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On the evolute offsets of ruled surfaces in Minkowski 3-space

Dae Won YOON*
Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Republic of Korea

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Abstract: In this paper, we classify evolute offsets of a ruled surface in Minkowski 3-space $\mathbb{L}^3$ with constant Gaussian curvature and mean curvature. As a result, we investigate linear Weingarten evolute offsets of a ruled surface in $\mathbb{L}^3$.

Key words: Linear Weingarten surface, involute-evolute offset, ruled surface

1. Introduction

The geometry of curves and surfaces in Euclidean 3-space $\mathbb{E}^3$ represented for many years a popular topic in the field of classical differential geometry. Increasing interest in the theory of curves has led to the development of special curves to be examined. A way for the characterizations and classifications for curves is the relationship between the Frenet curvatures of the curves. Some of the curves are offsets of curves, in particular, involute-evolute offsets, Bertrand offsets, Mannheim offsets etc. [1, 4, 8–10, 17, 18]. As the study of offsets of surfaces, many authors studied them for various aspects. Farouki [5] developed methods for the generation of parallel offsets for a certain class of surfaces. Ravani and Ku [15] generalized the theory of Bertrand offsets of curves for ruled and developable surfaces using lines instead of points as the geometric building blocks of space. In [6] Kasap and Kuruoğlu initiated the study of Bertrand offsets of ruled surfaces in Minkowski 3-space. Önder [11] studied dual geodesic trihedra of Bertrand offsets of timelike surfaces in dual Lorentzian space and found some relations between certain invariants of the offsets. As a result, he gave some characterizations of Bertrand offsets of timelike ruled surfaces in view of the dual geodesic trihedron. Another type of offsets of surfaces is Mannheim offsets. In [14] the authors investigated the properties of Mannheim offsets of developable ruled surfaces in terms of the geodesic curvature and arc-length of spherical indicatrix of the director spherical curve of the surfaces. Moreover, Önder and Uğurlu [12] obtained the relationships between invariants of Mannheim offsets of timelike surfaces, and they gave the conditions for these surface offsets to be developable. Recently, in [7] Kasap et al. studied involute-evolute offsets of ruled surfaces in Euclidean 3-space $\mathbb{E}^3$.

In this paper, we study offsets of ruled surfaces in Minkowski 3-space $\mathbb{L}^3$. We also study an evolute offset with constant Gaussian curvature and constant mean curvature and give examples. As the results, we classify a linear Weingarten evolute offset of ruled surfaces. A linear Weingarten surface is the surface having a linear equation between the Gaussian curvature and the mean curvature of a surface.

*Correspondence: dwyoon@gnu.ac.kr
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2. Preliminaries

The Minkowski 3-space $\mathbb{L}^3$ is a real space $\mathbb{R}^3$ provided with the standard flat metric given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{L}^3$ is said to be *space-like* if $\langle \mathbf{x}, \mathbf{x}\rangle > 0$ or $\mathbf{x} = 0$, *time-like* if $\langle \mathbf{x}, \mathbf{x}\rangle < 0$, and *null* if $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ and $\mathbf{x} \neq 0$. A time-like or null vector in $\mathbb{L}^3$ is said to be *causal*. Similarly, an arbitrary curve $\gamma = \gamma(s)$ is *space-like, time-like*, or *null* if all of its tangent vectors $\gamma'(s)$ are space-like, time-like, or null, respectively. Here "prime" denotes the derivative with respect to the parameter s.

We now define some typical surfaces in $\mathbb{L}^3$ as follows:

$$\mathbb{S}_1^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 | -x_1^2 + x_2^2 + x_3^2 = 1\},$$
$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 | -x_1^2 + x_2^2 + x_3^2 = -1\}.$$

We call $\mathbb{S}_1^2$ and $\mathbb{H}^2$ de Sitter 2-space and hyperbolic space, respectively.

Let $\gamma : I \longrightarrow \mathbb{L}^3$ be a space-like or time-like curve in Minkowski 3-space $\mathbb{L}^3$ parameterized by its arc-length s. Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame field along $\gamma(s)$.

