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## Further results on edge - odd graceful graphs

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**Abstract:** We present edge - odd graceful labeling for the following families of graphs: wheel  $W_n$ , for  $n \equiv 1, 2$  and  $3 \pmod{4}$ ;  $C_n \odot \bar{K}_{2m-1}$ ; even helms;  $P_n \odot \bar{K}_{2m}$  and  $K_{2,s}$ . Also we present two theorems of non edge - odd graceful graphs and an idea to label complete graphs.

**Key words:** Edge - odd graceful labeling, Cycles, Symmetrical tree, Corona

### 1. Introduction

Solairaju and Chithra [3] defined a graph  $G$  with  $q$  edges to be edge-odd graceful if there is a bijection  $f$  from the edges of the graph to  $\{1, 3, 5, \dots, 2q - 1\}$  such that, when each vertex is assigned the sum of all the edges incident to it  $\pmod{2q}$ , the resulting vertex labels are distinct.

They prove that the following graphs are odd-graceful: paths with at least 3 vertices; odd cycles; ladders  $P_n \times P_2$  ( $n \geq 3$ ); stars with an even number of edges; and crowns  $C_n \odot K_1$ . In [4] they prove the following graphs have edge-odd graceful labelings:  $P_n$  ( $n > 1$ ) with a pendant edge attached to each vertex (combs); the graph obtained by appending  $2n + 1$  pendant edges to each endpoint of  $P_2$  or  $P_3$ ; and the graph obtained by subdividing each edge of the star  $K_{1,2n}$ .

In this paper we try to give some further results: we present an edge - odd graceful labeling to each of these families: wheel  $W_n$  for  $n \equiv 1, 2$  and  $3 \pmod{4}$ ;  $C_n \odot \bar{K}_{2m-1}$ ; even helms;  $P_n \odot \bar{K}_{2m}$  and  $K_{2,s}$ . We prove that the trees of odd number of vertices and odd degrees can't be an edge - odd graceful graphs. Also we prove that the cycle  $C_n$  is not an edge - odd graceful graph when  $n$  is even. Finally we give a simple way to label complete graphs which provides edge - odd graceful labelings to a good number of complete graphs  $K_n$  within the range  $n \in \{4, 5, 6, \dots, 99, 100\}$ .

### 2. Main results

**Definition 2.1** [1, 3, 4] A graph  $G(V(G), E(G))$  with  $p$  vertices and  $q$  edges is said to have an edge - odd graceful labeling if there exists a bijection  $f$  from  $E(G)$  to  $\{1, 3, 5, \dots, 2q - 1\}$  so that the induced mapping  $f^+$  from  $V(G)$  to  $\{0, 1, 2, \dots, 2q - 1\}$ , given by:  $f^+(v) = \left(\sum_{u \in V(G)} f(vu)\right) \pmod{2q}$ , is an injective.

We call the graph that admits an edge - odd graceful labeling: an edge - odd graceful graph.

**Example 2.2** In Figure 1 we present an edge - odd graceful labeling of the Peterson graph.

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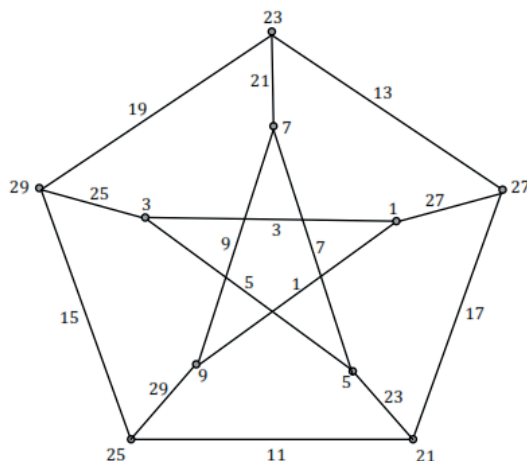


Figure 1. Peterson Graph

**Theorem 2.3** *The wheel  $W_n$  is an edge - odd graceful graph when  $n$  is even and  $n \equiv 1 \pmod 4$ .*

**Proof** Let the wheel  $W_n$  be given as indicated in Figure 2:

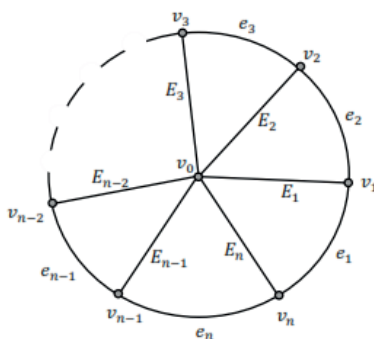


Figure 2.

