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## $h$ -Admissible Fourier integral operators

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**Abstract:** We study in this work a class of  $h$ -admissible Fourier integral operators. These operators are bounded (respectively compact) in  $L^2$  if the weight of the amplitude is bounded (respectively tends to 0).

**Key words:**  $h$ -Admissible Fourier integral operators, symbol and phase, boundedness and compactness

### 1. Introduction

The theory of  $h$ -pseudodifferential operators is well suited for investigating various problems connected with semiclassical elliptic differential equations. However, this theory fails to be adequate for studying semiclassical equations of hyperbolic type, and one is then forced to examine a wider class of operators, the so-called  $h$ -Fourier integral operators.

Since 1970, many efforts have been made by several authors in order to study this type of operator (see, e.g., [1, 4–9, 11]). The first works on Fourier integral operators deal with local properties. We note that Asada and Fujiwara ([1]) have studied for the first time a class of Fourier integral operators defined on  $\mathbb{R}^n$ .

The  $h$ -Fourier integral operators are represented by formulas of the type

$$(I(a, \phi; h)f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x, \theta, y)} a(x, \theta, y) f(y) dy d\theta, \quad (1.1)$$

$f \in S(\mathbb{R}^n)$  (the Schwartz space). The function  $a(x, \theta, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n)$  is called the amplitude, the function  $\phi(x, y, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n; \mathbb{R})$  is called the phase function, and  $h \in ]0, h_0]$  is a semiclassical parameter.

The purpose of this work is to generalize the notion of  $h$ -admissible operators defined in [13] by studying the  $h$ -admissible Fourier integral operators of the form (2.2) (below). On some conditions on  $a$  and  $\phi$ , we show that the  $h$ -admissible Fourier integral operators are well defined and they are continuous on  $S(\mathbb{R}^n)$  and on  $S'(\mathbb{R}^n)$  (the space of tempered distributions). We give also a result where it is shown that these types of operators are stable by composition.

The natural question is how these operators will be bounded on  $L^2$  or will be compact on  $L^2$ . It has been proved in [1] with some hypothesis on the phase function  $\phi$  and the amplitude  $a$  that all operators of the form

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(1.1) are bounded on  $L^2$ . The technique used there is based on the fact that the operators  $I(a, \phi; h) I^*(a, \phi; h)$ ,  $I^*(a, \phi; h) I(a, \phi; h)$  are  $h$ -pseudodifferential and it uses the Calderòn–Vaillancourt theorem [3] (here  $I(a, \phi; h)^*$  is the adjoint of  $I(a, \phi; h)$ ).

In this work, we apply the same technique of [1] to establish the boundedness and the compactness of the operators of the type

$$(F_h \psi)(x) = (2\pi h)^{-n} \int \int e^{\frac{i}{h}(S(x,\theta) - y\theta)} a(x, y) \psi(y) dy d\theta. \tag{1.2}$$

To this end, we give a brief and simple proof of a result of [1] in our framework.

We mainly prove the continuity of the operator  $F_h$  on  $L^2(\mathbb{R}^n)$  when the weight of the amplitude  $a$  is bounded. Moreover,  $F_h$  is compact on  $L^2(\mathbb{R}^n)$  if this weight tends to zero. Using the estimate given in [10, 13, 14] for  $h$ -pseudodifferential ( $h$ -admissible) operators, we also establish an  $L^2$ -estimate of  $\|F_h\|$ .

We note that if the amplitude  $a$  is just bounded, the Fourier integral operator  $F_h$  is not necessarily bounded on  $L^2(\mathbb{R}^n)$ . In Hasanov [6] and [2, 15] a class of unbounded Fourier integral operators with an amplitude in the Hörmander’s class  $S_{1,1}^0$  and in  $\bigcap_{0 < \rho < 1} S_{\rho,1}^0$  was constructed.