If $\gamma(s)$ is a space-like curve in $\mathbb{L}^3$, the Frenet formulae of $\gamma(s)$ are given by ([16])

$$\begin{aligned}
\gamma'(s) &= \mathbf{t}(s),\\
\mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s),\\
\mathbf{n}'(s) &= -\epsilon\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s),\\
\mathbf{b}'(s) &= \epsilon\tau(s)\mathbf{n}(s),
\end{aligned} \tag{2.1}$$

where $\langle \mathbf{t}, \mathbf{t}\rangle = 1, \langle \mathbf{n}, \mathbf{n}\rangle = \epsilon(=\pm 1), \langle \mathbf{b}, \mathbf{b}\rangle = -\epsilon$. Here the functions $\kappa(s)$ and $\tau(s)$ are the curvature function and torsion function of $\gamma(s)$.

If $\gamma(s)$ is a time-like curve in $\mathbb{L}^3$, the Frenet formulae of $\gamma(s)$ are given by ([16])

$$\begin{aligned}
\gamma'(s) &= \mathbf{t}(s),\\
\mathbf{t}'(s) &= \kappa(s)\mathbf{n}(s),\\
\mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s),\\
\mathbf{b}'(s) &= -\tau(s)\mathbf{n}(s),
\end{aligned} \tag{2.2}$$

where $\langle \mathbf{t}, \mathbf{t}\rangle = -1, \langle \mathbf{n}, \mathbf{n}\rangle = \langle \mathbf{b}, \mathbf{b}\rangle = 1$. Here $\kappa(s)$ and $\tau(s)$ are the curvature function and torsion function of a time-like curve $\gamma(s)$.

If $\gamma(s)$ is a space-like or time-like pseudospherical curve parametrized by arc-length s in $\mathbb{S}_1^2$ or $\mathbb{H}^2$, let $\mathbf{t}(s) = \gamma'(s)$ and $\mathbf{g}(s) = \gamma(s) \times \gamma'(s)$. Then we have a pseudoorthonormal frame $\{\gamma(s), \mathbf{t}(s), \mathbf{g}(s)\}$ along $\gamma(s)$. It is called the *pseudospherical Frenet frame* of the pseudospherical curve $\gamma(s)$. If γ is a space-like curve, then the vector $\mathbf{g}$ is time-like when γ is on $\mathbb{S}_1^2$, and the vector $\mathbf{g}$ is space-like when γ is on $\mathbb{H}^2$. Similarly, if the curve γ is time-like, then the vector $\mathbf{g}$ is space-like. The following theorem can be easily obtained.

Theorem 2.1 ([2, 3]). *Under the above notations, we have the following pseudospherical Frenet formulae of* γ*:*

(1) If γ *is a pseudospherical space-like curve,*

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= \epsilon\gamma(s) + \epsilon\kappa_g(s)\mathbf{g}(s), \\ \mathbf{g}'(s) &= -\kappa_g(s)\mathbf{t}(s). \end{aligned} \tag{2.3}$$

Here γ *is on* $\mathbb{H}^2$ *when* $\epsilon = 1$*, and* γ *is on* $\mathbb{S}^2_1$ *when* $\epsilon = -1$*.*

(2) If γ *is a pseudospherical time-like curve,*

$$\begin{aligned} \gamma'(s) &= \mathbf{t}(s), \\ \mathbf{t}'(s) &= \gamma(s) + \kappa_g(s)\mathbf{g}(s), \\ \mathbf{g}'(s) &= \kappa_g(s)\mathbf{t}(s). \end{aligned} \tag{2.4}$$

The function $\kappa_g(s)$ is called the *geodesic curvature* of the pseudospherical curve γ.

3. Evolute offset of ruled surfaces

In this section, we first define a ruled surface in Minkowski 3-space $\mathbb{L}^3$. Let I_1 and I_2 be some open intervals in the real line $\mathbb{R}$. Let $\mathbf{c} = \mathbf{c}(u)$ be a curve in $\mathbb{L}^3$ defined on I_1 and $\mathbf{e} = \mathbf{e}(u)$ a transversal vector field along $\mathbf{c}$. Then a parametrization of a ruled surface is given by

$$\varphi(u, v) = \mathbf{c}(u) + v\mathbf{e}(u), \quad u \in I_1, \quad v \in I_2. \tag{3.1}$$

For such a ruled surface, $\mathbf{c}$ and $\mathbf{e}$ are called the *base curve* and the *director curve*, respectively.