We define the labeling function  $f$  as follows:

$$f(e_i) = 4n - 2i + 1; i = 1, 2, \dots, n$$

$$f(E_i) = 2i - 1; i = 1, 2, 3, \dots, n.$$

We realize that:

$$f^+(v_i) = (f(e_i) + f(E_i) + f(e_{i+1})) \pmod{4n} = (4n + f(e_{i+1})) \pmod{4n} = f(e_{i+1}); i = 1, 2, 3, \dots, n-1, \text{ and}$$

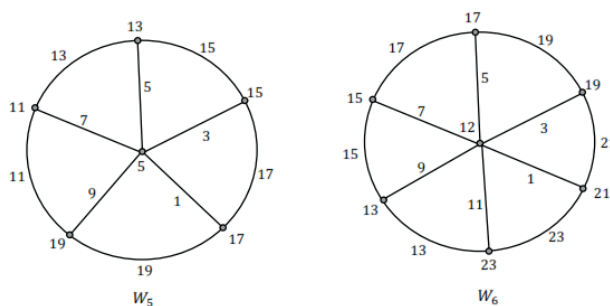
$$f^+(v_n) = (f(e_n) + f(E_n) + f(e_1)) \pmod{4n} = (4n + f(e_1)) \pmod{4n} = f(e_1).$$

$$f^+(v_0) = \sum_{i=1}^n f(E_i) \pmod{4n} = \sum_{i=1}^n (2i - 1) \pmod{4n} = n^2 \pmod{4n}$$

$$\implies f^+(v_0) \equiv \begin{cases} 0 \pmod{4n}, & n \equiv 0 \pmod{4} \\ 2n \pmod{4n}, & n \equiv 2 \pmod{4} \\ n \pmod{4n}, & n \equiv 1 \pmod{4} \end{cases}$$

We can see that in these three cases there is no repetition in the vertex labels which completes the proof.  $\square$

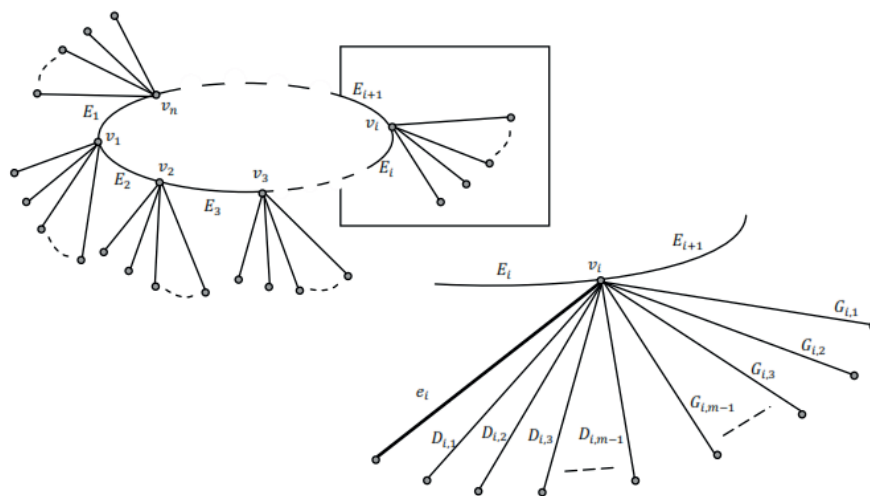
**Example 2.4** In Figure 3 we present the labeling of  $W_5$  and  $W_6$  according to Theorem 2.3 .



