# The inclusion theorems for variable exponent Lorentz spaces

ÖZNUR KULAK

## Recommended Citation

KULAK, ÖZNUR (2016) "The inclusion theorems for variable exponent Lorentz spaces," *Turkish Journal of Mathematics*: Vol. 40: No. 3, Article 10. https://doi.org/10.3906/mat-1502-23
Available at: https://journals.tubitak.gov.tr/math/vol40/iss3/10

# The inclusion theorems for variable exponent Lorentz spaces

**Öznur KULAK**[*]

Department of Banking and Finance, Görele Applied Sciences Academy, Giresun University,
Görele, Giresun, Turkey

**Abstract:** Let $(X,\Sigma,\mu)$ and $(X,\Sigma,\nu)$ be measure spaces. Assume that $L^{p_1(.),q_1(.)}(X,\mu)$ and $L^{p_2(.),q_2(.)}(X,\nu)$ are two variable exponent Lorentz spaces where $p,q \in P_0([0,l])$. In this paper we investigated the existence of the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ under what conditions for two measures $\mu$ and $\nu$ on $(X,\Sigma)$.

**Key words:** Inclusion, variable exponent Lorentz space

## 1. Introduction

Let $(X,\Sigma,\mu)$ be a measure space. The distribution function of $f$ is defined by

$$\lambda_f(y) = \mu(\{x \in X : |f(x)| > y\}) = \int\limits_{\{x \in X : |f(x)| > y\}} d\mu(x) \ \ [4,6].$$

The rearrangement function of $f$ is defined by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \le t\} = \sup\{y > 0 : \lambda_f(y) > t\}, t \ge 0 \ \ [4,6].$$

Moreover, the average function of $f^*$ is given by

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)\,ds.$$

Let $0 < l \le \infty$. We put

$$p_- = \inf_{x \in [0,l]} p(x), \qquad p^+ = \sup_{x \in [0,l]} p(x).$$

In this paper, we shall also use the notation

$$P_a = \{p : a < p_- \le p^+ < \infty\}, \ a \in \mathbb{R}.$$

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[*]Correspondence: oznur.kulak@giresun.edu.tr
2010 *AMS Mathematics Subject Classification:* 43A15.

The set $IP([0, \infty])$ is the family of $p \in L^\infty([0, \infty])$ such that there exist the limits $p(0) = \lim\limits_{x \to 0} p(x)$, $p(\infty) = \lim\limits_{x \to \infty} p(x)$ and we have

$$|p(x) - p(0)| \leq \frac{C}{\ln \frac{1}{|x|}}, \quad |x| \leq \frac{1}{2} \quad (C > 0)$$

and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)}, \quad |x| > 2 \quad (C > 0). \tag{1.1}$$

We also denote $IP_a([0, l]) = IP([0, l]) \cap P_a([0, l])$. If $l = \infty$, it is enough that the inequality (1.1) is satisfied [4].

Let $\Omega \subset X$. We denote $l = \mu(\Omega)$. Assume that $p, q \in P_0([0, l])$. The variable exponent Lorentz spaces $L^{p(.), q(.)}(\Omega, \mu)$ are defined as the set of all (equivalence classes) measurable functions $f$ on $X$ such that $J_{p,q}(f) < \infty$, where

$$J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} dt. \tag{1.2}$$

We use the notation

$$\|f\|_{L^{p(.), q(.)}(\Omega, \mu)}^1 = \inf\left\{\lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \leq 1\right\} \quad [2].$$

Let $p \in IP_0([0, l])$ and $q \in IP_1([0, l])$. If $l = \infty$, then the equality (1.2) is equivalent to the following sum:

$$\int_0^1 t^{\frac{q(0)}{p(0)} - 1} (f^*(t))^{q(t)} dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)} - 1} (f^*(t))^{q(t)} dt \quad [2].$$

If $l < \infty$, then the equality (1.2) is equivalent to the integral $\int_0^l t^{\frac{q(0)}{p(0)} - 1} (f^*(t))^{q(t)} dt$ [4]. The space $L^{p(.), q(.)}(\Omega, \mu)$ is a normed vector space with norm

$$\|f\|_{L^{p(.), q(.)}(\Omega, \mu)} = \inf\left\{\lambda > 0 : J_{p,q}(\frac{f}{\lambda}) \leq 1\right\}$$

such that $J_{p,q}(f) = \int_0^l t^{\frac{q(t)}{p(t)} - 1} (f^{**}(t))^{q(t)} dt$ [4].

For $0 \leq p \leq q \leq \infty$, the inclusion $l^p(X) \subset l^q(X)$ is known. In [13], the inclusion $L^p(\mu) \subset L^q(\mu)$ was characterized by all positive measures whenever $0 < p \leq q \leq \infty$. Then Romero [10] improved some results of [13]. Lastly, the more general inclusion $L^p(\mu) \subset L^q(\nu)$ was considered by [8], where $\mu$ and $\nu$ are two measures on $(X, \Sigma)$. Moreover, in [5], Gürkanlı considered inclusion theorems of Lorentz spaces. Embeddings for discrete weighted Lebesgue spaces with variable exponents were studied by Nekvinda [9]. In [1], the inclusion $L^{p(.)}(\mu) \subset L^{q(.)}(\nu)$ was considered by Aydın and Gürkanlı. In [2], Bandaliev considered embeddings between

variable exponent Lebesgue spaces with different measures. Also in this area, Diening et al. studied Lebesgue and Sobolev spaces with variable exponent $[3]$. Now, in the present paper, we investigate the existence of the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ under what conditions.

## 2. Main results

Let $(X, \Sigma, \mu)$ be a measure space. If two measures $\mu$ and $\nu$ are absolutely continuous with respect to each other ($\mu << \nu$ and $\nu << \mu$) then we denote this by $\mu \approx \nu$ $[11]$.

