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libo Zhao

yangming li

LU GONG

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On the finite p-groups with unique cyclic subgroup of given order

Libo ZHAO[1], **Yangming LI**[1], **Lü GONG**[2,*]
[1]Department of Mathematics, Guangdong University of Education, Guangzhou, P.R. China
[2]School of Sciences, Nantong University, Jiangsu, P.R. China

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Abstract: In this paper, we prove that if G is nonabelian and $|G| > p^4$, then G has a unique cyclic subgroup of order p^m with $m \geq 3$ if and only if G has a unique abelian subgroup of order p^3 if and only if G is a 2-group of maximal class.

Key words: p-group of maximal class, extra-special p-group

1. Introduction

All groups considered in this paper are finite p-groups. The terminology and the notation in this paper are standard. The Frattini subgroup, the commutator subgroup, and the center of a group G will be denoted by $\Phi(G)$, G', and $Z(G)$ respectively. We use $c(G)$ and G_i to denote the nilpotent class and the ith term of the lower central series of G, respectively. The number of subgroups of order p^m, abelian subgroups of order p^m, and cyclic subgroups of order p^m are denoted by $s_m(G)$, $a_m(G)$, and $c_m(G)$, respectively. For a subgroup H in G, the centralizer of H in G is denoted by $C_G(H)$.

There is much interest in investigating the structure of a group whenever the number of some kind of subgroups is given. For example, finite p-groups with exactly one minimal nonabelian subgroup of given structure of order p^3 are classified by [5]. In [3], finite p-groups with exactly one minimal nonabelian subgroup of index p are investigated. In this paper, we are interested in the finite p-groups with unique cyclic subgroup of given order.

In [1] and [2], the authors proved that

Theorem 1 ([1]) *Suppose that a 2-group G is neither cyclic nor of maximal class. If $n > 1$, then $c_n(G)$ is even.*

Theorem 2 ([2]) *Let G be a noncyclic p-group, $p > 2$, and $n > 0$. If $n > 1$, then p divides $c_n(G)$.*

By the above two theorems, we see that finite p-groups with unique cyclic subgroup of order p^m are cyclic groups or 2-groups of maximal class. In this paper, we give a direct elementary proof. Moreover, we proved the following theorem:

*Correspondence: lieningzai1917@126.com
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Theorem 3 *Let G be a nonabelian p-group and $|G| > p^4$. Then $c_m(G) = 1$ if and only if $a_3(G) = 1$ if and only if G is 2-group of maximal class, where $m \geq 3$.*

2. Preliminaries

We gathered all the results used in what follows.

Lemma 4 ([4] Theorem 2.2.13) *Let G be a finite p-group with unique subgroup of order p. Then*

(i) *G is cyclic if $p > 2$;*

(ii) *G is a cyclic group or generalized quaternion group when $p = 2$.*

Lemma 5 ([4] Exercise 2.2.3) *Let G be a nonabelian p-group. Then the number of abelian maximal subgroups of G is* 0, 1*, or* $1 + p$*.*

Lemma 6 ([4] Theorem 2.7.1) *Let G be a p-group with $|G| = p^n$ and $1 < m < n$. If $s_m(G) = 1$, then G is cyclic.*

Lemma 7 ([4] Theorem 2.7.2) *Let G be a p-group with $|G| = p^n$ and $1 < m \leq n$. If $s_m(G) = c_m(G)$, then*

(1) G is cyclic if $p^m \neq 4$;

(2) G is a cyclic group or generalized quaternion group if $p^m = 4$.

Lemma 8 ([4] Theorem 2.5.2) *Let G be a p-group of maximal class with $|G| = p^n$. Then*

(1) G_i is the unique normal subgroup of order p^{n-i};

(2) If $p > 2$ and $n > 3$, then G does not have a cyclic normal subgroup of order p^2.