**Figure 3.**

**Theorem 2.5** The corona [2]  $C_n \odot \overline{K}_{2m-1}$  is an edge - odd graceful graph.

**Proof** Let the vertex and the edge symbols be given as in Figure 4:



**Figure 4.**

We define the edge labeling function  $f$  :

$$f(E_i) = 2(i-1)m + 1, \quad f(e_i) = 2q - (2(i-1)m + 1), \quad i = 1, 2, 3, \dots, n.$$

$$f(D_{i,j}) = 2q - (2(i-1)m + 2j + 1), \quad j = 1, 2, 3, \dots, m-1$$

$$f(G_{i,j}) = 2(i - 1)m + 2j + 1, j = 1, 2, 3, \dots, m - 1.$$

We realize the following:  $f(D_{i,j}) + f(G_{i,j}) \equiv 0 \pmod{2q}$ ,  $j = 1, 2, 3, \dots, m - 1$ .

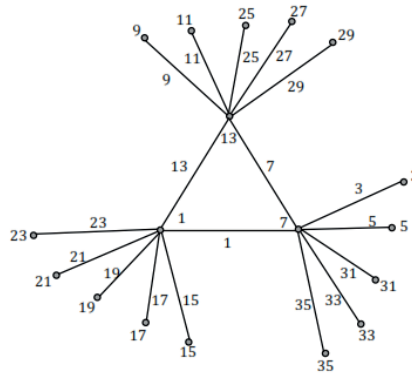
Also:  $f(E_i) + f(e_i) \equiv 0 \pmod{2q}$ , so verifying the vertex labelings we get that:

$$f^+(v_i) = \left( \sum_{j=1}^{m-1} f(D_{i,j}) + \sum_{j=1}^{m-1} f(G_{i,j}) + f(E_i) + f(e_i) + f(E_{i+1}) \right) \pmod{2q} = f(E_{i+1}), i = 1, 2, \dots, n -$$

1, and similarly we get:  $f^+(v_n) = f(E_1)$ .

Each pendant vertex takes the labels of its incident edge. Overall all vertex labels are edge labels so there are no repeated vertex labels, which completes the proof. □

**Example 2.6** In Figure 5: we give  $C_3 \odot K_5$  labeled according to Theorem 2.5.



**Figure 5.** edge - odd graceful labeling of  $C_3 \odot K_5$

**Theorem 2.7** The helm  $H_n$  [1] (i.e. the graph obtained from a wheel by attaching a pendant edge at each vertex of the  $n$ -cycle) is an edge - odd graceful for even  $n$ .

**Proof** Let the helm  $H_n$  be given as indicated in Figure 6:

We define now the labeling  $f$  as follows:

$$f(E_i) = 2i - 1, i = 1, 2, 3, \dots, n$$

$$f(D_i) = 6n - (2i - 1), i = 1, 2, 3, \dots, n$$

$$f(G_i) = 4n - (2i - 1), i = 1, 2, 3, \dots, n$$

Let us now verify  $f^+$  for the vertices:

$$\begin{aligned} f^+(v_i) &= (f(G_i) + f(D_i) + f(E_{i-1}) + f(D_{i-1})) \pmod{6n} = \\ &= 4n - 4i + 2 \pmod{6n} = 4n - 4i + 2, i = 2, 3, \dots, n. \end{aligned}$$

Similarly we find:  $f^+(v_1) = (f(G_1) + f(D_1) + f(E_n) + f(D_n)) \pmod{6n} = 4n - 2$ .

$$f(v_0) = \sum_{i=1}^n f(E_i) \pmod{6n} = \sum_{i=1}^n (2i - 1) \pmod{6n} = n^2 \pmod{6n}$$

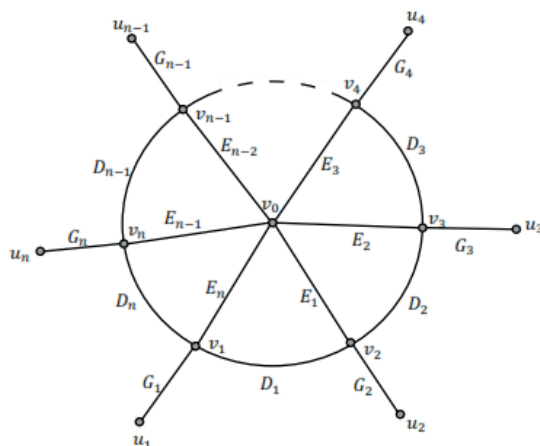


Figure 6.