**Lemma 1** *The inclusion* $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ *holds in the sense of equivalence classes if and only if* $\mu \approx \nu$ *and* $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ *in the sense of individual functions.*

**Proof** Assume that $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ holds in the sense of equivalence classes. Let $f \in L^{p_1(.),q_1(.)}(X,\mu)$ be any individual function. That means $f \in L^{p_1(.),q_1(.)}(X,\mu)$ in the sense of equivalence classes. Therefore, we have $f \in L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of equivalence classes from the assumption. Thus we obtain $f \in L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of individual functions. Therefore, we find the inclusion $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of individual functions. Let $E \in S$ with $\mu(E) = 0$. Then since $\chi_E = 0$ $\mu$-almost everywhere (a.e), we have

$$J_{p_1,q_1}(\chi_E) = \int_0^l t^{\frac{q_1(t)}{p_1(t)}-1}\left(\chi_E^*(t)\right)^{q_1(t)} dt = \int_0^l t^{\frac{q_1(t)}{p_1(t)}-1}\left(\chi_{[0,\mu(E))}(t)\right)^{q_1(t)} dt$$

$$= \int_0^{\mu(E)} t^{\frac{q_1(t)}{p_1(t)}-1} dt = 0$$

and we write $\chi_E \in L^{p_1(.),q_1(.)}(X,\mu)$. Therefore, $\chi_E$ is in the equivalence classes of $0 \in L^{p_1(.),q_1(.)}(X,\mu)$. Moreover, the equivalence classes of $0$ (with respect to $\mu$) are also an element of $L^{p_2(.),q_2(.)}(X,\nu)$. Thus $\chi_E$ is in the equivalence classes of $0 \in L^{p_2(.),q_2(.)}(X,\nu)$ with respect to $\nu$. This implies $\nu(E) = 0$. Therefore, $\nu << \mu$. Similarly, $\mu << \nu$ is proved. The proof of the other side is clear.

Throughout, we assume that $p, q \in P_0([0, l])$ unless the contrary is stated. $\qquad\square$

**Lemma 2** *a) Let* $\mu(X) = \infty$, $p, q \in IP_1([0, \infty])$, $q(\infty) > p(\infty)$ *and* $q(0) < p(0)$. *If* $(f_n)_{n \in \mathbb{N}}$ *convergences to* $f$ *in* $L^{p(.),q(.)}(X,\mu)$ *then* $(f_n)_{n \in \mathbb{N}}$ *convergences to* $f$ *in measure.*

*b) Let* $\mu(X) < \infty$ *and* $p, q \in IP_1([0, \infty])$. *If* $(f_n)_{n \in \mathbb{N}}$ *convergences to* $f$ *in* $L^{p(.),q(.)}(X,\mu)$ *then* $(f_n)_{n \in \mathbb{N}}$ *convergences to* $f$ *in measure.*

**Proof** a) Assume that $(f_n)_{n \in \mathbb{N}}$ convergences to $f$ in $L^{p(.),q(.)}(X,\mu)$. Then we write

$$J_{p,q}(f_n - f) \cong \int_0^1 t^{\frac{q(0)}{p(0)}-1}(f_n - f)^*(t)^{q(t)} dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1}(f_n - f)^*(t)^{q(t)} dt \to 0$$

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for $n \to \infty$. Since $q(\infty) > p(\infty)$ and $q(0) < p(0)$ , we have

$$\int\limits_0^\infty (f_n - f)^* (t)^{q(t)} \, dt \leq J_{p,q} (f_n - f) \to 0.$$

for $n \to \infty$. Then $(f_n - f)^*$ convergences to 0 in $L^{q(\cdot)} ([0, \infty])$. Thus we find that $(f_n - f)^*$ convergences to 0 in measure (with respect to measure on $[0, \infty]$) by [7]. Furthermore, since

$$\lambda_{(f_n-f)^*} (\varepsilon) = \mu \left( \{ t : (f_n - f)^* (t) > \varepsilon \} \right) = \mu \left( \{ x : (f_n - f) (x) > \varepsilon \} \right) = \lambda_{(f_n-f)} (\epsilon) \quad [6]$$

for all $\varepsilon > 0$, $f_n$ converges to $f$ in measure.

**b)** Assume that $(f_n)_{n \in \mathbb{N}}$ convergences to $f$ in $L^{p(\cdot),q(\cdot)} (X, \mu)$. Then since $l = \mu (X) < \infty$,

$$J_{p,q} (f_n - f) \cong \int\limits_0^l t^{\frac{q(0)}{p(0)} - 1} (f_n - f)^* (t)^{q(t)} \, dt \to 0 \tag{2.1}$$

holds for $n \to \infty$. In addition, $L^{p(\cdot),q(\cdot)} (X, \mu)$ is a Banach function space [4] and we have

$$\int\limits_X (f_n - f) (x) \, dx \leq C_X \, \|f_n - f\|_{L^{p(\cdot),q(\cdot)}(X,\mu)} . \tag{2.2}$$

Therefore by using (2.1) and (2.2), we obtain $(f_n)_{n \in \mathbb{N}}$ convergences to $f$ in $L^1 (X)$. Thus $f_n$ converges to $f$ in measure. □

**Theorem 1** *a) Let $p_i, q_i \in IP_1 ([0, \infty])$, $(i = 1, 2)$, $\mu (X) = \infty$, $q_i (\infty) > p_i (\infty)$, and $q_i (0) < p_i (0)$, $(i = 1, 2)$. Then the inclusion*

$$L^{p_1(\cdot),q_1(\cdot)} (X, \mu) \subset L^{p_2(\cdot),q_2(\cdot)} (X, \nu)$$

*holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists $C > 0$ such that*

$$\|f\|^1_{L^{p_2(\cdot),q_2(\cdot)}(X,\nu)} \leq C \, \|f\|^1_{L^{p_1(\cdot),q_1(\cdot)}(X,\mu)}$$

*for all $f \in L^{p_1(\cdot),q_1(\cdot)} (X, \mu)$.*

*b) Let $p_i, q_i \in IP_1 ([0, l])$, $(i = 1, 2)$ and $l = \mu (X) < \infty$. Then the inclusion*

$$L^{p_1(\cdot),q_1(\cdot)} (X, \mu) \subset L^{p_2(\cdot),q_2(\cdot)} (X, \nu)$$