**2. A general class of  $h$ -admissible Fourier integral operators**

We are interesting in giving a sense of the integrals of type

$$(I(a, \phi; h) f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\frac{i}{h}\phi(x,\theta,y)} a(x, \theta, y) f(y) dy d\theta, \tag{2.1}$$

with  $f \in S(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n, h \in ]0, h_0]$ .

Suppose that the function  $\phi$  satisfies the following conditions:

(H<sub>1</sub>)  $\phi : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $C^\infty$  application ( $\phi$  is a real function)

(H<sub>2</sub>)  $\forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha,\beta,\gamma} \geq 0$ ;

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma \phi(x, \theta, y)| \leq C_{\alpha,\beta,\gamma} \lambda^{2 - (|\alpha| + |\beta| + |\gamma|)}(x, \theta, y),$$

where  $\lambda(x, \theta, y) = (1 + |x|^2 + |y|^2 + |\theta|^2)^{1/2}$ .

(H<sub>3</sub>) There exist real numbers  $K_1, K_2 > 0$  such that

$$K_1 \lambda(x, \theta, y) \leq \lambda(\partial_y \phi, \partial_\theta \phi, y) \leq K_2 \lambda(x, \theta, y), \forall (x, \theta, y) \in \mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n.$$

(H<sub>3</sub><sup>\*</sup>) There exist real numbers  $K_1^*, K_2^* > 0$  such that

$$K_1^* \lambda(x, \theta, y) \leq \lambda(x, \partial_\theta \phi, \partial_x \phi) \leq K_2^* \lambda(x, \theta, y), \forall (x, \theta, y) \in \mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n.$$

For any open subset  $\Omega$  of  $\mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n, \mu \in \mathbb{R}$  and  $\rho \in [0, 1]$ , we set

$$\Gamma_\rho^\mu(\Omega) = \left\{ a \in C^\infty(\Omega) : |\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma a| \leq C_{\alpha,\beta,\gamma} \lambda^{\mu - \rho(|\alpha| + |\beta| + |\gamma|)}(x, \theta, y) \right\}.$$

When  $\Omega = \mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n$ , we denote  $\Gamma_\rho^\mu(\Omega) = \Gamma_\rho^\mu$ .

In order to generalize the notion of  $h$ -admissible operators (cf.[13]), we give the following definitions

**Definition 1** We call  $h$ -admissible symbol of weight  $(\mu, \rho)$ , every  $C^\infty$  application  $a(h)$  of  $]0, h_0]$  in  $\Gamma_\rho^\mu$ , such that

$$\forall N \in \mathbb{N}, a(h) = \sum_{j=0}^N h^j a_j + h^{N+1} r_{N+1}(h),$$

where  $a_j \in \Gamma_\rho^{\mu-2\rho j}$ , and  $\{r_{N+1}(h), h \in ]0, h_0]\}$  is bounded in  $\Gamma_\rho^{\mu-2\rho(N+1)}$ .

**Definition 2** We call  $h$ -admissible Fourier integral operator, every  $C^\infty$  application  $A$  of  $]0, h_0]$  in  $\mathcal{L}(S(\mathbb{R}^n), \mathbf{L}^2(\mathbb{R}^n))$  ( $\mathcal{L}(E, F)$  is the space of bounded linear mapping from  $E$  to  $F$ ), for which there exists a sequence  $(a_j)_j \in \Gamma_0^\mu$  satisfying

$$A(h) = \sum_{j=0}^N h^j I(a_j, \phi; h) + h^{N+1} R_{N+1}(h), \text{ for } N \in \mathbb{N} \text{ and } N \text{ large enough,} \tag{2.2}$$

where

$$(I(a_j, \phi; h) f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{\frac{i}{h} \phi(x, \theta, y)} a_j(x, \theta, y) f(y) dy d\theta,$$

$$\sup_{h \in ]0, h_0]} \|R_{N+1}(h)\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{R}^n))} < \infty.$$