$$\implies f(v_0) = \begin{cases} 0 \pmod{6n}, & n \equiv 0 \pmod{6} \\ 2n \pmod{6n}, & n \equiv 2 \pmod{6} \\ 4n \pmod{6n}, & n \equiv 4 \pmod{6} \end{cases}$$

In all cases  $f(v_0)$  is not congruent to  $f^+(v_i)$  modulo  $(6n)$ ,  $\forall i = 1, 2, 3, \dots, n$  which completes the proof.  $\square$

**Example 2.8** The graph  $H_6$  labeled according to Theorem 2.7 is presented in Figure 7.

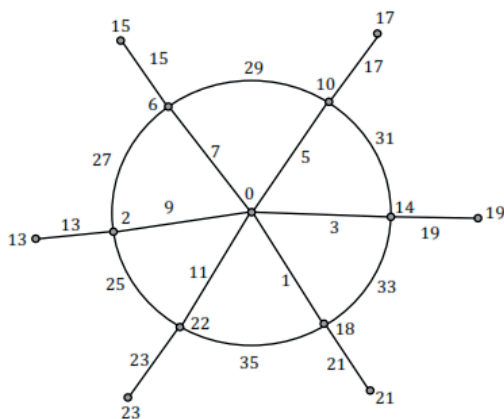


Figure 7.

**Theorem 2.9** The graph  $P_n \odot \overline{K}_{2m}$ ,  $n \geq 3$  is an edge - odd graceful graph.

**Proof** Let the graph  $P_n \odot \overline{K}_{2m}$  be given as indicated in Figure 8.

With  $q = 2nm + n - 1$ , we define the edge labeling function  $f$ :

$$f(D_{i,j}) = 2q - (2(i - 1)m + 2j - 1), \quad j = 1, 2, 3, \dots, m, \quad i = 1, 2, 3, \dots, n$$

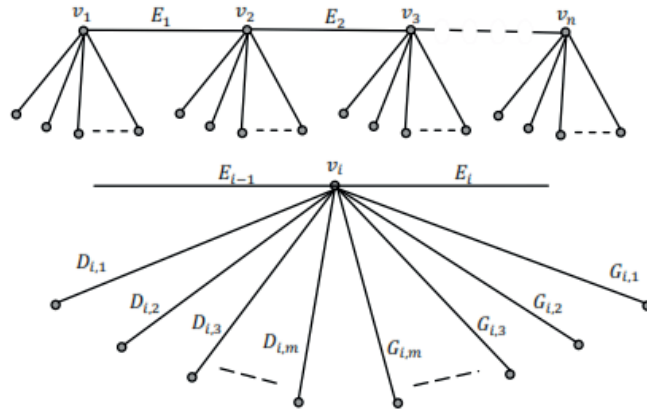


Figure 8.

$$f(G_{i,j}) = 2(i-1)m + 2j - 1, \quad j = 1, 2, 3, \dots, m, i = 1, 2, 3, \dots, n$$

$$= f(E_i) = 2nm + 2i - 1, \quad i = 1, 2, 3, \dots, n - 1$$

We consider the labeling of vertices, so we find:

$$f^+(v_1) = \left( \sum_{j=1}^m f(D_{1,j}) + \sum_{j=1}^m f(G_{1,j}) + f(E_1) \right) \text{ mod}(2q) = f(E_1)$$

$$f^+(v_n) = \left( \sum_{j=1}^m f(D_{n,j}) + \sum_{j=1}^m f(G_{n,j}) + f(E_{n-1}) \right) \text{ mod}(2q) = f(E_{n-1})$$

$$f^+(v_i) = \left( \sum_{j=1}^m f(D_{i,j}) + \sum_{j=1}^m f(G_{i,j}) + f(E_{i-1}) + f(E_i) \right) \text{ mod}(2q) =$$

