 0$ and thus necessarily $D \cong \mathbb{Z}_2$. □

To establish an analogy with the classical case we set $\overline{Z}(R) := \{x^{n_x-1} | x \in \mathrm{Nil}(R)^*\}$ $\overline{Z}(R)^2 := \{x^{n_x-1}y^{n_y-1} | x, y \in \mathrm{Nil}(R)^*\}$.

Corollary 3.4 *Let R be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. Then $\overline{\Gamma}(R)$ is complete if and only if $Z(R) = \mathrm{Nil}(R)$ and $\overline{Z}(R)^2 = \{0\}$.*

In [7, Theorem 2.2], the diameter of $\Gamma(R)$ was studied when $Z(R) = \mathrm{Nil}(R) \neq \{0\}$. For our case, we have the following result, which is slightly different from [7, Theorem 2.2].

Theorem 3.5 *Let R be a ring with $Z(R) = \mathrm{Nil}(R) \neq \{0\}$. Then $\mathrm{diam}(\overline{\Gamma}(R)) \leq 2$ and exactly one of the following three cases must occur.*

1. $|Z(R)^*| = 1$. *Then* R *is isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[X]/(X^2)$, *and* $\operatorname{diam}(\overline{\Gamma}(R)) = 0$.

2. $|Z(R)^*| \geq 2$ *and* $Z(R)^2 = \{0\}$. *Then* $\overline{\Gamma}(R)$ *is a complete graph, and* $\operatorname{diam}(\overline{\Gamma}(R)) = 1$.

3. $|Z(R)^*| \geq 2$ *and* $Z(R)^2 \neq \{0\}$. *If* $\overline{Z}(R)^2 = 0$ *then* $\overline{\Gamma}(R)$ *is a complete graph, and* $\operatorname{diam}(\overline{\Gamma}(R)) = 1$. *If not* $\operatorname{diam}(\overline{\Gamma}(R)) = 2$.

Proof 1. If $|Z(R)^*| = 1$ in this case $\overline{\Gamma}(R) = \Gamma(R)$, and then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [17, Proposition 2.2].

2. If $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$, then $xy = 0$ for all $x, y \in Z(R)$. Thus, $\overline{\Gamma}(R)$ is a complete graph and $\operatorname{diam}(\overline{\Gamma}(R)) = 1$.

3. By Corollary 3.4, $\overline{\Gamma}(R)$ is a complete graph; hence $\operatorname{diam}(\overline{\Gamma}(R)) = 1$. If not, there is x, $y \in Z(R)^*$ such that $x^{n_x-1}y^{n_y-1} \neq 0$ and hence $xy \notin \{0, x, y\}$, so $x - xy - y$ is a path between x and y of length 2. □

Now we study the diameter of the graph of finite direct products of rings.

Proposition 3.6 *Let* $R = \prod_{i=1}^{n} R_i$ *where* $(R_i)_{1 \leq i \leq n}$ *is a finite family of rings with* $n \in \mathbb{N}^* \backslash \{1\}$.

1. *If* $n = 2$, *we have the following assertions:*

   (a) $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\overline{\Gamma}(R)) = 1$ *if and only if* $R_1 \cong R_2 \cong \mathbb{Z}_2$.

   (b) *If* R_1 *and* R_2 *are integral domains with* $|R_1| \geq 3$ *or* $|R_2| \geq 3$, *then* $\Gamma(R) = \overline{\Gamma}(R)$ *and* $\operatorname{diam}(\Gamma(R)) = 2$. *In this case* $\Gamma(R)$ *is a complete bipartite graph.*

   (c) *If at least one of* R_1 *and* R_2 *contains a nonnilpotent zero divisor, then* $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\overline{\Gamma}(R)) = 3$.

   (d) *If at least one of* R_1 *and* R_2 *is not integral domains such that all zero divisors are nilpotent in each ring with nonzero zero divisors, then* $\operatorname{diam}(\Gamma(R)) = 3$ *and* $\operatorname{diam}(\overline{\Gamma}(R)) = 2$.