Suppose that a director curve $\mathbf{e}$ is a pseudospherical curve such that

$$\langle \mathbf{e}(u), \mathbf{e}(u)\rangle = \epsilon_1 = \pm 1, \quad \langle \mathbf{e}'(u), \mathbf{e}'(u)\rangle = \epsilon_2 = \pm 1, \quad \langle \mathbf{c}'(u), \mathbf{e}'(u)\rangle = 0. \tag{3.2}$$

In this case, the parameter u is arc-length of the pseudospherical curve $\mathbf{e}$. A curve $\mathbf{e}$ can be regarded as a vector and it is called the *pseudospherical indicatrix vector* of $\varphi(u, v)$. $\mathbf{c}$ is said to be the *striction curve* of $\varphi(u, v)$.

From now on, we shall often not write the parameter u explicitly in our formulae. We put $\mathbf{t} = \mathbf{e}'$ and $\mathbf{g} = \mathbf{e} \times \mathbf{e}'$. Then the set $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ is the pseudospherical Frenet frame of $\mathbf{e}$ and the vectors $\mathbf{t}$ and $\mathbf{g}$ are said to be the *pseudocentral normal* and the *pseudoasymptotic normal* of $\varphi(u, v)$, respectively ([13]). For the pseudospherical Frenet frame $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$, the following equations hold:

$$\begin{aligned} \mathbf{e}' &= \mathbf{t}, \\ \mathbf{t}' &= \epsilon_1\epsilon_2(-\mathbf{e} + J\mathbf{g}), \\ \mathbf{g}' &= \epsilon_2 J\mathbf{t}, \end{aligned} \tag{3.3}$$

where $J = \langle \mathbf{e}'', \mathbf{e}' \times \mathbf{e}\rangle$ denotes the geodesic curvature κ_g of a pseudospherical curve $\mathbf{e}$.

On the other hand, the derivative of the striction curve $\mathbf{c}$ is given by

$$\mathbf{c}' = \epsilon_1 F\mathbf{e} - \epsilon_1\epsilon_2 Q\mathbf{g}, \tag{3.4}$$

where $F = \langle \mathbf{c}', \mathbf{e}\rangle$ and $Q = \langle \mathbf{c}', \mathbf{e}\times\mathbf{e}'\rangle$. The function Q is called the *parameter of distribution* of $\varphi(u,v)$. The functions $J, F,$ and Q of $\varphi(u,v)$ are called *structure functions* of a ruled surface $\varphi(u,v)$ in Minkowski 3-space $\mathbb{L}^3$.

On the other hand, the parameter u is arc-length parameter of the curve $\mathbf{e}$, but usually it is not arc-length parameter of the curve $\mathbf{c}$. By (3.4), we have

Proposition 3.1 *Let $\varphi(u,v)$ be a ruled surface satisfying (3.2) in $\mathbb{L}^3$. If the parameter u is also arc-length parameter of the striction curve $\mathbf{c}$ of $\varphi(u,v)$, the structure functions F and Q of $\varphi(u,v)$ satisfy $|F^2 - \epsilon_2 Q^2| = 1$.*

Now we compute the Gaussian curvature and the mean curvature of a ruled surface $\varphi(u,v)$ in $\mathbb{L}^3$. From (3.3) and (3.4) the coefficients of the first fundamental form of $\varphi(u,v)$ are given by

$$E = \epsilon_1 F^2 - \epsilon_1\epsilon_2 Q^2 + \epsilon_2 v^2,\ F = \langle \mathbf{c}', \mathbf{e}\rangle,\ G = \epsilon_1.$$

The unit normal vector $\mathbf{u}$ of $\varphi(u,v)$ is written as

$$\mathbf{u} = \frac{1}{D}(\epsilon_2 Q\mathbf{t} - v\mathbf{g}),$$

where $D = \sqrt{|EG - F^2|} = \sqrt{|Q^2 - \epsilon_1 v^2|}$. This leads to the coefficients $L, M,$ and N of the second fundamental form as

$$L = \frac{1}{D}(\epsilon_1 Q(F - QJ) - Q'v + Jv^2),\ M = \frac{Q}{D},\ N = 0.$$