To give a meaning to the right-hand side of (2.1), we consider  $g \in S(\mathbb{R}_x^n \times \mathbb{R}_\theta^N \times \mathbb{R}_y^n)$ ,  $g(0) = 1$ . If  $a \in \Gamma_0^\mu$ , we define

$$a_p(x, \theta, y) = g\left(\frac{x}{p}, \frac{\theta}{p}, \frac{y}{p}\right) a(x, \theta, y), \quad p > 0.$$

**Theorem 1** If  $\phi$  satisfies  $(H_1), (H_2), (H_3)$ , and  $(H_3^*)$ , and if  $a \in \Gamma_0^\mu$ , then

1. For all  $f \in S(\mathbb{R}^n)$ ,  $\lim_{p \rightarrow \infty} [(I(a_p, \phi; h) f)](x)$  exists for every  $x \in \mathbb{R}^n$  and is independent of the choice of the function  $g$ . We then set

$$(I(a, \phi; h) f) := \lim_{p \rightarrow \infty} (I(a_p, \phi; h) f).$$

2.  $I(a, \phi; h) \in \mathcal{L}(S(\mathbb{R}^n))$  and  $I(a, \phi; h) \in \mathcal{L}(S'(\mathbb{R}^n))$ .

**Proof** Let  $\delta \in C_0^\infty(\mathbb{R}^n)$  such that  $\text{supp} \delta \subseteq [-1, 2]$  and  $\delta \equiv 1$  on  $[0, 1]$ .

For all  $\varepsilon > 0$ , we set

$$\omega_\varepsilon(x, \theta, y) = \delta\left(\frac{|\partial_y \phi|^2 + |\partial_\theta \phi|^2}{\varepsilon \lambda(x, \theta, y)^2}\right).$$

The hypothesis  $(H_3)$  implies that there exists  $\gamma > 0$  such that we have on the support of  $\omega_\varepsilon$

$$\lambda(x, \theta, y) \leq \gamma \left[ (1 + |y|^2)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \lambda(x, \theta, y) \right].$$

Therefore, there exists  $\varepsilon_0$  and a constant  $\gamma_0$ , such that for all  $\varepsilon \leq \varepsilon_0$  we have the inequality

$$\lambda(x, \theta, y) \leq \gamma_0 \left(1 + |y|^2\right)^{\frac{1}{2}}.$$

on the support of  $\omega_\varepsilon$ .

In the sequel, we fix  $\varepsilon = \varepsilon_0$ . Then it is immediate that  $I(\omega_{\varepsilon_0} a_p, \phi; h) f$  is an absolutely convergent integral and we have

$$\lim_{p \rightarrow \infty} I(\omega_{\varepsilon_0} a_p, \phi; h) f = I(\omega_{\varepsilon_0} a, \phi; h) f. \tag{2.3}$$

Using  $(H_2)$  we prove also that  $I(\omega_{\varepsilon_0} a, \phi; h) f$  is a continuous operator from  $S(\mathbb{R}^n)$  into itself. To study  $\lim_{p \rightarrow \infty} I((1 - \omega_{\varepsilon_0}) a_p, \phi; h) f$ , we introduce the operator

$$L = \frac{h}{i} \frac{\left(\sum_{j=1}^n (\partial_{y_j} \phi) \frac{\partial}{\partial y_j} + \sum_{j=1}^N (\partial_{\theta_j} \phi) \frac{\partial}{\partial \theta_j}\right)}{|\partial_y \phi|^2 + |\partial_\theta \phi|^2}.$$

Clearly we have

$$L(e^{i\phi}) = e^{i\phi}. \tag{2.4}$$

Let  $\Omega_0$  be the open subset of  $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$  defined by

$$\Omega_0 = \left\{ (x, \theta, y), |\partial_y \phi|^2 + |\partial_\theta \phi|^2 > \frac{\varepsilon_0}{2} \lambda(x, \theta, y)^2 \right\}.$$

We need the following lemma.