2. *If* $n \geq 3$, $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\overline{\Gamma}(R)) = 3$.

Proof **Case** $n = 2$**.** The proof of both (a) and (b) is trivial.

We prove assertion (c). Suppose that R_1 contains a nonnilpotent zero divisor z. Then there is an element $z' \in R_1$ such that $zz' = 0$. Then using the following path, in both $\Gamma(R)$ and $\overline{\Gamma}(R)$, $(1,0)-(0,1)-(z',0)-(z,1)$, and the fact that there is no vertex adjacent to both $(1,0)$ and $(z,1)$, we conclude that $\operatorname{diam}(\Gamma(R)) = \operatorname{diam}(\overline{\Gamma}(R)) = 3$.

(d). We prove only the case where, for instance, R_1 is not integral domains such that all zero divisors are nilpotent and R_2 is integral domains. First, using the same path as above, we have $\operatorname{diam}(\Gamma(R)) = 3$. However, in $\overline{\Gamma}(R)$, $d((1,0),(z,1)) = 1$ for every $z \in Z(R_1)^* = \operatorname{Nil}(R_1)$. Now we have: $Z(R)^* = T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4$ where $T_1 = \{(a,0) | a \textit{ is regular}\}$, $T_2 = \{(b,0) | b \in Z(R_1)\}$, $T_3 = \{(0,x) | x \in R_2\}$, and $T_4 = \{(a,x) | a \in Z(R_1), x \in R_2\}$. A simple study of the distance between any two elements shows that $\operatorname{diam}(\overline{\Gamma}(R)) = 2$.

Case $n \geq 3$**.** Note that $(0,1,1,...,1) - (1,0,0,...,0) - (0,0,...,0,1) - (1,1,...,1,0)$ is a shortest path between $(0,1,1,...,1)$ and $(1,1,1,...,0)$. □

Proposition 3.6 helps to determine the diameter of $\mathbb{Z}_n$ in some cases.

Proposition 3.7 *For a positive integer $n \in \mathbb{N}^*$, the following assertions hold true:*

1. *If $n = 2^2$, then* $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 0$.

2. *If either $n = 2^m$ with $m > 2$ or $n = p^m$ with p is an odd prime and $m \geq 2$, then* $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 1$. *In this case $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete graph.*

3. *If $n = p^\alpha q^\beta$ with p and q are distinct primes, then* $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 2$. *In this case,*

   - *If $p = 2$ and $\alpha = \beta = 1$, then $\overline{\Gamma}(\mathbb{Z}_n)$ is a star graph.*
   - *If either ($p = 2$, $\alpha = 2$ and $\beta = 1$) or $n = pq$, then $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.*

4. *If $n = \prod_{i=1}^{k} P_i^{\alpha_i}$ is the prime factorization of n with $p_i \neq p_j$, for $i \neq j$, and $k \geq 3$, then* $\mathrm{diam}(\overline{\Gamma}(R)) = 3$.

Proof 1. Let $n = 2^2$; then $Z(\mathbb{Z}_4)^* = \{\overline{2}\}$ and hence $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_4)) = 0$.

2. Let $n = 2^m$ with $m > 2$ or $n = p^m$ where p is an odd prime and $m \geq 2$. Then all zero divisors of $\mathbb{Z}_n$ are multiples of $\overline{2}$ for the first and $\overline{p}$ for the second. It is clear that all zero divisors are adjacent to each other, so $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 1$ and $\overline{\Gamma}(R)$ is a complete graph.

3. Let $n = p^\alpha q^\beta$ with p and q are distinct primes. Then, by Proposition 3.6, $\mathrm{diam}(\overline{\Gamma}(\mathbb{Z}_n)) = 2$.