*holds in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists $C > 0$ such that*

$$\|f\|^1_{L^{p_2(\cdot),q_2(\cdot)}(X,\nu)} \leq C \, \|f\|^1_{L^{p_1(\cdot),q_1(\cdot)}(X,\mu)}$$

*for all $f \in L^{p_1(\cdot),q_1(\cdot)} (X, \mu)$.*

**Proof a)** Suppose that $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ holds in the sense of equivalence classes. We define the unit operator $I(f) = f$ from $L^{p_1(.),q_1(.)}(X,\mu)$ into $L^{p_2(.),q_2(.)}(X,\nu)$. Now we show that $I$ is closed. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence such that $f_n \to f$ in $L^{p_1(.),q_1(.)}(X,\mu)$ and $I(f_n) = f_n \to g$ in $L^{p_2(.),q_2(.)}(X,\nu)$. Thus, by Lemma 2, $(f_n)_{n\in\mathbb{N}}$ convergences to $f$ in measure (with respect to $\mu$). Hence there exists subsequence $(f_{n_i})_{n_i\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $(f_{n_i})_{n_i\in\mathbb{N}}$ pointwise converges to $f$, $\mu$-almost everywhere (a.e.). Moreover, since $(f_n)_{n\in\mathbb{N}}$ convergences to $g$ in $L^{p_2(.),q_2(.)}(X,\nu)$, it is easy to show that $(f_{n_i})_{n_i\in\mathbb{N}}$ convergences to $g$ in $L^{p_2(.),q_2(.)}(X,\nu)$. Then $(f_{n_i})_{n_i\in\mathbb{N}}$ convergences to $g$ in measure (with respect to $\nu$). Thus we find a subsequence $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}} \subset (f_{n_i})_{n_i\in\mathbb{N}}$ such that $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ convergences to $g$ pointwise $\nu$-a.e. Let $M$ be a set of the points such that $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ does not convergence to $g$ pointwise. Hence $\nu(M) = 0$. From the assumption $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\nu)$ in the sense of equivalence classes and so we write $\mu \approx \nu$ by Lemma 1. Thus $\nu(M) = \mu(M) = 0$. Hence $\left(f_{n_{i_k}}\right)_{n_{i_k}\in\mathbb{N}}$ convergences to $g$ pointwise $\mu$-a.e. Consequently using the following inequality

$$|f(x) - g(x)| \leq \left|f(x) - f_{n_{i_k}}(x)\right| + \left|f_{n_{i_k}}(x) - g(x)\right|,$$

we have $f = g$ $\mu$-a.e. and $f = g$ $\nu$-a.e. That means $I$ is closed. By the closed graph theorem, there exists $C > 0$ such that

$$\|f\|^1_{L^{p_2(.),q_2(.)}(X,\nu)} \leq C \|f\|^1_{L^{p_1(.),q_1(.)}(X,\mu)}.$$

The proof of the other direction is easy.

In this Theorem, (b) can be proved easily by using the technique of the proof in (a). $\qquad\square$

**Lemma 3 a)** If $\nu(E) \leq \mu(E)$ for all $E \in \Sigma$, then the inequality

$$\|f\|^1_{L^{p(.),q(.)}(X,\nu)} \leq \|f\|^1_{L^{p(.),q(.)}(X,\mu)}$$

holds for all $f \in L^{p(.),q(.)}(X,\mu)$.

**b)** Let $p \in IP_0([0,l]), 1 \leq q < \infty$. If there exists $M > 0$ such that $\nu(E) \leq M\mu(E)$ for all $E \in \Sigma$, then the inequality

$$\|f\|^1_{L^{p(.),q}(X,\nu)} \leq M \|f\|^1_{L^{p(.),q}(X,\mu)}$$

holds for all $f \in L^{p(.),q}(X,\mu)$.

**Proof a)** Let $\nu(E) \leq \mu(E)$ for all $E \in \Sigma$. From [5], we have $f^{*,\nu}(t) \leq f^{*,\mu}(t)$ ($f^{*,\nu}$ and $f^{*,\mu}$ are the rearrangements of $f$ with respect to the measures $\nu$ and $\mu$ respectively) for all $t \geq 0$. This implies

$$\int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{*,\nu}\right)(t)^{q(t)}\,dt \leq \int_0^l t^{\frac{q(t)}{p(t)}-1}\left(f^{*,\mu}\right)(t)^{q(t)}\,dt.$$

where $l = \mu(X)$. Thus we have

$$\|f\|^1_{L^{p(.),q(.)}(X,\nu)} \leq \|f\|^1_{L^{p(.),q(.)}(X,\mu)}.$$

**b)** Let $l = \mu(X) = \infty$. Assume that there exists $M > 0$ such that $\nu(E) \leq M\mu(E)$ for all $E \in \Sigma$. If we take $k = M\mu$, then $k$ is a measure. Then it is known that $f^{*,k}(t) = f^{*,\mu}\left(\frac{t}{M}\right) \geq f^{*,\nu}(t)$ by [5]. Therefore, if we set $\frac{t}{M} = u$, then

$$J_{p,q}^{\nu}(f) = \int_0^{\infty} t^{\frac{q}{p(t)}-1} f^{*,\nu}(t)^q \, dt \leq J_{p,q}^{k}(f) = \int_0^{\infty} t^{\frac{q}{p(t)}-1} f^{*,k}(t)^q \, dt$$

$$= \int_0^{\infty} t^{\frac{q}{p(t)}-1} f^{*,\mu}\left(\frac{t}{M}\right)^q \, dt$$

$$\cong \int_0^1 t^{\frac{q}{p(0)}-1} f^{*,\mu}\left(\frac{t}{M}\right)^q \, dt + \int_1^{\infty} t^{\frac{q}{p(\infty)}-1} f^{*,\mu}\left(\frac{t}{M}\right)^q \, dt$$

$$= M^{\frac{q}{p(0)}} \int_0^1 u^{\frac{q}{p(0)}-1} f^{*,\mu}(u)^q \, du + M^{\frac{q}{p(\infty)}} \int_1^{\infty} u^{\frac{q}{p(\infty)}-1} f^{*,\mu}(u)^q \, du$$

$$\lesssim M_0 \int_0^{\infty} u^{\frac{q}{p(t)}-1} f^{*,\mu}(u)^q \, du = M_0 J_{p,q}^{\mu}(f)$$

where $M_0 = \text{maks}\left\{M^{\frac{q}{p(0)}}, M^{\frac{q}{p(\infty)}}\right\}$. Thus we have $\|f\|_{L^{p(\cdot),q}(X,\nu)}^1 \leq M \|f\|_{L^{p(\cdot),q}(X,\mu)}^1$.