Lemma 9 ([4] Theorem 2.5.5) *G is a 2-group of maximal class if and only if $|G : G'| = 4$.*

Lemma 10 ([4] Theorem 2.5.6) *Let G be a nonabelian p-group with $p > 2$. If G has an abelian maximal subgroup, then G is of maximal class if and only if $|G : G'| = p^2$.*

Lemma 11 ([4] Theorem 2.5.7) *Let G be a nonabelian p-group. If G has a subgroup A of order p^2 such that $C_G(A) = A$, then G is of maximal class.*

Lemma 12 ([4] Theorem 7.1.6) *Let G be an extra-special p-group. Then $|G| = p^{2m+1}$ for some integer m.*

3. Finite p-groups with $c_m(G) = 1$

Firstly, we have the following Lemmas.

Lemma 13 *Let G be a metacyclic p-group with $p > 2$ and $H \leq G$. Then H is abelian if $|H| \leq |G/G'|$ and H is nonabelian if $|H| > |G/G'|$.*

Proof Suppose $G = \langle a\rangle\langle b\rangle$ and $G/\langle a\rangle \cong \langle \bar{b}\rangle$. For any subgroup H of G, assume that $H/H \cap \langle a\rangle \cong H\langle a\rangle/\langle a\rangle = \langle \bar{b}^{p^j}\rangle$ and $H \cap \langle a\rangle = \langle a^{p^i}\rangle$. Thus, $|H| = |G|/p^{i+j}$ and $H = \langle a^{p^i}, b^{p^j}\rangle$. Since $p > 2$, we see that $[a^{p^i}, b^{p^j}] = 1$ if and only if $[a, b]^{p^{i+j}} = 1$. Hence, $H' = 1$ if and only if $p^{i+j} \geq |G'|$. By $|H| = |G|/p^{i+j}$, we see $p^{i+j} = |G|/|H|$. Then H is abelian if and only if $|H| \leq |G/G'|$. □

Lemma 14 *Let G be a p-group with $|G| \geq p^4$. Then G has an abelian normal subgroup of order p^3.*

Proof Take a subgroup N of order p^2 that is normal in G. Since "N/C", $G/C_G(N) \lesssim \mathrm{Aut}(N)$. Thus, $|G/C_G(N)||p$, $|C_G(N)| \geq p^3$. By $C_G(N) \trianglelefteq G$, we see that there exists a normal subgroup M such that $N \leq M \leq C_G(N)$ and $|M| = p^3$. Then $M = \langle N, c\rangle$, where $c \in C_G(N)$. Thus, M is abelian. Therefore, M is desired. □

Proposition 15 *Let G be a finite p-group and $n > 3$. If $c_n(G) = 1$, then $c_3(G) = 1$.*

Proof Suppose, by way of contradiction, that $C_{p^n} \cong \langle a\rangle \leq G$, $C_{p^3} \cong \langle x\rangle \leq G$, and $x \notin \langle a\rangle$. Since $\langle a^p\rangle char \langle a\rangle \trianglelefteq G$, $\langle a^p\rangle \trianglelefteq \langle a\rangle\langle x\rangle$. By $\langle a\rangle/\langle a^p\rangle \trianglelefteq \langle a,x\rangle/\langle a^p\rangle$ and $o(\langle a\rangle/\langle a^p\rangle) = p$, we see $[a,x] \in \langle a^p\rangle$. Thus, we may assume $[a,x] = a^{ip}$. If $p > 2$, then

$$(ax^{-1})^{p^3} = a^{p^3}[a,x]^{\binom{p^3}{2}}[a,x,x]^{\binom{p^3}{3}}...x^{-p^3} = a^{vp^3}$$