If $n = 2q$, then $Z(\mathbb{Z}_n)^* = \{\overline{2h}/0 < h < q\} \cup \{\overline{q}\}$. Hence, $\overline{q}$ is adjacent to every other vertex and $d(\overline{2h}, \overline{2h'}) = 2$ with $0 < h < q$ and $0 < h' < q$. Therefore, $\overline{\Gamma}(\mathbb{Z}_{2q})$ is a star graph.

If $n = 4q$, then $Z(\mathbb{Z}_n)^* = \{\overline{2h}/0 < h < 2q\} \cup \{\overline{q}\}$. The two following sets $A := \{\overline{2h}/0 < h < 2q$ and $h \neq q\}$ and $B := \{\overline{kq}/\ 0 < k < 4\}$ form a partition of $Z(\mathbb{Z}_n)^*$ and show that $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.

If $n = pq$, then $Z(\mathbb{Z}_n)^* = \{\overline{hp}/0 < h < q\} \cup \{\overline{kq}/0 < k < p\}$. The two following sets $A =: \{\overline{hp}/0 < h < q\}$ and $B := \{\overline{kq}/k < p\}$ form a partition of $Z(\mathbb{Z}_n)^*$ and shows that $\overline{\Gamma}(\mathbb{Z}_n)$ is a complete bipartite graph.

4. Follows from Proposition 3.6. □

4. Cycles in extended graphs of rings

In this section, we study the girth of $\overline{\Gamma}(R)$.

Since $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$ and by [10, Theorem 2.4], we have $\mathrm{gr}(\overline{\Gamma}(R)) \in \{3, 4, \infty\}$.

In the classical case there are some examples of rings R such that $\mathrm{gr}(\Gamma(R)) = \infty$. The following result shows that when $\overline{\Gamma}(R) \neq \Gamma(R)$, we have $\mathrm{gr}(\overline{\Gamma}(R)) \in \{3, 4\}$.

Theorem 4.1 *Let R be a ring. If $\overline{\Gamma}(R) \neq \Gamma(R)$, then $\overline{\Gamma}(R)$ contains a cycle.*

Proof Since $\overline{\Gamma}(R) \neq \Gamma(R)$ and by Theorem 2.1, there is either a nilpotent element x with $n_x \geq 3$ or an element $x \in Z(R)\backslash \mathrm{Nil}(R)$ such that $\mathrm{Ann}(x) \neq \mathrm{Ann}(x^2)$. For the first case, we have $x - (x + x^{n_x - 1}) - x^{n_x - 1} - x$ is a cycle of length 3. For the second case, there exists $y \in Z(R)^*$ such that $yx^2 = 0$ and $yx \neq 0$. If $y^2 = 0$

then $y-(x+y)-xy-y$ is a cycle of length 3. If not $x-yx-x^2-y-x$ is a cycle of length 4. □

Corollary 4.2 *If R contains a nilpotent element of index greater than or equal to three, then* $\mathrm{gr}(\overline{\Gamma}(R))=3$.

Corollary 4.3 *If there are elements x and z of $Z(R)^*$ such that $x\notin \mathrm{Nil}(R)$, $z^2=0$, $zx\neq 0$, and $zx^2=0$ then* $\mathrm{gr}(\overline{\Gamma}(R))=3$.

In [7, Theorem 2.11], the girth of $\Gamma(R)$ is studied when $Z(R)=\mathrm{Nil}(R)\neq\{0\}$. For our case, we have the following slightly different result.

Theorem 4.4 *Let R be a ring with $Z(R)=\mathrm{Nil}(R)\neq\{0\}$. Then exactly one of the following three cases must occur.*

1. *If $|Z(R)^*|=1$, then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and $\mathrm{gr}(\overline{\Gamma}(R))=\infty$.*
2. *If $|Z(R)^*|=2$, then R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and $\mathrm{gr}(\overline{\Gamma}(R))=\infty$.*
3. *If $|Z(R)^*|=3$, then R is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, $\mathbb{Z}_4[X]/(2X,X^2-2)$, $\mathbb{Z}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2,X)^2$, $\mathbb{Z}_4[X]/(X^2+X+1)$, or $\mathbb{F}_4[X]/(X^2)$, and $\mathrm{gr}(\overline{\Gamma}(R))=3$.*
4. *If $|Z(R)^*|\geq 4$, then $\mathrm{gr}(\overline{\Gamma}(R))=3$.*