Thus, using the data described above, the Gaussian curvature K and the mean curvature H of $\varphi(u,v)$ are given respectively by

$$\begin{aligned} K &= \frac{Q^2}{D^4}, \\ H &= \frac{1}{2D^3}(\epsilon_1 Jv^2 - \epsilon_1 Q'v - Q(QJ + F)). \end{aligned} \tag{3.5}$$

Definition 3.2 *Let $\varphi(u,v)$ and $\varphi^*(u,v)$ be two ruled surfaces in $\mathbb{L}^3$. A surface $\varphi(u,v)$ is said to be an involute offset of $\varphi^*(u,v)$ if there exists a one-to-one correspondence between their rulings such that the pseudocentral normal of $\varphi(u,v)$ and the pseudospherical indicatrix vector of $\varphi^*(u,v)$ are linearly dependent at the striction points of their corresponding rulings. In this case, $\varphi^*(u,v)$ is said to be an evolute offset of $\varphi(u,v)$.*

Let $\varphi^*(u,v)$ be an evolute offset of a ruled surface $\varphi(u,v)$ satisfying (3.2) in Minkowski 3-space $\mathbb{L}^3$. Then the surface $\varphi^*(u,v)$ can be written as

$$\varphi^*(u,v) = \mathbf{c}^*(u) + v\mathbf{e}^*(u) = \mathbf{c}(u) + (R(u) + v)\mathbf{t}(u), \tag{3.6}$$

where R is the distance between the corresponding striction points of $\varphi(u,v)$ and $\varphi^*(u,v)$. By using (3.3) and (3.4) the coefficients of the first fundamental form of $\varphi^*(u,v)$ are

$$\begin{aligned} E^* &= \epsilon_1 F^2 - \epsilon_1\epsilon_2 Q^2 + 2\epsilon_1\epsilon_2(JQ-F)(R+v) + \epsilon_1(1-\epsilon_2 J^2)(R+v)^2 + \epsilon_2 {R'}^2, \\ F^* &= \epsilon_2 R', \\ G^* &= \epsilon_2. \end{aligned}$$

Moreover, the unit normal vector $\mathbf{u}^*$ of $\varphi^*(u,v)$ is given by

$$\mathbf{u}^* = \frac{1}{D^*}[(\epsilon_2 Q - \epsilon_2 J(R+v))\mathbf{e} + (-\epsilon_2 F + (R+v))\mathbf{g}],$$

where $D^* = \sqrt{|E^*G^* - {F^*}^2|} = \sqrt{|(Q-J(R+v))^2 - \epsilon_2(-\epsilon_2 F + R + v)^2|}$. From this, we get the coefficients of the second fundamental form as follows:

$$\begin{aligned} L^* &= \frac{1}{D^*}[\epsilon_2(F' - 2\epsilon_2 R')(Q - J(R+v)) - (-\epsilon_2 F + R + v)(2R'J - Q' + (R+v)J')], \\ M^* &= \frac{1}{D^*}(\epsilon_2 FJ - Q), \\ N^* &= 0. \end{aligned}$$

By a direct computation, we can show that the Gaussian curvature K^* and the mean curvature H^* of $\varphi^*(u,v)$ are given by

$$K^* = -\frac{1}{{D^*}^4}(\epsilon_2 FJ - Q)^2 \tag{3.7}$$

and

$$H^* = \frac{1}{2{D^*}^3}H_1^*, \tag{3.8}$$

where $H_1^* = -\epsilon_2 J'v^2 + (FJ' - F'J - 2\epsilon_2 RJ' + \epsilon_2 Q')v + (F'Q - FQ' - F'JR + FJ'R + \epsilon_2 RQ' - \epsilon_2 R^2 J')$.

From (3.7), we have

Theorem 3.3 *Let $\varphi^*(u,v)$ be an evolute offset of a ruled surface $\varphi(u,v)$ satisfying (3.2). Then $\varphi^*(u,v)$ is flat if and only if the structure functions $Q, J,$ and F of $\varphi(u,v)$ satisfy $Q = \epsilon_2 JF$.*