Similarly the Lemma is proved for $l = \mu(X) < \infty$. $\qquad\square$

**Lemma 4** *Let $p, q \in IP_1([0,l])$, and $l = \mu(X) < \infty$.*

*a)* *If $\mu \approx \nu$ and there exists $M > 0$ such that $\nu(E) \leq M\mu(E)$ for all $E \in \Sigma$ then the inclusion $L^1(X,\mu) \subset L^1(X,\nu)$ holds.*

*b)* *If the inclusion $L^1(X,\mu) \subset L^1(X,\nu)$ holds then the inclusion $L^{p(\cdot),q(\cdot)}(X,\mu) \subset L^{p(\cdot),q(\cdot)}(X,\nu)$ holds.*

**Proof** **a)** It is known by $[5]$.

**b)** Take any $f \in L^{p(\cdot),q(\cdot)}(X,\mu)$ is given. Since $\chi_{[0,\infty]} t^{\frac{q(t)}{p(t)}-1} f^*(t)^{q(t)} \in L^1(\mu)$ and $L^1(\mu) \subset L^1(\nu)$. Thus we obtain $\chi_{[0,\infty]} t^{\frac{q(t)}{p(t)}-1} f^*(t)^{q(t)} \in L^1(\nu)$. That means $f \in L^{p(\cdot),q(\cdot)}(X,\nu)$. $\qquad\square$

**Theorem 2** *Let $p, q \in IP_1([0,l])$ and $l = \mu(X) < \infty$. Then the inclusion $L^{p(\cdot),q(\cdot)}(X,\mu) \subset L^{p(\cdot),q(\cdot)}(X,\nu)$ holds if and only if $\mu \approx \nu$ and there exists $M > 0$ such that $\nu(E) \leq M\mu(E)$ for all $E \in \Sigma$.*

**Proof** $\implies$ By Theorem 1, there exists $M > 0$ such that

$$\|f\|_{L^{p(\cdot),q(\cdot)}(X,\nu)}^1 \leq M \|f\|_{L^{p(\cdot),q(\cdot)}(X,\mu)}^1 \tag{2.3}$$

for all $f \in L^{p(.),q(.)}(X,\mu)$. Moreover,

$$J_{p,q}^{\nu}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1} \left((\chi_E)^{*,\nu}(t)\right)^{q(t)} dt = \int_0^{\nu(E)} t^{\frac{q(0)}{p(0)}-1} dt = \frac{p(0)}{q(0)} \nu(E)^{\frac{q(0)}{p(0)}}$$

holds. Similarly, we have

$$J_{p,q}^{\mu}(f) \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q(0)}{p(0)}}.$$

Therefore, we write

$$\|\chi_E\|_{L^{p(.),q(.)}(X,\nu)}^1 \cong \frac{p(0)}{q(0)} \nu(E)^{\frac{q(0)}{p(0)}} \tag{2.4}$$

and

$$\|\chi_E\|_{L^{p(.),q(.)}(X,\mu)}^1 \cong \frac{p(0)}{q(0)} \mu(E)^{\frac{q(0)}{p(0)}}. \tag{2.5}$$

Thus, from (2.3), (2.4), and (2.5), we have

$$\nu(E) \leq M\mu(E).$$

$\Longleftarrow$ From Lemma 4, the proof is clear. $\qquad\square$

**Theorem 3** *Let* $p_i, q_i \in IP_1([0,l])$, $(i=1,2), l = \mu(X) < \infty$, *and* $q_1(0)p_2(0) > q_2(0)p_1(0)$. *If* $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$ *then there exists a constant* $m \geq 0$ *such that* $\mu(E) \geq m$ *for every* $\mu$-*nonnull set* $E \in \Sigma$.

**Proof** By Theorem 1, there exists $C > 0$ such that

$$\|f\|_{L^{p_2(.),q_2(.)}(X,\mu)}^1 \leq C \|f\|_{L^{p_1(.),q_1(.)}(X,\mu)}^1$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Let $E \in \Sigma$ be a $\mu$-nonnull set. Since $\mu(E) < \infty$, we have

$$J_{p_1,q_1}(\chi_E) \cong \int_0^l t^{\frac{q_1(0)}{p_1(0)}-1} (\chi_E^*(t))^{q_1(t)} dt = \int_0^{\mu(E)} t^{\frac{q_1(0)}{p_1(0)}-1} dt = \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}.$$

and so $\|\chi_E\|_{L^{p_1(.),q_1(.)}(X,\nu)}^1 \cong \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}$ holds. Similarly, we have $\|\chi_E\|_{L^{p_2(.),q_2(.)}(X,\nu)}^1 \cong \frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}}$.
Then we write

$$\frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \leq C \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}}$$

$$\frac{1}{C} \frac{q_1(0)p_2(0)}{p_1(0)q_2(0)} \leq \mu(E)^{\frac{q_1(0)}{p_1(0)} - \frac{q_2(0)}{p_2(0)}}.$$

If we set $m = \left(\frac{1}{C} \frac{q_1(0)p_2(0)}{p_1(0)q_2(0)}\right)^{\frac{p_1(0)p_2(0)}{q_1(0)p_2(0)-q_2(0)p_1(0)}}$, we obtain $\mu(E) \geq m$ for every $\mu$-nonnull set $E \in \Sigma$. $\qquad\square$