Proof All assertions follow from [7, Theorem 2.11] except the following cases: for $R\cong\mathbb{Z}_8$, $2-4-6-2$ is a cycle of length 3. For $R\cong\mathbb{Z}_2[X]/(X^3)$, $X-X^2-(X^2+X)-X$ is a cycle of length 3. Finally, for $R\cong\mathbb{Z}_4[X]/(2X,X^2-2)$, $2-X-(X-2)-2$ is a cycle of length 3. □

[4, Theorems 2.3 and 2.5] allow us to establish situations where $\mathrm{gr}(\overline{\Gamma}(R))=4$. Namely, we have the two following results.

Theorem 4.5 *Let R be a commutative ring with $\mathrm{Nil}(R)\neq 0$ and $\mathrm{gr}(\Gamma(R))=4$. Then $\overline{\Gamma}(R)\neq\Gamma(R)$ implies that $\mathrm{gr}(\overline{\Gamma}(R))=4$ and $\overline{\Gamma}(R)$ is a complete bipartite graph.*

Proof From [4, Theorem 2.3], $R\cong D\times B$ where D is an integral domain with $|D|\geq 3$ and $B=\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then $Z(R)=A\cup B$ where $A=\{(a,x)|a\in D \text{ and } x\in\{0;2\}\}$ and $B=\{(0,n)|n\in\mathbb{Z}_4\}$. One can show that all elements of A are connected to all elements of B in $\overline{\Gamma}(R)$ such that $\overline{\Gamma}(R)$ is a complete bipartite graph, and therefore $\mathrm{gr}(\overline{\Gamma}(R))=4$. □

Theorem 4.6 *Let R be a commutative ring with $\mathrm{Nil}(R)\neq 0$ and $\mathrm{gr}(\Gamma(R))=\infty$. Then exactly one of the following holds:*

1. *$\overline{\Gamma}(R)=\Gamma(R)$ is a singleton or a star graph. In this case $\mathrm{gr}(\overline{\Gamma}(R))=\infty$.*
2. *$\Gamma(R)=\overline{K}^{1,3}$ (i.e. $R\cong\mathbb{Z}_2\times\mathbb{Z}_4$ or $R\cong\mathbb{Z}_2\times\mathbb{Z}_2[X]/(X^2)$). In this case $\mathrm{gr}(\overline{\Gamma}(R))=4$ and $\overline{\Gamma}(R)\neq\Gamma(R)$.*

Proof Flows from [4, Theorem 2.5] and, when $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, we have the following cycle $(1,0) - (0,2) - (1,2) - (0,1) - (1,0)$ in $\overline{\Gamma}(R)$ of length 4 . □

In [10, Theorem 3.5], the notion of complemented graph is used to characterize when the total quotient ring of a reduced ring R is von Neumann regular. In Proposition 4.8 below we attempt to give a similar result. In fact, when $\overline{\Gamma}(R) \neq \Gamma(R)$, we only show that $\overline{\Gamma}(R)$ is complemented is a sufficient condition so that $T(R)$ be zero-dimensional. The converse remains an open problem. For that, we need the following definition and lemma.

Let us say, as in [20], that distinct vertices x and y of $\overline{\Gamma}(R)$ are orthogonal, written $x \perp_{\overline{\Gamma}(R)} y$, if x and y are adjacent and there is no vertex z of $\overline{\Gamma}(R)$ that is adjacent to both x and y, i.e. the edge $x - y$ is not a part of any triangle of $\overline{\Gamma}(R)$. We say that $\overline{\Gamma}(R)$ is complemented if for each vertex x of $\overline{\Gamma}(R)$, there is a vertex y of $\overline{\Gamma}(R)$ (called a complement of x) such that $x \perp_{\overline{\Gamma}(R)} y$.