with $(v,p) = 1$. Hence, $C_{p^n} \cong \langle ax^{-1}\rangle \neq \langle a\rangle$, a contradiction. Now, assume $p = 2$. If $x^2 \notin \langle a\rangle$, then it is easy to see $C_{2^n} \cong \langle ax^{-2}\rangle \neq \langle a\rangle$. When $2|i$, there exists $\langle ax^{-1}\rangle$ such that $C_{2^n} \cong \langle ax^{-1}\rangle \neq \langle a\rangle$. For $x^2 \in \langle a\rangle$ and $(2,i) = 1$, setting $x^2 = a^{j2^{n-2}}$ with $(j,2) = 1$, $1 = [a^{j2^{n-2}}, x] = [a,x]^{j2^{n-2}} = a^{ij2^{n-1}}$. Thus, $a^{2^{n-1}} = 1$, a contradiction. The proof is complete. □

Proposition 16 *Let G be a nonabelian p-group with $|G| \geq p^4$. If $c_3(G) = 1$, then $a_3(G) = 1$.*

Proof Assume the contrary; there exists a subgroup N such that $N \cong C_{p^2} \times C_p$ or $C_p \times C_p \times C_p$. Suppose the unique cyclic subgroup of order p^3 is $M = \langle a\rangle$. Now we divide our analysis into two cases: (1) $N \cong C_{p^2} \times C_p$ and (2) $N \cong C_p \times C_p \times C_p$.

Case 1: $C_{p^2} \times C_p \cong N = \langle x\rangle \times \langle y\rangle$.

Since $\langle a^p\rangle$ *char* $\langle a\rangle \trianglelefteq G$, we may assume that $[a,x] = a^{ip}$ and $[a,y] = a^{jp}$ for some integers i and j.

(1.1) $x \in \langle a\rangle$. We see $y \notin \langle a\rangle$ from $y \notin \langle x\rangle$. If $p|j$, then $(ay^{-1})^p = a^p[a,y]^{\binom{p}{2}}y^{-p} = a^{pv}$ with $(v,p) = 1$. Thus, $C_{p^3} \cong \langle ay^{-1}\rangle \neq \langle a\rangle$. Therefore, $(p,j) = 1$. By $x \in \langle a\rangle$, we may assume $x = a^{rp}$ with $(r,p) = 1$. Then

$$1 = [x,y] = [a^{pr}, y] = [a,y]^{pr} = a^{p^2jr} \neq 1,$$

a contradiction.

(1.2) Let $x \notin \langle a\rangle$. Since $N = \langle x\rangle \times \langle y\rangle = \langle x^{-1}y\rangle \times \langle y\rangle$, we may assume that $x^{-1}y \notin \langle a\rangle$ by (1.1). If $x^p \notin \langle a\rangle$, then $C_{p^3} \cong \langle ay^{-p}\rangle \neq \langle a\rangle$. Therefore, $x^p = a^{kp^2}$ with $(k,p) = 1$. It is easy to see $(ij,p) = 1$ from $c_3(G) = 1$. If $p > 2$, then

$$1 = [a,x^p] = [a,x]^p[a,x,x]^{\binom{p}{2}} = a^{ip^2},$$

which contradicts $o(a) = p^3$. When $p = 2$,

$$(ax^{-1}y)^2 = a^2[a,xy]x^2 = a^2[a,y][a,x][a,x,y]x^2 = a^2a^{2(i+j)}a^{4ij}a^{4k} = a^{2v}$$

with $(v,2) = 1$. Thus, $C_{p^3} \cong \langle ax^{-1}y\rangle \neq \langle a\rangle$, a contradiction.

Case 2: $C_p \times C_p \times C_p \cong N = \langle x\rangle \times \langle y\rangle \times \langle z\rangle$.