**Lemma 1** For all integer  $q \geq 0$ , and  $b \in C^\infty(\mathbb{R}_y^n \times \mathbb{R}_\theta^N)$ , we have

$$({}^tL)^q((1 - \omega_{\varepsilon_0}) b) = \sum_{|\alpha|+|\beta| \leq q} g_{\alpha, \beta}^q \partial_y^\alpha \partial_\theta^\beta ((1 - \omega_{\varepsilon_0}) b),$$

where the  $g_{\alpha, \beta}^q$  are in  $\Gamma_0^{-q}(\Omega_0)$  and depend only on  $\phi$ . Recall that  ${}^tL$  designates the transpose of  $L$ .

We prove the lemma by recurrence. It is obvious for  $q = 0$ . Now we see easily that

$${}^tL = \sum_j F_j \frac{\partial}{\partial y_j} + \sum_j G_j \frac{\partial}{\partial \theta_j} + H, \tag{2.5}$$

where  $F_j \in \Gamma_0^{-1}(\Omega_0)$ ,  $G_j \in \Gamma_0^{-1}(\Omega_0)$ , and  $H \in \Gamma_0^{-2}(\Omega_0)$  (which results from the hypothesis  $(H_2)$ ). Therefore, the recurrence is immediately proved.  $\square$

For all integer  $q \geq 0$ , we have from (2.4)

$$I((1 - \omega_{\varepsilon_0}) a_p, \phi; h) f(x) = \int \int e^{\frac{i}{h} \phi(x, \theta, y)} ({}^tL)^q((1 - \omega_{\varepsilon_0}) a_p, f) dy d\theta. \tag{2.6}$$

Now  $({}^tL)^q((1 - \omega_{\varepsilon_0}) a_p f)$  described (when  $p$  varies) a bound of  $\Gamma_0^{\mu-q}$ , and

$$\lim_{p \rightarrow \infty} ({}^tL)^q((1 - \omega_{\varepsilon_0}) a_p f)(x, \theta, y) = ({}^tL)^q((1 - \omega_{\varepsilon_0}) a f)(x, \theta, y), \tag{2.7}$$

for all  $(x, \theta, y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$ .

Finally, for all  $s > n + N$  we have

$$\int \int \lambda^{-s}(x, \theta, y) d\theta dy \leq \gamma_s \lambda^{n \neq N-s}(x). \tag{2.8}$$

It results so from (2.6)–(2.8) and using Lebesgue’s theorem we have

$$\lim_{p \rightarrow \infty} I((1 - \omega_{\varepsilon_0}) a_p, \phi; h) f(x) = \int \int e^{\frac{i}{h} \phi(x, \theta, y)} ({}^t L)^q ((1 - \omega_{\varepsilon_0}) a, f; h) dy d\theta, \tag{2.9}$$

where  $q$  satisfies  $q > n + N + \mu$ . From (2.3) and (2.9) we can prove the first part of the theorem.

Now let us show that  $I((1 - \omega_{\varepsilon_0}) a, \phi)$  is continuous. Taking account of (2.5) and (2.9), we get

$$I((1 - \omega_{\varepsilon_0}) a, \phi; h) f(x) = \sum_{|\gamma| \leq q} \int \int e^{\frac{i}{h} \phi(x, \theta, y)} b_\gamma^{(q)}(x, \theta, y) \partial_y^\gamma f(y) dy d\theta, \tag{2.10}$$

with  $b_\gamma^{(q)} \in \Gamma_0^{\mu-q}$ . On the other hand, we have

$$x^\alpha \partial_x^\beta \left( e^{\frac{i}{h} \phi}(x, \theta, y) b_\gamma^{(q)}(x, \theta, y) \right) \in \Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.11}$$