$$= (f(E_{i-1}) + f(E_i)) \text{ mod}(2q) = (4nm + 4i - 4) \text{ mod}(2q), \quad i = 2, 3, \dots, n - 1$$

hence  $f^+(v_i) = \begin{cases} 4nm + 4i - 4, & i = 2, 3, \dots, \frac{n+1}{2} \\ 4\left(i - \frac{n+1}{2}\right), & i = \frac{n+3}{2}, \frac{n+5}{2}, \dots, n - 1 \end{cases}$ , when  $n$  is odd

or it follows that:  $f^+(v_i) = \begin{cases} 4nm + 4i - 4, & i = 2, 3, \dots, \frac{n}{2} \\ 4\left(i - \frac{n}{2}\right) - 2, & i = \frac{n+2}{2}, \frac{n+4}{2}, \dots, n - 1 \end{cases}$ , when  $n$  is even.

In both cases all these labels are even and different numbers. □

Regarding the pendant vertices we can see that each of them takes the label of its edge, so they are all odd and different numbers.

**Example 2.10** In Figure 9: we give  $P_4 \odot K_4$  labeled according to Theorem 2.9.

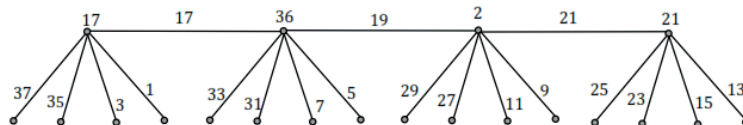


Figure 9. edge - odd graceful labeling of  $P_4 \odot K_4$

**Theorem 2.11** The complete bipartite graph  $K_{2,s}$  is an edge - odd graceful graph when  $s$  is odd.

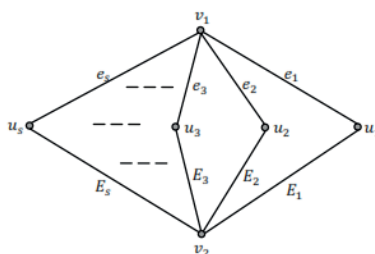


Figure 10.

**Proof** Let the complete bipartite graph \$K\_{2,s}\$ be given as indicated in Figure 10.

We define the labeling function \$f\$ as follows:

$$f(e_i) = 2i - 1, \quad i = 1, 2, 3, \dots, s$$

$$f(E_i) = 2s + 2i - 1, \quad i = 1, 2, 3, \dots, s$$

Considering the vertex labels we find:

$$f(u_i) = (2s + 4i - 2) \pmod{4s} = \begin{cases} 2s + 4i - 2, & i = 1, 2, \dots, \frac{s-1}{2} \\ 4\left(i - \frac{s+1}{2}\right), & i = \frac{s+1}{2}, \frac{s+3}{2}, \dots, s \end{cases}$$

$$f(v_1) = \left(\sum_{i=1}^s 2i - 1\right) \pmod{4s} = s^2 \pmod{4s} = \begin{cases} s, & s \equiv 1 \pmod{4} \\ 3s, & s \equiv 3 \pmod{4} \end{cases}$$

$$f(v_2) = \left(\sum_{i=1}^s 2s + 2i - 1\right) \pmod{4s} = 3s^2 \pmod{4s} = \begin{cases} 3s, & s \equiv 1 \pmod{4} \\ s, & s \equiv 3 \pmod{4} \end{cases}$$

Hence there are no repeated vertex labels. □

**Example 2.12** Figure 11 presents \$K\_{2,3}\$ and \$K\_{2,5}\$ labeled according to Theorem 2.11

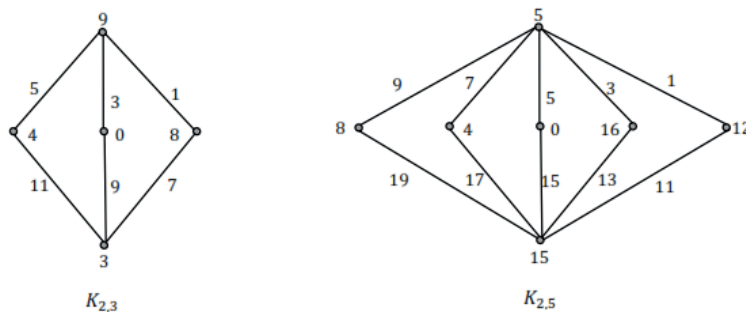


Figure 11.