Remark 3.4 *Let $\varphi(u,v) = \mathbf{e}'(u) + v\mathbf{e}(u)$ be a ruled surface with $\mathbf{c}(u) = \mathbf{e}'(u), \langle \mathbf{e}(u), \mathbf{e}(u)\rangle = 1$ and $\langle \mathbf{e}'(u), \mathbf{e}'(u)\rangle = -1$. Then we have $F = 1$ and $Q = -J$. In this case, the surface $\varphi(u,v)$ has a nonzero Gaussian curvature, but the evolute offset $\varphi^*(u,v)$ of $\varphi(u,v)$ has a zero Gaussian curvature, that is, it is a flat surface.*

Example 3.5 *We consider $\mathbf{e}(u) = (x(u), y(u), z(u))$ with $\langle \mathbf{e}(u), \mathbf{e}(u)\rangle = 1$ and $\langle \mathbf{e}'(u), \mathbf{e}'(u)\rangle = -1$. Then the following relations hold:*

$$-x^2 + y^2 + z^2 = 1, \tag{3.9}$$

$$-{x'}^2 + {y'}^2 + {z'}^2 = -1. \tag{3.10}$$

We now try to solve the above equations. From (3.9), we may put $x = x(u)$ *and* $y = y(u)$ *by*

$$\begin{aligned} x(u) &= \sqrt{1-z^2}\sinh\theta(u), \quad 1-z^2>0 \\ y(u) &= \sqrt{1-z^2}\cosh\theta(u), \end{aligned} \tag{3.11}$$

and then determine the function $\theta = \theta(u)$ *satisfying (3.10). By using (3.10) and (3.11) we have*

$$\theta'^2 = \frac{z'^2 + z^2 - 1}{(1-z^2)^2}.$$

We assume that $z'^2 + z^2 - 1 > 0$ *(when* $z'^2 + z^2 - 1 = 0$, θ *is constant). Then the function* $\theta(u)$ *is of the form*

$$\theta(u) = \pm\int_0^u \frac{\sqrt{z'(t)^2 + z(t)^2 - 1}}{1 - z(t)^2}dt \tag{3.12}$$

and without loss of generality we may assume that the signature is positive. Since $z'^2 + z^2 > 1$, *we take* $z(u) = \sqrt{2}\cos u$. *Then we have*

$$\theta(u) = -\tanh^{-1}(\tan u).$$

From this, the spherical curve $\mathbf{e}(u)$ *can be expressed as*

$$\mathbf{e}(u) = \left(-\sqrt{1-2\cos^2 u}\sinh(\tanh^{-1}(\tan u)), \sqrt{1-2\cos^2 u}\cosh(\tanh^{-1}(\tan u)), \sqrt{2}\cos u\right). \tag{3.13}$$

Thus, the ruled surface $\varphi(u,v) = \mathbf{e}'(u) + v\mathbf{e}(u)$ *has a nonzero Gaussian curvature, but its evolute offset* $\varphi^*(u,v)$ *has a zero Gaussian curvature.*

Theorem 3.6 *Let* $\varphi^*(u,v)$ *be an evolute offset of a ruled surface* $\varphi(u,v)$ *satisfying (3.2) in* $\mathbb{L}^3$. *Then an evolute offset* $\varphi^*(u,v)$ *has a zero mean curvature if and only if the structure functions satisfy* $Q = \epsilon_2 JF$ *and* $J = constant$.

Proof If $\varphi^*(u,v)$ has a zero mean curvature, then from (3.8) we have

$$\begin{aligned} FQ' &= F'Q, \\ Q' &= \epsilon_2 F'J, \\ J' &= 0, \end{aligned} \tag{3.14}$$

which imply we can show that J is constant and $Q = \epsilon_2 JF$. The converse assertion is trivial. Hence the theorem is proved. □

Now we will construct an evolute offset with zero mean curvature. From Theorem 3.6 and (3.3) we have the following ordinary differential equation

$$\mathbf{e}''' = \epsilon_1(J^2 - \epsilon_2)\mathbf{e}'. \tag{3.15}$$

Case 1. $\epsilon_1(J^2 - \epsilon_2) = k^2$ for some real number k.