**Theorem 4** *Let $p, q \in IP_1([0,l])$ and $l = \mu(X) < \infty$ .*

**a)** *If $q_2(.) \leq q_1(.)$, $q_1(0) \leq p_1(0)$, and $q_2(0) \geq p_2(0)$ then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

*holds.*

**b)** *If $q(.) \leq p(.)$ and $q(0) = p(0)$, then the inclusion*

$$L^{p(.),p(.)}(X,\mu) \subset L^{p(.),q(.)}(X,\mu)$$

*holds.*

**c)** *If $p(.) \leq q(.)$ and $q(0) = p(0)$, then the inclusion*

$$L^{p(.),q(.)}(X,\mu) \subset L^{p(.),p(.)}(X,\mu)$$

*holds.*

**d)** *If $q(0) \geq p(0)$ then the inclusion*

$$L^{q(.),q(.)}(X,\mu) \subset L^{p(.),q(.)}(X,\mu)$$

*holds.*

**e)** *If $q(0) \leq p(0)$, then the inclusion*

$$L^{p(.),q(.)}(X,\mu) \subset L^{q(.),q(.)}(X,\mu)$$

*holds.*

**Proof**   **a)** Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Then we have

$$\infty > J_{p_1,q_1}(f) \cong \int_0^l t^{\frac{q_1(0)}{p_1(0)}-1}(f^*(t))^{q_1(t)}\, dt \geq l^{\frac{q_1(0)}{p_1(0)}-1}\int_0^l (f^*(t))^{q_1(t)}\, dt.$$

Therefore, we obtain $f^* \in L^{q_1(.)}([0,l])$. Moreover, since $q_2(.) \leq q_1(.)$, we write $L^{q_1(.)}([0,l]) \subset L^{q_2(.)}([0,l])$ from [7]. That means $f^* \in L^{q_2(.)}([0,l])$. From this result, we have

$$J_{p_2,q_2}(f) \cong \int_0^l t^{\frac{q_2(0)}{p_2(0)}-1}(f^*(t))^{q_2(t)}\, dt \leq l^{\frac{q_2(0)}{p_2(0)}-1}\int_0^l (f^*(t))^{q_2(t)}\, dt < \infty.$$

Thus we find that $f \in L^{p_2(.),q_2(.)}(X,\mu)$.

**b)** Take any $f \in L^{p(.),p(.)}(X,\mu)$. That means $f^* \in L^{p(.)}([0,l])$. Again since $q(.) \leq p(.)$, we know that $L^{p(.)}([0,l]) \subset L^{q(.)}([0,l])$ from [7]. Therefore, we have

$$J_{p,q}(f) \cong \int_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt = \int_0^l (f^*(t))^{q(t)}\, dt < \infty.$$

Thus we obtain $f \in L^{p(.),q(.)}(X,\mu)$.

**c)** This hypothesis is proved easily using the technique in (b).

**d)** Take any $f \in L^{q(.),q(.)}(X,\mu)$. Assume that $l = \mu(X) \geq 1$. Then since $J_{q,q}(f) \cong \int\limits_0^l (f^*(t))^{q(t)}\, dt < \infty$,

and since $q(0) \geq p(0)$, we have

$$J_{p,q}(f) \cong \int\limits_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt + \int\limits_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt$$

$$\leq \int\limits_0^1 (f^*(t))^{q(t)}\, dt + l^{\frac{q(0)}{p(0)}-1}\int\limits_1^l (f^*(t))^{q(t)}\, dt < \infty$$

Therefore, $f \in L^{p(.),q(.)}(X,\mu)$. Now let $l < 1$ and so we have

$$J_{p,q}(f) \cong \int\limits_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt < \int\limits_0^l (f^*(t))^{q(t)}\, dt < \infty.$$

Thus similarly $f \in L^{p(.),q(.)}(X,\mu)$.

**e)** Take any $f \in L^{p(.),q(.)}(X,\mu)$. Assume that $l = \mu(X) \geq 1$. Then since $J_{p,q}(f) \cong \int\limits_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt < \infty$, we have

$$\int\limits_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt \leq J_{p,q}(f) < \infty$$

and

$$\int\limits_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt \leq J_{p,q}(f) < \infty.$$

In addition, since $q(0) \leq p(0)$, we have

$$J_{q,q}(f) \cong \int\limits_0^l (f^*(t))^{q(t)}\, dt \leq \int\limits_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt + \int\limits_1^l t^{\frac{q(0)}{p(0)}-1}t^{-\left(\frac{q(0)}{p(0)}-1\right)}(f^*(t))^{q(t)}\, dt$$

$$\leq \int\limits_0^1 t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt + l^{-\left(\frac{q(0)}{p(0)}-1\right)}\int\limits_1^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt < \infty$$

Therefore, $f \in L^{q(.),q(.)}(X,\mu)$. Now let $l < 1$ and we have

$$J_{q,q}(f) \cong \int\limits_0^l (f^*(t))^{q(t)}\, dt < \int\limits_0^l t^{\frac{q(0)}{p(0)}-1}(f^*(t))^{q(t)}\, dt < \infty.$$

Thus similarly $f \in L^{q(.),q(.)}(X,\mu)$. $\qquad\square$

**Theorem 5** *If $\mu(X) < \infty$, $1 \leq q_2(.) \leq q_1(.)$, $\frac{1}{p_1(.)} + \frac{1}{q_2(.)} = 1$ and $\frac{1}{p_2(.)} + \frac{1}{q_1(.)} = 1$ then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

*holds.*

**Proof** Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Then we have

$$t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) = t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) \, t t^{-1}$$

$$= t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) \, t^{\left(\frac{1}{p_1(.)} + \frac{1}{q_2(.)}\right)} t^{-\left(\frac{1}{p_2(.)} + \frac{1}{q_1(.)}\right)} = t^{\left(\frac{1}{p_1(t)} - \frac{1}{q_1(t)}\right)} f^*(t).$$