We deduce from (2.10) and (2.11) that, for all  $q > n + N + \mu + |\alpha| + |\beta|$ , there exists a constant  $C_{\alpha, \beta, q}$  such that

$$|x^\alpha \partial_x^\beta I((1 - \omega_{\varepsilon_0}) a, \phi; h) f(x)| \leq C_{\alpha, \beta, q} \sup_{\substack{x \in \mathbb{R}^n \\ |\gamma| \leq q}} |\partial_x^\gamma f(x)|,$$

which proves the continuity of  $I((1 - \omega_{\varepsilon_0}) a, \phi)$ . □

### 3. Composition of two h-admissible Fourier integral operators

**Theorem 2** *Let  $\phi_1, \phi_2$  be two phases satisfying  $(H_1), (H_2)$ , and  $(H_3)$ . Set*

$$\phi(x, \theta, z) = \phi_1(x, \theta_1, y) + \phi_2(y, \theta_2, z), \tag{3.1}$$

with  $\theta_1 \in \mathbb{R}^{N_1}, \theta_2 \in \mathbb{R}^{N_2}, x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n, \theta = (\theta_1, y, \theta_2)$ . Then  $\phi$  verifies  $(H_1), (H_2), (H_3)$  and, for all  $a_1 \in \Gamma_0^{\mu_1}, a_2 \in \Gamma_0^{\mu_2}$ , we have

$$I(a_1, \phi_1; h) I(a_2, \phi_2; h) = I(a_1 \times a_2, \phi; h), \tag{3.2}$$

with

$$(a_1 \times a_2)(x, \theta, z) = a_1(x, \theta_1, y) a_2(y, \theta_2, z).$$

**Proof**  $(H_1)$  and  $(H_2)$  are immediate to verify for  $\phi$ . Therefore, we will prove the hypothesis  $(H_3)$ .

The left side of the inequality  $(H_3)$  is evident; it suffices to show that  $\phi$  satisfies the right side of the inequality:

there exists  $K > 0$  such that

$$\lambda(x, \theta_1, y, \theta_2, z) \leq K \lambda(z, \partial_z \phi_2, \partial_y \phi_1 + \partial_y \phi_2, \partial_{\theta_1} \phi_1, \partial_{\theta_2} \phi_2). \tag{3.3}$$

Applying the property  $(H_3)$  to  $\phi_1$  and  $\phi_2$  we get that there exists  $C > 0$  such that

$$\lambda(x, \theta_1, y, \theta_2, z) \leq C \lambda(\partial_y \phi_1, \partial_{\theta_1} \phi_1, y, \partial_{\theta_2} \phi_2, \partial_z \phi_2, z). \tag{3.4}$$

We have also

$$\lambda(y) \leq C' \lambda(\partial_{\theta_2} \phi_2, \partial_z \phi_2, z), \tag{3.5}$$

from  $(H_3)$  applied to  $\phi_2$ , and

$$|\partial_y \phi_2| \leq C'' \lambda(y, \theta_2, z) \leq C''' (\lambda(\partial_{\theta_2} \phi_2, \partial_z \phi_2, z)), \tag{3.6}$$

from  $(H_2)$  and  $(H_3)$  applied to  $\phi_2$ .

Finally, we note that

$$|\partial_y \phi_1| \leq |\partial_y \phi_1 + \partial_y \phi_2| + |\partial_y \phi_2|. \tag{3.7}$$

The inequalities (3.4)–(3.7) imply (3.3).

Now let us show the composition formulas. For this, let us introduce the sequences of functions ( $i = 1, 2$ )

$$\chi_p^i(x, \theta_i, y) = \exp\left(-p^{-1}(|x|^2 + |\theta_i|^2 + |y|^2)\right); (x, \theta_i, y) \in \mathbb{R}^n \times \mathbb{R}^{N_i} \times \mathbb{R}^n$$

It is clear that (3.2) is satisfied for

$$a_p^1 = a_1 \chi_p^1, a_p^2 = a_2 \chi_p^2.$$

However,

$$\chi_p^1(x, \theta_1, y) \chi_p^2(y, \theta_2, z) = \exp\left(-p^{-1}(|x|^2 + 2|y|^2 + |\theta_1|^2 + |\theta_2|^2 + |z|^2)\right).$$

Hence, it results that

$$\lim_{p \rightarrow \infty} (I(a_p^1 a_p^2, \phi; h) f)(x) = (I(a_1 a_2, \phi) f; h)(x), \tag{3.8}$$

for all  $f \in S(\mathbb{R}^n)$ .