Lemma 4.7 *Let R be a ring. If there are orthogonal elements $x, y \in Z(R)^*$ and there are $n, m \in \mathbb{N}^* \backslash \{1\}$ such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$, then $x^n + y^m$ is a regular element of R.*

Proof Suppose that $z(x^n + y^m) = 0$ for some $z \in R\backslash\{0\}$. Let $t = zx^n = -zy^m$, and then $tx^n = ty^m = 0$. If $t = x$ then $x^{n+1} = 0$. With $t = zx^n$, we get $x^n = zx^{2n-1} = 0$, a contradiction since $x^n \neq 0$. Similarly, we prove that $t \neq y$. Also, if $t \neq 0$, then t is adjacent to both x and y, a contradiction since $x \perp_{\overline{\Gamma}(R)} y$. Then $t = 0$ and so $zx^n = -zy^m = 0$. Then it remains to prove that $z \neq x$ and $z \neq y$. Indeed, if $z = x$ then $x^{n+1} = 0$. Then x is adjacent to x^2 (since $x^{n+1} = x^2 x^{n-1}$ and $x^2 \neq x$). Now with $xy^m = 0$ we have $x^2 y^m = 0$. Then x^2 is adjacent to both x and y, a contradiction. Similarly, we prove that $z \neq y$. □

Proposition 4.8 *Let R be a ring with $\overline{\Gamma}(R) \neq \Gamma(R)$. If $\overline{\Gamma}(R)$ is complemented, then $T(R)$ is zero-dimensional.*

Proof First, note that all nilpotent elements have index 2 (since $\overline{\Gamma}(R)$ is complemented). Then by Theorem 2.1, there is an element $x_0 \in Z(R)\backslash \mathrm{Nil}(R)$ such that $\mathrm{Ann}(x) \neq \mathrm{Ann}(x^2)$. This implies that there is $z_0 \in Z(R)$ such that $z_0 x_0 \neq 0$ and $z_0 x_0^2 = 0$. Also, note that from Corollary 4.3 $z_0 \notin \mathrm{Nil}(R)$.

Now, to show that $T(R)$ is zero-dimensional, it is sufficient to show that each nonminimal prime ideal Q of R contains a regular element of R. Let $P \subset Q$ be distinct prime ideals of R. Then there is $x \in Q\backslash P$. Note that $x \notin \mathrm{Nil}(R)$. We have the following possible situations:

<u>Case 1 (x is adjacent to x_0):</u> Then $x_0 \in Q$ and, by Lemma 4.7, there exist α, $\beta \in \mathbb{N}^*$ such that $x_0^\alpha + x^\beta$ is a regular element of R that belongs to Q.

<u>Case 2 (x is adjacent to z_0):</u> The proof is the same as above.

<u>Case 3 ($x_0 \notin P$):</u> Then $xx_0 \in Q\backslash P$. With xx_0 adjacent to z_0 and by Lemma 4.7, $x^2 x_0^2 + z_0$ is regular and belongs to Q.

<u>Case 4 ($x_0 \in P$):</u> If $z_0 \notin P$, then $xz_0 \in Q\backslash P$ and so xz_0 is adjacent to x_0. By Lemma 4.7, $xz_0 + x_0^2$ is regular and belongs to Q. If $z_0 \in P$, then $x_0^2 + z_0$ is regular and belongs to Q since $x_0^2 z_0 = 0$. □

Now we study the girth of the graph of finite direct products of rings.

Proposition 4.9 *Let $R = \prod_{i=1}^{n} R_i$ where $(R_i)_{1 \leq i \leq n}$ is a finite family of rings with $n \in \mathbb{N}^* \backslash \{1\}$.*

1. *If $n = 2$, we have the following assertions:*

   (a) $\operatorname{gr}(\Gamma(R)) = \operatorname{gr}(\overline{\Gamma}(R)) = \infty$ *if and only if* R_1 *and* R_2 *are integral domains and at least one is isomorphic to* $\mathbb{Z}_2$.