Therefore, write $t^{\left(\frac{1}{p_2(t)} - \frac{1}{q_2(t)}\right)} f^*(t) \in L^{q_1(.)}([0,l]) \subset L^{q_2(.)}([0,l])$. That means $f \in L^{p_2(.),q_2(.)}(X,\mu)$. $\qquad\square$

**Theorem 6** *Let $p_i, q_i \in IP_1([0,l])$, $(i = 1,2)$, $\mu(X) < \infty$, $q_2(.) \geq q_1(.)$, $q_2(0) \geq p_2(0)$, and $q_1(0) \leq p_1(0)$. If there exists a constant $m > 0$ such that $\mu(E) \geq m$ for every $\mu$-nonnull set $E \in \Sigma$ then the inclusion*

$$L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$$

*holds.*

**Proof** Take any $f \in L^{p_1(.),q_1(.)}(X,\mu)$. Define that the set $E_n = \{x \in X : |f(x)| > n\}$ for every $n \in \mathbb{N}$. Since $q_1(0) \leq p_1(0)$, we write $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{q_1(.),q_1(.)}(X,\mu)$ from Theorem 4. Therefore, there exists $C > 0$ such that

$$\|f\|_{L^{q_1(.),q_1(.)}(X,\mu)}^1 \leq C \|f\|_{L^{p_1(.),q_1(.)}(X,\mu)}^1 \tag{2.6}$$

for all $f \in L^{p_1(.),q_1(.)}(X,\mu)$. On the other hand, since $|f(x)| > n > 1$ for all $x \in E_n$, we have $|f^*(t)| > 1$ for all $t \in [0,\mu(E_n)]$. Thus if we set $\frac{t}{2} = u$, then we have

$$n^{q_1-} \mu(E_n) \leq \int_{E_n} |f|^{q_1-} \, d\mu = \int_X |f\chi_{E_n}|^{q_1-} \, d\mu = \int_0^\infty \left|(f\chi_{E_n})^*\right|(t)^{q_1-} \, dt$$

$$\leq \int_0^\infty |f^*|\left(\frac{t}{2}\right)^{q_1-} \chi_{[0,\mu(E_n)]}\left(\frac{t}{2}\right) dt = 2\int_0^\infty |f^*|(u)^{q_1-} \chi_{[0,\mu(E_n)]}(u) \, du$$

$$= 2\int_0^{\mu(E_n)} |f^*|(u)^{q_1-} \, du \leq 2\int_0^{\mu(E_n)} |f^*|(u)^{q_1(u)} \, du \leq 2\int_0^l |f^*|(u)^{q_1(u)} \, du$$

for every $n \in \mathbb{N}$. Then using the inequality (2.6), we obtain

$$n^{q_1-} \mu(E_n) \leq 2C \int_0^l t^{\frac{q_1(0)}{p_1(0)} - 1} |f^*|(u)^{q_1(u)} \, du < \infty \tag{2.7}$$

for every $n \in \mathbb{N}$. By the hypothesis, either $\mu(E_n) = 0$ or $\mu(E_n) \geq m$. Since the sequence $(E_n)$ is nonincreasing and $\overset{\infty}{\underset{n=1}{\cap}} E_n = \varnothing$, we have $\mu(E_n) \to 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $|f(x)| \leq n_0$, $\mu - a.e.$ for all $x \in X$, and so we write $|f^*(t)| \leq n_0$, $\mu - a.e.$ for all $t \in [0, l]$. Therefore, we have

$$
\int_0^l |f^*|(t)^{q_2(t)} \, dt = \int_0^l |f^*|(t)^{q_2(t) - q_1(t)} |f^*|(t)^{q_1(t)} \, dt \leq \int_0^l n_0^{q_2(t) - q_1(t)} |f^*|(t)^{q_1(t)} \, dt.
$$

Therefore, we write

$$
\int_0^l |f^*|(t)^{q_2(t)} \, dt < n_0^{q_2^+ - q_1^-} \int_0^l |f^*|(t)^{q_1(t)} \, dt < \infty. \tag{2.8}
$$

Thus we have $f \in L^{q_2(\cdot), q_2(\cdot)}(X, \mu)$. That means $L^{p_1(\cdot), q_1(\cdot)}(X, \mu) \subset L^{q_2(\cdot), q_2(\cdot)}(X, \mu)$. Similarly, using the inequalities (2.8), we write $L^{q_1(\cdot), q_1(\cdot)}(X, \mu) \subset L^{q_2(\cdot), q_2(\cdot)}(X, \mu)$. Lastly using $q_2(0) \geq p_2(0)$, we obtain

$$
L^{p_1(\cdot), q_1(\cdot)}(X, \mu) \subset L^{q_1(\cdot), q_1(\cdot)}(X, \mu) \subset L^{q_2(\cdot), q_2(\cdot)}(X, \mu) \subset L^{p_2(\cdot), q_2(\cdot)}(X, \mu)
$$

from Theorem 2.4. $\qquad\square$

**Lemma 5** *Hölder inequality for variable exponent Lorentz spaces:*

Let $1 \leq q(.) \leq q^+ < \infty$, $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$, and $\frac{1}{q(t)} + \frac{1}{q'(t)} = 1$. If $f \in L^{p(\cdot), q(\cdot)}(X, \mu)$ and $g \in L^{p'(\cdot), q'(\cdot)}(X, \mu)$ then $fg \in L^1(X, \mu)$ and there exists $C > 0$ such that

$$
\int_X |f(x) g(x)| \, d\mu(x) \leq C \, \|f\|_{L^{p(\cdot), q(\cdot)}(X, \mu)} \, \|g\|_{L^{p'(\cdot), q'(\cdot)}(X, \mu)}.
$$