On the other hand, the proof of theorem 1 shows that there exists, for all  $l \in \mathbb{N}$  and  $j = 1, 2$ , an integer  $M_{j,l}$  and a constant  $C_{j,l} > 0$ , such that, for all  $f$  in  $S(\mathbb{R}^n)$  and  $p \geq 1$ , we have

$$\|I(a_p^j, \phi_j; h) f\|_{B^l} \leq C_{l,j} \|f\|_{B^{M_{j,l}}}, \tag{3.9}$$

where  $B^l(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n), x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n), |\alpha| + |\beta| \leq l\}$ .

We deduce first from (3.9) that, for all fixed  $f_0$  in  $S(\mathbb{R}^n)$ ,  $g_p = I(a_p^2, \phi_2; h) f_0$  describes a bounded of  $S(\mathbb{R}^n)$  when  $p$  varies.  $S(\mathbb{R}^n)$  being Montel space, we can extract a subsequence, suppose that  $g_p$  converges in  $S(\mathbb{R}^n)$  to  $g = I(a_1, \phi_2; h) f_0$ , but we have

$$\begin{aligned} \|I(a_p^1, \phi_1; h) g_p - I(a_1, \phi_1; h) g\|_{B^l} &\leq \|I(a_p^1, \phi_1; h) (g_p - g)\|_{B^l} \\ &+ \|(I(a_p^1, \phi_1; h) - I(a_1, \phi_1; h)) g\|_{B^l}. \end{aligned} \tag{3.10}$$

Even re-extracting a subsequence, we can suppose that

$$I(a_p^1, \phi_1; h) g \rightarrow I(a_1, \phi_1; h) g, \text{ in } S(\mathbb{R}^n). \tag{3.11}$$

It follows from (3.9)–(3.11) that, for all  $l$ , leaves to extract a subsequence, we have

$$I(a_p^1, \phi_1; h) I(a_p^2, \phi_2; h) f_0 \rightarrow I(a_1, \phi_1; h) I(a_2, \phi_2; h) f_0 \text{ in } B^l. \tag{3.12}$$

□

#### 4. About the particular case

In this section we shall be interested in a particular case on the phase function  $\phi$ , which is very important in applications for solving Cauchy problems [12]. Let

$$\phi(x, y, \theta) = S(x, y) - y\theta,$$

Suppose that  $S$  satisfies

(G<sub>1</sub>)  $S \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_\theta^n; \mathbb{R})$  ( $S$  is a real function)

(G<sub>2</sub>) For all  $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ , there exist  $C_{\alpha, \beta} > 0$ , such that

$$\left| \partial_x^\alpha \partial_\theta^\beta S(x, \theta) \right| \leq C_{\alpha, \beta} \lambda(x, \theta)^{(2-|\alpha|-|\beta|)}.$$

(G<sub>3</sub>) There exists  $\delta_0 > 0$  such that

$$\inf_{x, \theta \in \mathbb{R}^n} \left| \det \frac{\partial^2 S}{\partial x \partial \theta}(x, \theta) \right| \geq \delta_0.$$

**Lemma 2** *If  $S$  satisfies (G<sub>1</sub>), (G<sub>2</sub>), and (G<sub>3</sub>), then  $S$  satisfies the following inequalities:*

*There exist  $C_1, C_2 > 0$ , such that*

$$\begin{cases} |x| \leq C_1 \lambda(\theta, \partial_\theta S), & \text{for all } (x, \theta) \in \mathbb{R}^{2n}, \\ |\theta| \leq C_2 \lambda(x, \partial_x S), & \text{for all } (x, \theta) \in \mathbb{R}^{2n}. \end{cases} \tag{4.1}$$

*There also exists  $C_3 > 0$  such that for all  $(x, \theta), (x', \theta') \in \mathbb{R}^{2n}$ ,*

$$|x - x'| + |\theta - \theta'| \leq C_3 [ |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(x', \theta')| + |\theta - \theta'| ]. \tag{4.2}$$