### 3. Two theorems of non edge - odd graceful graphs

**Theorem 3.1** *The tree in which all vertices have odd degrees is not edge - odd graceful.*

**Proof** When a vertex in a given tree has an odd degree then it must have an odd label, since its label is an odd summation of odd numbers modulo  $2q$ , but among the possible odd numbers of the set:  $\{0, 1, 2, 3, \dots, 2q-1\}$  we have only  $q$  different odd numbers, hence they will not be enough to cover the labels of  $q + 1$  vertices of this tree without repetition.  $\square$

**Example 3.2** *The two trees shown in Figure 12 are not edge - odd graceful.*

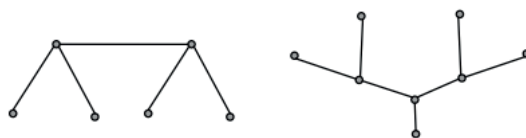


Figure 12.

**Theorem 3.3** *The cycle  $C_n$  is not an edge - odd graceful graph when  $n$  is even.*

**Proof** Suppose that an even cycle  $C_n$  admits an edge - odd graceful labeling, then the edge labels are the elements of the set:  $\{1, 3, 5, \dots, 2n - 1\}$ , so by the definition of the edge - odd graceful labeling we can deduce that the vertex labels must be exactly  $0, 2, 4, \dots, 2n - 2 \pmod{2n}$ . Now the summation of all vertex labels could be calculated twice:

- 1)  $\sum_{v \in V(C_n)} f^*(v) = \left(2 \sum_{e \in E(C_n)} f(e)\right) \equiv 2n^2 \equiv 0 \pmod{2n}$ , (since every edge label is calculated twice when we pass through all vertices)
- 2)  $\sum_{v \in V(C_n)} f^*(v) = \left(\sum_{i=0}^{2n-2} i\right) \equiv n(n - 1) \equiv n \pmod{2n}$

The difference in answers in 1) and 2) leads to a contradiction.  $\square$

### 4. Some edge - odd graceful complete graphs

**Introductory Example 4.1** *Let  $A_4, A_5$  and  $A_6$  be the following given matrices.*

$$A_4 = \begin{pmatrix} v_1 & 1 & 3 & 5 \\ v_2 & 1 & 7 & 9 \\ v_3 & 3 & 7 & 11 \\ v_4 & 5 & 9 & 11 \end{pmatrix}, A_5 = \begin{pmatrix} v_1 & 1 & 3 & 5 & 7 \\ v_2 & 1 & 9 & 11 & 13 \\ v_3 & 3 & 9 & 15 & 17 \\ v_4 & 5 & 11 & 15 & 19 \\ v_5 & 7 & 13 & 17 & 19 \end{pmatrix},$$

$$A_6 = \begin{pmatrix} v_1 & 1 & 3 & 5 & 7 & 9 \\ v_2 & 1 & 11 & 13 & 15 & 17 \\ v_3 & 3 & 11 & 19 & 21 & 23 \\ v_4 & 5 & 13 & 19 & 25 & 27 \\ v_5 & 7 & 15 & 21 & 25 & 29 \\ v_6 & 9 & 17 & 23 & 27 & 29 \end{pmatrix}.$$

We will define a labeling of the complete graph  $K_n$  "  $g^+$ " according to the matrix  $A_n$ ,  $n = 4, 5$  and  $6$ , as follows:

The label of the vertex  $v_i$  "  $g^+(v_i)$ ",  $i = 1, 2, \dots, n$ , is the summation of the numbers in its row, where these numbers are given as the labels of the edges that are incident to  $v_i$ . With realizing that:  $g^+(v_1) = \sum_{j=1}^{n-1} 2j - 1 = (n - 1)^2$  and

$$g^+(v_i) = g^+(v_{i-1}) + (n - i)(2n - 2i) + 2(i - 2), \quad i = 2, 3, \dots, n.$$