Let $\epsilon_2 = 1$. Without loss of generality, we may assume $\mathbf{e}'(0) = (0,1,0)$. Thus, $\mathbf{e}'''(u) = k^2\mathbf{e}'(u)$ implies

$$\mathbf{e}'(u) = (B_1 \sinh ku, \cosh ku + B_2 \sinh ku, B_3 \sinh ku)$$

for some constants B_1, B_2, and B_3. Since $\epsilon_2 = 1$, we have $B_1^2 - B_3^2 = 1$ and $B_2 = 0$. From this, we can obtain

$$\mathbf{e}(u) = \left(\frac{B_1}{k}\cosh ku + D_1, \frac{1}{k}\sinh ku, \frac{B_3}{k}\cosh ku + D_3\right) \tag{3.16}$$

for some constants D_1, D_3 satisfying $D_3^2 - D_1^2 = \frac{1}{k^2} + \epsilon_1$, $B_1 D_1 = B_3 D_3$ and $B_1^2 - B_3^2 = 1$. We now change the coordinates by $\bar{x}, \bar{y}, \bar{z}$ such that $\bar{x} = B_1 x - B_3 z$, $\bar{y} = y$, $\bar{z} = -B_3 x + B_1 z$, that is,

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} B_1 & 0 & -B_3 \\ 0 & 1 & 0 \\ -B_3 & 0 & B_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With respect to the coordinates $(\bar{x}, \bar{y}, \bar{z})$, $\mathbf{e}(u)$ turns into

$$\mathbf{e}(u) = \left(\frac{1}{k}\cosh ku, \frac{1}{k}\sinh ku, D\right) \tag{3.17}$$

for a constant $D = B_1 D_3 - B_3 D_1$ with $D^2 = \frac{1}{k^2} + \epsilon_1$. By (3.4) and (3.17), the striction curve $\mathbf{c}$ can be expressed as

$$\begin{aligned} \mathbf{c}(u) = \Big(&\epsilon_1(\frac{1}{k} - JD)\int F(u)\cosh ku du, \epsilon_1(\frac{1}{k} - JD)\int F(u)\sinh ku du, \\ &\epsilon_1(D - \frac{J}{k})\int F(u) du\Big) + \mathbf{D}_0 \end{aligned} \tag{3.18}$$

for some constant vector $\mathbf{D}_0$. Thus, up to a rigid motion the evolute offset $\varphi^*(u,v)$ of the ruled surface $\varphi(u,v)$ given by (3.17) and (3.18) has the parametrization of the form

$$\begin{aligned} \varphi^*(u,v) = \Big(&\epsilon_1(\frac{1}{k} - JD)\int F(u)\cosh ku du + (R(u) + v)\sinh ku, \\ &\epsilon_1(\frac{1}{k} - JD)\int F(u)\sinh ku du + (R(u)+v)\cosh ku, \epsilon_1(D - \frac{J}{k})\int F(u) du\Big). \end{aligned} \tag{3.19}$$

Next let $(\epsilon_1, \epsilon_2) = (1, -1)$. We now consider an initial condition $\mathbf{e}'(0) = (1,0,0)$ of the ODE (3.15). Quite similarly as we did, we obtain

$$\mathbf{e}(u) = \left(\frac{1}{k}\sinh ku, \frac{B_2}{k}\cosh ku + D_2, \frac{B_3}{k}\cosh ku + D_3\right)$$

satisfying $B_2^2 + B_3^2 = 1, B_2 D_2 + B_3 D_3 = 0$ and $D_2^2 + D_3^2 = 1 - \frac{1}{k^2}$.
If we adopt the coordinates transformation such that

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & B_2 & B_3 \\ 0 & -B_3 & B_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

with respect to the new coordinates $(\bar{x}, \bar{y}, \bar{z})$, the vector $\mathbf{e}(u)$ becomes

$$\mathbf{e}(u) = \left(\frac{1}{k}\sinh ku, \frac{1}{k}\cosh ku, D\right) \tag{3.20}$$

and the striction curve is given by

$$\begin{aligned}\mathbf{c}(u) = \Bigg((&\frac{1}{k} - JD)\int F(u)\sinh ku du, (\frac{1}{k} - JD)\int F(u)\cosh ku du,\\ &(D - \frac{J}{k})\int F(u) du\Bigg) + \mathbf{D}_0,\end{aligned} \tag{3.21}$$

where $D = B_2D_3 - B_3D_2$ with $D^2 = 1 - \frac{1}{k^2}$ and $\mathbf{D}_0$ is a constant vector.