Since $|M \cap N| \leq p$, we may assume that $y, z \notin \langle a\rangle$ and $yz \notin \langle a\rangle$. By $\langle a^p\rangle \trianglelefteq MN$, $[a, y] = a^{jp}$ and $[a, z] = a^{kp}$. It is easy to see that $(jk, p) = 1$. If $p > 2$, then $C_{p^3} \cong \langle ay^{-1}\rangle \neq \langle a\rangle$. Therefore, $p = 2$. However, there exists ayz such that $(ayz)^2 = a^2[a, yz] = a^2a^{2(k+j)}a^{4kj}$. Hence, $\langle ayz\rangle \cong \langle a\rangle$, a contradiction. □

Proposition 17 *Let G be a p-group with $|G| \geq p^4$. Then $a_3(G) = 1$ and $c_3(G) = 1$ if and only if G is a cyclic group or 2-group of maximal class.*

Proof If G is abelian, then G is cyclic from Lemma 6. Now assume G is nonabelian. If there exists a subgroup $A \cong C_p \times C_p$, then $C_G(A) = A$ from $a_3(G) = 1$ and $c_3(G) = 1$. Thus, we see G is a p-group of maximal class by Lemma 11. If all the subgroups of order p^2 are cyclic, then it follows from Lemma 7 that G is a generalized quaternion group, which is also of maximal class. Hence, G is a 2-group of maximal class from Lemma 8.

Since G is a 2-group of maximal class, it is easy to check that $c_3(G) = 1$ and $a_3(G) = 1$. □

Proposition 18 *Let G be a p-group with $|G| \geq p^4$. Then $a_3(G) = 1$ and $c_3(G) = 0$ if and only if G is of maximal class of order p^4 with $p > 2$.*

Proof If G is of maximal class of order p^4, then $d(G) = 2$. Therefore, the number of maximal subgroups is $1 + p$. By Lemma 14, we see that $a_3(G) \geq 1$. It follows that $a_3(G) = 1$ or $1 + p$ from Lemma 5. If $a_3(G) = 1 + p$, then G is minimal nonabelian and $c(G) = 2$, a contradiction. Therefore, $a_3(G) = 1$ and then $c_3(G) = 0$ or 1. If $c_3(G) = 1$, then there exists a cyclic normal subgroup of order p^2, which contradicts Lemma 8. Thus, $c_3(G) = 0$.

Conversely, we see that G is nonabelian by Lemma 6. First we prove that the groups of order p^4 satisfying $a_3(G) = 1$ and $c_3(G) = 0$ are p-groups of maximal class with $p > 2$. In this case $|Z(G)| = p$. If not, $|Z(G)| = p^2$. We see $G/Z(G) \cong C_p \times C_p$ from G is nonabelian. Then the number of abelian subgroups of order p^3 containing $Z(G)$ is $1+p$, a contradiction. By Lemmas 9 and 10, we need to prove $|G'| = p^2$. Assume that $|G'| = p$. If $d(G) = 2$, then G is a minimal nonabelian p-group. Hence, $|Z(G)| = p^2$, which is impossible. Therefore, $d(G) = 3$ and $G' = \Phi(G)$; therefore, G is an extra-special p-group. Again, we have a contradiction, Lemma 12, because $|G| = p^4$. Thus, $|G'| = p^2$ and G is of maximal class. Now, if G is a 2-group of maximal class, then the abelian subgroup of order p^3 is cyclic. Therefore, $p > 2$.

Next, noting that the property is inherited by subgroups, we only need to prove that any group of order p^5 $(p > 2)$ does not satisfy $a_3(G) = 1$ and $c_3(G) = 0$. If there exists a group G of order p^5 that satisfies the property, then for each maximal subgroup M of G, M has an abelian subgroup of order p^3 by Lemma 14. Thus, M satisfies $a_3(G) = 1$ and $c_3(G) = 0$. Thus, M is of class 3 by the above paragraph. Therefore, $c(G) = 3$ or 4.