We make some calculations and discuss each one of these defined labelings as follows:

- 1) For  $A_4$  and  $K_4$  we see that:  $g^+(v_3) = g^+(v_1) + 12 \implies g^+(v_3) \equiv g^+(v_1) \pmod{2q = 12}$ , so the defined labeling in this case is not edge - odd graceful labeling.
- 2) For  $A_5$  and  $K_5$  we see that:  $g^+(v_5) = g^+(v_1) + 40 \implies g^+(v_5) \equiv g^+(v_1) \pmod{2q = 20}$ , also the defined labeling in this case is not edge - odd graceful labeling.
- 3) For  $A_6$  and  $K_6$  we see there are no identical vertex labels modulo  $2q = 30$ , i.e. the labeling  $g^+$  represents an edge - odd graceful labeling of  $K_6$ .

**Note 4.2** *In the following proposition we are going to generalize what is given in the previous example.*

**Proposition 4.3** *Let  $g$  be a labeling of the complete graph  $K_n$ :*

$g : E(K_n) \longrightarrow \{1, 3, 5, \dots, 2q - 1 = n(n - 1) - 1\}$ , for which the vertices have their incident labeled edges according to the following matrix:

$$\begin{pmatrix} v_1 & 1 & 3 & 5 & 7 & \dots & 2n - 5 & 2n - 3 \\ v_2 & 1 & 2n - 1 & 2n + 1 & 2n + 3 & \dots & 4n - 9 & 4n - 7 \\ v_3 & 3 & 2n - 1 & 4n - 5 & 4n - 3 & \dots & 6n - 15 & 6n - 13 \\ v_4 & 5 & 2n + 1 & 4n - 5 & 6n - 5 & \dots & 8n - 23 & 8n - 21 \\ v_5 & 7 & 2n + 3 & 4n - 3 & 6n - 5 & & 10n - 33 & 10n - 31 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} & 2n - 5 & 4n - 9 & 6n - 15 & 8n - 23 & \dots & 2n^2 - (n^2 + n + 5) & 2n^2 - (n^2 + n + 1) \\ v_n & 2n - 3 & 4n - 7 & 6n - 13 & 8n - 21 & \dots & 2n^2 - (n^2 + n + 3) & 2n^2 - (n^2 + n + 1) \end{pmatrix}$$

The numbers in the row in front of the vertex  $v_i$  are the  $n - 1$  labels of the edges that are incident to it,  $i = 1, 2, \dots, n$ . Let now  $g^+$  be the mapping which is defined as follows:  $g^+(v_i)$  is equal to the summation over all labels in its row.

We can see the following facts:

$$g^+(v_1) = \sum_{j=1}^{n-1} 2j - 1 = (n - 1)^2$$

$$g^+(v_i) = g^+(v_{i-1}) + (n - i)(2n - 2i) + 2(i - 2), \quad i = 2, 3, \dots, n.$$

We will test now whether or not  $2q = n(n - 1)$  divides:

- 1)  $(n - i)(2n - 2i) + 2(i - 2), \quad i = 2, 3, \dots, n$
- 2)  $\sum_{j=2}^i (n - j)(2n - 2j) + 2(j - 2), \quad i = 3, \dots, n.$

If  $n(n - 1)$  does not divide these numbers, then we will not find different vertices with the same label "as in case 3) in Introductory Example 4.1" and then the previous labeling will be an edge - odd graceful labeling, and  $g^+ \equiv f^+$ .

We do these tests using *Mathematica* until we reach  $K_{100}$ , so we get that:  $K_n$  is edge - odd graceful graph for  $n =$

$n =$	6	9	13	15	16	18	19	21	22	27
$n =$	28	30	31	33	34	36	37	39	40	42
$n =$	43	45	46	48	49	51	52	54	55	57
$n =$	58	60	61	64	66	67	69	70	72	73
$n =$	75	76	78	79	81	82	84	85	90	91
$n =$	93	94	96	97	99	100	*	*	*	*

**Note 4.4** Proposition 4.3 suggests that we can find complete graphs of some orders which are edge - odd graceful, which means that there is no upper bound for the number of edges for these orders.

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