**Proof** The mappings  $\mathbb{R}^n \ni \theta \rightarrow f_x(\theta) = \partial_x S(x, \theta)$ ,  $\mathbb{R}^n \ni x \rightarrow g_\theta(x) = \partial_\theta S(x, \theta)$  and  $\mathbb{R}^{2n} \ni (x, \theta) \rightarrow h_2(x, \theta) = (\theta, \partial_\theta S(x, \theta))$  are global diffeomorphisms of  $\mathbb{R}^n$ . From (G<sub>2</sub>) and (G<sub>3</sub>), it follows that  $\| (f_x^{-1})' \|$  and  $\| (g_\theta^{-1})' \|$  are uniformly bounded on  $\mathbb{R}^n$  and  $\| (h_2^{-1})' \|$  is uniformly bounded on  $\mathbb{R}^{2n}$ . Thus (G<sub>3</sub>) and Taylor’s theorem lead to the following estimate:

There exist  $M, N > 0$ , such that for all  $(x, \theta), (x', \theta') \in \mathbb{R}^{2n}$ ,

$$|\theta| = |f_x^{-1}(f_x(\theta)) - f_x^{-1}(f_x(0))| \leq M |\partial_x S(x, \theta) - \partial_x S(x, 0)| \leq C_4 \lambda(x, \partial_x S),$$

with  $C_4 > 0$ ;

$$|x| = |g_\theta^{-1}(g_\theta(\theta)) - g_\theta^{-1}(g_\theta(0))| \leq N |\partial_\theta S(x, \theta) - \partial_\theta S(0, \theta)| \leq C_5 \lambda(\partial_\theta S, \theta),$$



with  $C_5 > 0$ ;

$$|(x, \theta) - (x', \theta')| = |h_2^{-1}(h_2(x, \theta)) - h_2^{-1}(h_2(x', \theta'))| \leq C_5 |(\theta, \partial_\theta S(x, \theta)) - (\theta', \partial_\theta S(x', \theta'))|.$$

□

**Lemma 3** *Let us assume that  $S$  satisfies  $(G_1)$ ,  $(G_2)$ , and  $(G_3)$ . Then the function  $\phi(x, y, \theta) = S(x, \theta) - y\theta$  satisfies  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$ , and  $(H_3^*)$ .*

**Proof**  $(H_1)$  and  $(H_2)$  are trivially satisfied.

From (4.1) we have

$$\lambda(x, y, \theta) \leq \lambda(x, \theta) + \lambda(y) \leq C_6 (\lambda(\theta, \partial_\theta S) + \lambda(y)), \quad C_6 > 0.$$

Moreover, we have  $\partial_{y_j} \phi = -\theta_j$ ; and  $\partial_{\theta_j} \phi = \partial_{\theta_j} S - y_j$ , and so

$$\lambda(\theta, \partial_\theta S) = \lambda(\partial_y \phi, \partial_\theta \phi + y) \leq 2\lambda(\partial_y \phi, \partial_\theta \phi, y),$$

which finally gives for some  $C_7 > 0$ ,

$$\lambda(x, \theta, y) \leq C_6 (2\lambda(\partial_y \phi, \partial_\theta \phi, y)) \leq \frac{1}{C_7} \lambda(\partial_y \phi, \partial_\theta \phi, y).$$

The second inequality in  $(H_3)$  is a consequence of (4.1).