   (b) *If* R_1 *and* R_2 *are integral domains with* $|R_1| \geq 3$ *and* $|R_2| \geq 3$, *then* $\Gamma(R) = \overline{\Gamma}(R)$ *and* $\operatorname{gr}(\Gamma(R)) = 4$.

   (c) *If at least one of* R_1 *and* R_2 *is not integral domains, then* $\operatorname{gr}(\Gamma(R)) = \operatorname{gr}(\overline{\Gamma}(R)) = 3$.

2. *If $n \geq 3$, then* $\operatorname{gr}(\Gamma(R)) = \operatorname{gr}(\overline{\Gamma}(R)) = 3$.

Proof **Case** $n = 2$**.** The proof of both (a) and (b) is trivial.

We prove assertion (c). Suppose that R_1 contains a zero divisor z. Then there is an element $z' \in R_1$ such that $zz' = 0$. Then $(z, 0) - (z', 1) - (0, 1) - (z, 0)$ is a cycle of length 3. Thus, $\operatorname{gr}(\Gamma(R)) = \operatorname{gr}(\overline{\Gamma}(R)) = 3$.
Case $n = 3$**.** $(1, 0, 0, ..., 0) - (0, 1, 0, ..., 0) - (0, 0, 1, ..., 0) - (1, 0, ..., 0, 0)$ is a cycle of length 3. Thus, $\operatorname{gr}(\Gamma(R)) = \operatorname{gr}(\overline{\Gamma}(R)) = 3$. □

Proposition 4.9 can be used to determine the girth of $\mathbb{Z}_n$ in some cases.

Proposition 4.10 *For a positive integer $n \in \mathbb{N}^*$, the following assertions hold true:*

1. *If $n = 2^2$ or $n = 3^2$ or $n = 2p$ with p is an odd prime, then* $\operatorname{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = \infty$.

2. *If $n = pq$ or $n = 4p$ with p and q are odd primes, then* $\operatorname{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = 4$.

3. *We have* $\operatorname{gr}(\overline{\Gamma}(\mathbb{Z}_n)) = 3$ *if one of the three following assertions holds true:*

   (a) $n = p^m$ *with* $m > 2$ *and* p *is prime,*

   (b) $n = p^2$ *with* $p > 3$ *is prime,*

   (c) $n = \prod_{i=1}^{k} P_i^{\alpha_i}$ *is the prime factorization of* n *with* $p_i \neq p_j$, *for* $i \neq j$, *and* $k \geq 3$.

Proof 1. If $n = 2^2$ or $n = 3^2$, then $\mid Z(R)^* \mid= 1$ or $\mid Z(R)^* \mid= 2$ respectively, and hence $\operatorname{gr}(\overline{\Gamma}(R)) = \infty$. If $n = 2p$, then $\overline{\Gamma}(R)$ is a star graph (see Proposition 3.7), and hence $\operatorname{gr}(\overline{\Gamma}(R)) = \infty$.

2. If $n = pq$ or $n = 4p$ with p and q are odd primes, then $\overline{\Gamma}(R)$ is a complete bipartite graph (see Proposition 3.7). Thus, $\operatorname{gr}(\overline{\Gamma}(R)) = 4$.

3. For both assertions (a) and (b), we have that $\overline{\Gamma}(R)$ is complete (see Proposition 3.7), and hence $\operatorname{gr}(\overline{\Gamma}(R)) = 3$. For the third assertion, we have the following cycle of length 3: $\overline{p_1^{\alpha_1} p_3^{\alpha_3}} - \overline{p_2^{\alpha_2} p_3^{\alpha_3}} - \overline{p_1^{\alpha_1} p_2^{\alpha_2}} - \overline{p_1^{\alpha_1} p_3^{\alpha_3}}$. □

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