Case (i) $c(G) = 3$. If $Z(G) \geq p^2$, then there exists A such that $|A| = p^2$ and $A \leq Z(G)$. By the hypothesis, G/A has the unique subgroup of order p, and $p > 2$. Thus, G/A is cyclic since Lemma 4. Therefore, G is abelian, a contradiction, and so $|Z(G)| = p$, $|G_3| = p$. By Lemma 14, we see $d(G) = 2$. Write $\bar{G} = G/G_3$. Then $|(\bar{G})'| = p$ or p^2. If $|(\bar{G})'| = p^2$, then $\bar{G}$ is of maximal class by Lemma 10, which contradicts

$c(G) = 3$. Thus, $|(\bar{G})'| = p$. If $\bar{G}$ is metacyclic, then G is metacyclic. We may get a contradiction from Lemma 13 and $a_3(G) = 1$. Therefore,

$$\bar{G} \cong M_p(2,1,1) = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^{p^2} = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle.$$

Assume $G_3 = \langle x \rangle \cong C_p$. Since $c_3(G) = 0$, $a^{p^2} = 1$. Thus, $\langle a, x \rangle \cong C_{p^2} \times C_p$. By $Z(G) = G_3$, we see $a^p \notin Z(G)$. Since $[a^p, c] = [a, c]^p [a, c, a]^{\binom{p}{2}} = 1$, $[a^p, b] = c^p \neq 1$. Thus, $\langle c, a^p \rangle \cong C_{p^2} \times C_p$. However, $\langle c, a^p \rangle \neq \langle a, x \rangle$, a contradiction.

Case (ii) $c(G) = 4$. We see $G_3 \cong C_p \times C_p$ from Lemma 8. Assume that $G_4 = \langle z \rangle$ and $G_3 = \langle z \rangle \times \langle y \rangle$. It is easy to see that $G/G_3 = \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^p = \bar{b}^p = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c} \rangle$, and $G = \langle a, b \rangle$. $G' = \langle c, z, y \rangle$ is the unique abelian subgroup of order p^3. Since $[\langle a, b \rangle, \langle z, y \rangle] = G_4$, we have $[a, y] = z^i$, $[b, y] = z^j$ and at least one of i and j cannot be divided exactly by p. Then $\langle a^{-j} b^i, y, z \rangle$ is another abelian subgroup of order p^3 of G, a contradiction. The proof is complete. □

By the above propositions, we easily get the following theorem.

Theorem 19 *Let G be a nonabelian p-group with $|G| > p^4$. Then the following conclusions are equivalent:*

(1) $c_m(G) = 1$ where $m \geq 3$

(2) $a_3(G) = 1$

(3) G is a 2-group of maximal class.

Proof If (1), then (2) by Propositions 15 and 16. When (2) holds, we see (3) by Propositions 17 and 18. If G is a 2-group of maximal class, then G is isomorphic to one of the following three types of groups by Theorem 2.5.3 in [4]:

(a) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1} \rangle$, $n \geq 3$;

(b) $\langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1} \rangle, n \geq 3$;

(c) $\langle a, b | a^{2^{n-1}} = b^2 = 1, a^b = a^{-1+2^{n-2}} \rangle, n \geq 4$.

By calculation, we see that cyclic subgroups of order $\geq 2^3$ are in $\langle a \rangle$. Therefore, $c_m(G) = 1$ where $m \geq 3$. □

Theorem 20 *Let G be a finite p-group. Then $c_2(G) = 1$ if and only if G is a cyclic group or dihedral group.*

Proof If G is abelian, then G is cyclic. Assume that G is nonabelian and M is the unique cyclic subgroup of order p^2. If $C_G(M) = M$, then G is of maximal class by Lemma 11. For $C_G(M) > M$, we see $C_G(M)$ is cyclic from Lemma 6. Since any cyclic subgroup of order p^3 contains M and lies in $C_G(M)$, we have $a_3(G) = 1$. Therefore, G is of maximal class. By Lemma 8, G is a 2-group. It is easy to check that only the dihedral group in 2-groups of maximal class satisfies $c_2(G) = 1$. Conversely, the conclusion is obvious. □

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