**Proof** Let $f \in L^{p(\cdot), q(\cdot)}(X, \mu)$ and $g \in L^{p'(\cdot), q'(\cdot)}(X, \mu)$. Then we set $\frac{t}{2} = u$

$$
\int_X |f(x) g(x)| \, d\mu(x) = \int_0^\infty |(fg)^*(t)| \, dt \leq \int_0^\infty f^*\left(\frac{t}{2}\right) g^*\left(\frac{t}{2}\right) dt
$$

$$
= 2 \int_0^\infty f^*(u) g^*(u) \, dt = 2 \int_0^\infty t^{1-1} f^*(u) g^*(u) \, dt = 2 \int_0^\infty t^{\left(\frac{1}{p(t)} + \frac{1}{p'(t)}\right)} t^{-\left(\frac{1}{q(t)} + \frac{1}{q'(t)}\right)} f^*(u) g^*(u) \, dt.
$$

$$
= 2 \int_0^\infty t^{\left(\frac{1}{p(t)} - \frac{1}{q(t)}\right)} f^*(u) \, t^{\left(\frac{1}{p'(t)} - \frac{1}{q'(t)}\right)} g^*(u) \, dt
$$

Furthermore, by using the Hölder inequality for variable exponent Lebesgue spaces in $[7, 12]$, the inequalities $f^* \leq f^{**}$ and $g^* \leq g^{**}$, there exists $C_1 > 0$ such that

$$
\int_X |f(x) g(x)| \, d\mu(x) \leq 2 \int_0^\infty t^{\left(\frac{1}{p(t)} - \frac{1}{q(t)}\right)} f^*(u) \, t^{\left(\frac{1}{p'(t)} - \frac{1}{q'(t)}\right)} g^*(u) \, dt
$$

$$\leq 2C_1 \left\| t^{\left(\frac{1}{p(t)} - \frac{1}{q(t)}\right)} f^* \right\|_{L^{q(\cdot)}([0,\infty))} \left\| t^{\left(\frac{1}{p'(t)} - \frac{1}{q'(t)}\right)} g^* \right\|_{L^{q(\cdot)}([0,\infty))} = C \left\| f \right\|_{L^{p(\cdot),q(\cdot)}(X,\mu)} \left\| h \right\|_{L^{p'(\cdot),q'(\cdot)}(X,\mu)}$$

where $C = 2C_1$. $\qquad\square$

**Theorem 7** *a) Let $1 \leq q(.) \leq q^+ < \infty$. If $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$ and $\frac{1}{q(t)} + \frac{1}{q'(t)} = 1$ then the inclusion*

$$L^{p(\cdot),q(\cdot)}(X,\mu).L^{p'(\cdot),q'(\cdot)}(X,\mu) \subset L^1(X,\mu)$$

*holds.*

*b) Let $p \in IP_0([0,l])$, $q \in IP_1([0,l])$, $q(t) = \{ \begin{smallmatrix} c, & x \in \Omega \\ d, & x \notin \Omega \end{smallmatrix}$, and $c, d \geq 1$ such that $\Omega \subset [0,\infty)$. If $q(0) > p(0)$ and $q(\infty) \geq p(\infty)$ then*

$$L^1(X,\mu) \subset L^{p(\cdot),q(\cdot)}(X,\mu).L^{p'(\cdot),q'(\cdot)}(X,\mu)$$

*holds such that $X = [0,\infty]$ and $\mu(x) = dx$.*

**Proof** a) Let $f \in L^{p(\cdot),q(\cdot)}(X,\mu)$ and $h \in L^{p'(\cdot),q'(\cdot)}(X,\mu)$. From Lemma 5, there exists $C > 0$ such that

$$\int_X |f(x)h(x)| \, d\mu(x) \leq C \left\| f \right\|_{L^{p(\cdot),q(\cdot)}(X,\mu)} \left\| h \right\|_{L^{p'(\cdot),q'(\cdot)}(X,\mu)}.$$

Thus we obtain $f.h \in L^1(X,\mu)$. That means $L^{p(\cdot),q(\cdot)}(X,\mu).L^{p'(\cdot),q'(\cdot)}(X,\mu) \subset L^1(X,\mu)$.

b) Take any $g \in L^1(X,\mu)$. Define that

$$A_1 = \{x : 0 < |g(x)| < \infty\},$$

$$A_2 = \{x : |g(x)| = 0\},$$

and

$$(A_1 \cup A_2)^c = \{x : |g(x)| = \infty\}$$

such that $A_1 \cup A_2 \cup (A_1 \cup A_2)^c = [0,\infty]$. Now define that the functions

$$f(x) = \left\{ \begin{array}{ll} |g(x)|, & x \in A_1 \\ 0 & x \in A_2 \\ \infty, & x \in (A_1 \cup A_2)^c \end{array} \right.$$

and

$$h(x) = \left\{ \begin{array}{ll} \frac{|g(x)|}{|g(x)|^{q(x)}}, & x \in A_1 \\ 0 & x \in A_2 \\ \infty, & x \in (A_1 \cup A_2)^c \end{array} \right..$$

We also have $|g| = |fh|$. Since $g \in L^1(X,\mu)$, we know that $|g(x)| < \infty$ (a.e.). Thus we have $\mu((A_1 \cup A_2)^c) = 0$. On the other hand, we have

$$J_{p,q}(f) \cong \int_0^1 t^{\frac{q(0)}{p(0)} - 1} (f^*(t))^{q(t)} \, dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)} - 1} (f^*(t))^{q(t)} \, dt$$

$$\leq \int_0^1 (f^*(t))^{q(t)} \, dt + \int_1^\infty (f^*(t))^{q(t)} \, dt$$

$$= \int_\Omega \left( g^*(t)^{\frac{1}{c}} \right)^c dt + \int_{\mathbb{R}^+ \setminus \Omega} \left( g^*(t)^{\frac{1}{d}} \right)^d dt$$

$$= \int_X (g(x))^{q(x)} \, dx < \infty$$

and similarly $J_{p',q'}(h) < \infty$. Thus we obtain $f \in L^{p(.),q(.)}(X,\mu)$ and $h \in L^{p'(.),q'(.)}(X,\mu)$. Therefore, from the inequality $|g| = |fh|$, we find that $g \in L^{p(.),q(.)}(X,\mu).L^{p'(.),q'(.)}(X,\mu)$. Thus the proof is completed. □