Thus, up to a rigid motion, the evolute offset $\varphi^*(u,v)$ of the ruled surface $\varphi(u,v)$ given by (3.20) and (3.21) has the parametrization of the form

$$\begin{aligned}\varphi^*(u,v) = \Bigg(&(\frac{1}{k} - JD)\int F(u)\sinh ku du + (R(u)+v)\cosh ku,\\ &(\frac{1}{k} - JD)\int F(u)\cosh ku du + (R(u)+v)\sinh ku, (D - \frac{J}{k})\int F(u) du\Bigg).\end{aligned} \tag{3.22}$$

Case 2. $\epsilon_1(J^2 - \epsilon_2) = -k^2$ for some real number k.
Let $\varepsilon_2 = 1$. We may give the initial condition by $\mathbf{e}'(0) = (0,1,0)$ for the ordinary differential equation $\mathbf{e}''' + k^2\mathbf{e}' = 0$. Under such initial condition, a vector $\mathbf{e}$ is given by

$$\mathbf{e}(u) = \left(-\frac{B_1}{k}\cos ku + D_1, \frac{1}{k}\sin ku, -\frac{B_3}{k}\cos ku + D_3\right), \tag{3.23}$$

where $B_1, B_3, D_1,$ and D_3 are some constants satisfying $B_3^2 - B_1^2 = 1$, $B_1D_1 = B_3D_3$, and $D_1^2 - D_3^2 = \frac{1}{k^2} - \epsilon_1$.

If we take another coordinate system $(\bar{x}, \bar{y}, \bar{z})$ such that

$$\bar{x} = -B_3x + B_1z, \quad \bar{y} = y, \quad \bar{z} = B_1x - B_3z,$$

then a vector $\mathbf{e}$ takes the form

$$\mathbf{e}(u) = \left(D, \frac{1}{k}\sin ku, \frac{1}{k}\cos ku\right), \tag{3.24}$$

where $D = B_1D_3 - B_3D_1$ satisfying $D^2 = \frac{1}{k^2} - \epsilon_1$. Therefore, the striction curve $\mathbf{c}$ is determined by

$$\begin{aligned}\mathbf{c}(u) = \Bigg(&\epsilon_1(D + \frac{J}{k})\int F(u) du, \epsilon_1(\frac{1}{k} - JD)\int F(u)\sin ku du,\\ &\epsilon_1(\frac{1}{k} - JD)\int F(u)\cos ku du\Bigg) + \mathbf{D}_0,\end{aligned} \tag{3.25}$$

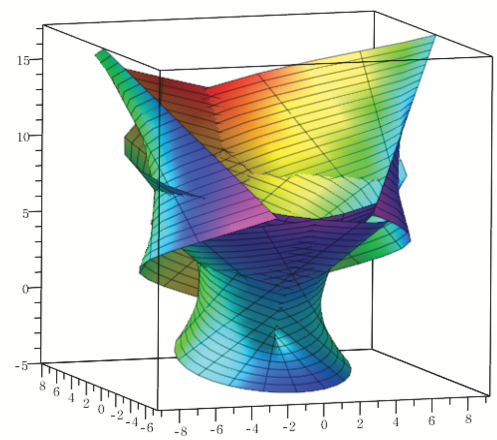

Figure 1.

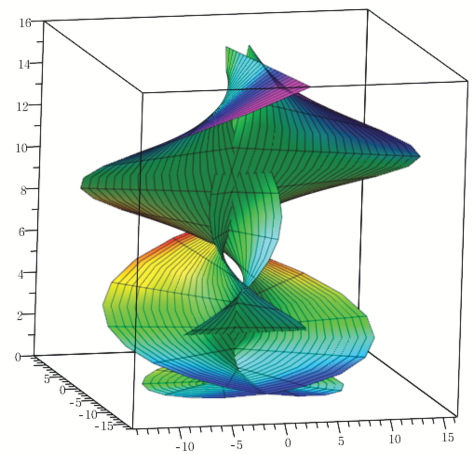

Figure 2.

where $\mathbf{D}_0$ is a constant vector. Thus, up to a rigid motion the parametrization of the evolute offset $\varphi^*(u,v)$ of the ruled surface $\varphi(u,v)$ given by (3.24) and (3.25) can be expressed as

$$\begin{aligned}\varphi^*(u,v) = \Big(&\epsilon_1(D+\frac{J}{k})\int F(u)du, \epsilon_1(\frac{1}{k}-JD)\int F(u)\sin ku du + (R(u)+v)\cos ku,\\ &\epsilon_1(\frac{1}{k}-JD)\int F(u)\cos ku du - (R(u)+v)\sin ku\Big).\end{aligned} \tag{3.26}$$

For specific functions $F(u)=u$ and $R(u)=\cos u$, the ruled surface $\varphi(u,v)$, generated by (3.24) and (3.25), is shown in Figure 1 and its evolute offset $\varphi^*(u,v)$, given by (3.26), is shown in Figure 2.