**Theorem 8** *Let $p_i \in IP_0([0,l])$, $q_i \in IP_1([0,l])$, $(i = 1,2)$, $l = \mu(X) < \infty$. If $L^{p_1(.),q_1(.)}(X,\mu) \subset L^{p_2(.),q_2(.)}(X,\mu)$ such that $q_{2-} < q_1^+ \frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}$ then any collection of disjoint measurable sets of positive measure is a finite element.*

**Proof** Assume that $(E_n)$ is a sequence of disjoint measurable sets such that $\mu(E_n) \neq 0$ for infinite one $n$. Thus, since $\overset{\infty}{\underset{n=1}{\cup}} E_n \subset X$, we have $\mu\left( \overset{\infty}{\underset{n=1}{\cup}} E_n \right) < \infty$. Therefore, we write $\sum_{n=1}^\infty \mu(E_n) < \infty$. That means $\lim_{n\to\infty} \mu(E_n) = 0$. Then there exists subsequence $(E_{n_k})_{n_k}$ such that $E = \overset{\infty}{\underset{k=1}{\cup}} E_{n_k}$, $E_{n_k} \cap E_{n_j}$ $(k \neq j)$, $\mu(E) < \infty$, and $\mu(E_{n_k}) = 2^{-kq_1^+ \frac{p_1(0)}{q_1(0)}} \mu(E)$ for every $k \in \mathbb{N}$. Define that

$$f(x) = \sum_{k=1}^\infty \left( 2^k k^{-2} \right) \chi_{E_{n_k}}(x).$$

Then since $2q_1^+ > 1$

$$J_{p_1,q_1}(f) \cong \int_0^{\mu(E)} t^{\frac{q_1(0)}{p_1(0)}-1} (f^*(t))^{q_1(t)} \, dt = \sum_{i=1}^\infty \int_0^{\mu\left(E_{n_i}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} (f^*(t))^{q_1(t)} \, dt$$

$$= \sum_{i=1}^\infty \int_0^{\mu\left(E_{n_i}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} \left( \left( \sum_{k=1}^\infty \left( 2^k k^{-2} \right) \chi_{E_{nk}} \right)^* (t) \right)^{q_1(t)} \, dt$$

$$\leq \int_0^{\mu\left(E_{n_1}\right)} t^{\frac{q_1(0)}{p_1(0)}-1} 2^{q_1^+} \, dt + \int_0^{\mu\left(E_{n_2}\right)} \left( 2^2 2^{-2} \right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} \, dt + \int_0^{\mu\left(E_{n_3}\right)} \left( 2^3 3^{-2} \right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} \, dt + \ldots$$

$$+ \int_0^{\mu(E_{n_m})} \left( 2^m m^{-2} \right)^{q_1^+} t^{\frac{q_1(0)}{p_1(0)}-1} \, dt + \ldots$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \left(2^2 2^{-2}\right)^{q_1^+} \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(2^3 3^{-2}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_1^+} \mu(E_{n_k})^{\frac{q_1(0)}{p_1(0)}}$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(\frac{8}{9}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_1^+} 2^{-kq_1^+} \mu(E)^{\frac{q_1(0)}{p_1(0)}}$$

$$= \frac{p_1(0)}{q_1(0)} \left\{ 2^{q_1^+} \mu(E_{n_1})^{\frac{q_1(0)}{p_1(0)}} + \mu(E_{n_2})^{\frac{q_1(0)}{p_1(0)}} + \left(\frac{8}{9}\right)^{q_1^+} \mu(E_{n_3})^{\frac{q_1(0)}{p_1(0)}} \right\}$$

$$+ \frac{p_1(0)}{q_1(0)} \mu(E)^{\frac{q_1(0)}{p_1(0)}} \sum_{k=4}^{\infty} k^{-2q_1^+} < \infty$$

holds. Thus, we have $f \in L^{p_1(.),q_1(.)}(X,\mu)$. On the other hand, we find that

$$J_{p_2,q_2}(f) \cong \int_0^{\mu(E)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(f^*(t)\right)^{q_2(t)} dt = \sum_{k=1}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(f^*(t)\right)^{q_2(t)} dt$$

$$\geq \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(\sum_{k=1}^{\infty} \left(2^k k^{-2}\right) \chi_{E_{nk}}\right)^* (t)^{q_2(t)} dt$$

$$= \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(2^k k^{-2}\right)^{q_2(t)} dt, \quad \left(2^k k^{-2} > 1 \text{ for } k \geq 4\right)$$

$$\geq \sum_{k=4}^{\infty} \int_0^{\mu\left(E_{n_k}\right)} t^{\frac{q_2(0)}{p_2(0)}-1} \left(2^k k^{-2}\right)^{q_2-} dt = \frac{p_2(0)}{q_2(0)} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_2-} \mu(E_{n_k})^{\frac{q_2(0)}{p_2(0)}}$$

$$= \frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \sum_{k=4}^{\infty} \left(2^k k^{-2}\right)^{q_2-} 2^{-kq_1^+\left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)}$$

$$= \frac{p_2(0)}{q_2(0)} \mu(E)^{\frac{q_2(0)}{p_2(0)}} \sum_{k=4}^{\infty} k^{-2q_2-} 2^{k\left((q_2-)-q_1^+\left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)\right)}.$$

If we say that $b_k = k^{-2q_2-} 2^{k\left((q_2-)-q_1^+\left(\frac{p_1(0)q_2(0)}{q_1(0)p_2(0)}\right)\right)}$, then we have $\lim_{k \to \infty} b_k = 0$. Thus, we find that $f \notin L^{p_2(.),q_2(.)}(X,\mu)$. However, from the assumption, we must obtain $f \in L^{p_2(.),q_2(.)}(X,\mu)$. Therefore, we find that any collection of disjoint measurable sets of positive measure is a finite element. $\square$

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