Case 3. $J^2-\epsilon_2=0$.

In this case $\epsilon_2=1$ and $J=\pm1$, which imply $Q=\pm F$. It contradicts the definition of D^*. Thus, there is no minimal evolute offset $\varphi^*(u,v)$ satisfying $\mathbf{e}'''=0$.

Consequently, we have

Theorem 3.7 *Let $\varphi^*(u,v)$ be an evolute offset of a ruled surface $\varphi(u,v)$ satisfying (3.2) in $\mathbb{L}^3$. Then $\varphi^*(u,v)$ has a zero mean curvature if and only if $\varphi^*(u,v)$ is part of a surface of the form (3.19), (3.22), or (3.26).*

If a ruled surface $\varphi(u,v)$ is minimal, then $J=F=0$ and $Q'=0$. Thus, the following theorem holds:

Theorem 3.8 *An evolute offset of a minimal ruled surface in $\mathbb{L}^3$ is minimal.*

4. Linear Weingarten offsets of ruled surfaces

In this section, we study a linear Weingarten offset of a ruled surface $\varphi(u,v)$ in Minkowski 3-space $\mathbb{L}^3$.

Let $\varphi^*(u,v)$ be an evolute offset of a ruled surface $\varphi(u,v)$ in $\mathbb{L}^3$. If $\varphi^*(u,v)$ satisfies the linear Weingarten surface equation

$$aK^* + bH^* = c, \tag{4.1}$$

where a, b, c are constant with $(a,b,c) \neq (0,0,0)$, then from (3.7) and (3.8) we have

$$b^2 {D^*}^2 {H_1^*}^2 - \Big(2a(\epsilon_2 JF - Q)^2 + 2c{D^*}^4\Big)^2 = 0. \tag{4.2}$$

On the other hand, (4.2) is a polynomial in v with functions of u as coefficients. Thus, all the coefficients must be zero. The coefficient of the highest degree v^8 of the left hand side of (4.2) is

$$-4c^2(J^2 - \epsilon_2)^4.$$

From this, $c = 0$ or $J^2 = 1$ in other words, $\epsilon_2 = 1$.

Case 1. $c = 0$.

In this case, (4.2) can be rewritten as

$$b^2 {D^*}^2 {H_1^*}^2 - 4a^2(\epsilon_2 JF - Q)^4 = 0. \tag{4.3}$$

Moreover, the coefficient of the term v^6 in (4.3) must be zero, that is

$$b^2(J^2 - \epsilon_2){J'}^2 = 0,$$

which yields $bJ' = 0$.

If J is constant, the coefficient of v^4 in (4.3) is $b^2(J^2-\epsilon_2)(\epsilon_2 Q' - F'J)$. Therefore, we get $Q' = \epsilon_2 F'J$, which implies from the coefficients of v^2, v^1 and v^0 we infer that $Q = \epsilon_2 JF$. According to Theorem 3.6, $\varphi^*(u,v)$ is minimal.

If $b = 0$, from (4.3) the structure functions satisfy $Q = \epsilon_2 JF$ because of $a \neq 0$. Thus, the surface $\varphi^*(u,v)$ is flat.

Case 2. $J^2 = 1$.

In this case, we can find the coefficient of the highest degree of the left-hand side of (4.2), and from this we have $Q = \pm F$. It is a contradiction according to Case 3 in Section 3.

Consequently, we have

Theorem 4.1 *Let $\varphi^*(u,v)$ be an evolute offset of a ruled surface $\varphi(u,v)$ in $\mathbb{L}^3$. If $\varphi^*(u,v)$ is a linear Weingarten surface, then $\varphi^*(u,v)$ is either flat or minimal.*

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