By a similar argument we can show  $(H_3^*)$ . □

When  $\theta = \theta'$  in (4.2), there exists  $C_3 > 0$ , such that all  $(x, x', \theta) \in \mathbb{R}^{3n}$ ,

$$|x - x'| \leq C_3 |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(x', \theta)|. \tag{4.3}$$

**Proposition 1** *If  $S$  satisfies  $(G_1)$  and  $(G_2)$ , then there exists a constant  $\varepsilon_0 > 0$  such that the phase function  $\phi$  given in (4.1) belongs to  $\Gamma_1^2(\Omega_{\phi, \varepsilon_0})$  where*

$$\Omega_{\phi, \varepsilon_0} = \left\{ (x, \theta, y) \in \mathbb{R}^{3n}; |\partial_\theta S(x, \theta) - y|^2 < \varepsilon_0 (|x|^2 + |y|^2 + |\theta|^2) \right\}.$$

**Proof** We have to show that: there exists  $\varepsilon_0 > 0$ , such that for all  $\alpha, \beta, \gamma \in \mathbb{N}^n$ , there exist  $C_{\alpha, \beta, \gamma} > 0$ :

$$\left| \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x, \theta, y) \right| \leq C_{\alpha, \beta, \gamma} \lambda(x, \theta, y)^{(2-|\alpha|-|\beta|-|\gamma|)}, \quad (x, \theta, y) \in \Omega_{\phi, \varepsilon_0}. \tag{4.4}$$

If  $|\gamma| = 1$ , then

$$\left| \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x, \theta, y) \right| = \left| \partial_x^\alpha \partial_\theta^\beta (-\theta) \right| = \begin{cases} 0 & \text{if } |\alpha| \neq 0 \\ \left| \partial_\theta^\beta (-\theta) \right| & \text{if } \alpha = 0 \end{cases};$$

If  $|\gamma| > 1$ , then

$$\left| \partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x, \theta, y) \right| = 0.$$

Hence, the estimate (4.4) is satisfied.

If  $|\gamma| = 0$ , then for all  $\alpha, \beta \in \mathbb{N}^n; |\alpha| + |\beta| \leq 2$ , there exists  $C_{\alpha, \beta} > 0$  such that

$$\left| \partial_x^\alpha \partial_\theta^\beta \phi(x, \theta, y) \right| = \left| \partial_x^\alpha \partial_\theta^\beta S(x, \theta) - \partial_x^\alpha \partial_\theta^\beta (y\theta) \right| \leq C_{\alpha, \beta} \lambda(x, \theta, y)^{(2-|\alpha|-|\beta|)}.$$

If  $|\alpha| + |\beta| > 2$ , one has  $\partial_x^\alpha \partial_\theta^\beta \phi(x, \theta, y) = \partial_x^\alpha \partial_\theta^\beta S(x, \theta)$ . In  $\Omega_{\phi, \varepsilon_0}$  we have

$$|y| = |\partial_\theta S(x, \theta) - y - \partial_\theta S(x, \theta)| \leq \sqrt{\varepsilon_0} \left( |x|^2 + |y|^2 + |\theta|^2 \right)^{\frac{1}{2}} + C_8 \lambda(x, \theta), \tag{4.5}$$

with  $C_8 > 0$ . For  $\varepsilon_0$  sufficiently small, we obtain a constant  $C_9 > 0$  such that

$$|y| \leq C_9 \lambda(x, \theta), \quad \forall (x, \theta, y) \in \Omega_{\phi, \varepsilon_0}.$$

This inequality leads to the equivalence

$$\lambda(x, \theta, y) \simeq \lambda(x, \theta) \text{ in } \Omega_{\phi, \varepsilon_0}; \tag{4.6}$$

thus the assumption  $(G_2)$  and (4.6) give the estimate (4.4). □

Using (4.6), we give the following result.

**Proposition 2** *If  $(x, \theta) \rightarrow a(x, \theta)$  belongs to  $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$ , then  $(x, \theta, y) \rightarrow a(x, \theta)$  belongs to  $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n) \cap \Gamma_k^m(\Omega_{\phi, \varepsilon_0})$ ,  $k \in \{0, 1\}$ .*

**5.  $L^2$ -boundedness and  $L^2$ -compactness of  $F_h$**

We have the following results concerning the  $L^2$ -boundedness and  $L^2$ -compactness of the  $h$ -admissible Fourier integral operator defined by

$$(F_h \psi)(x) = (2\pi h)^{-n} \int \int e^{\frac{i}{h}(S(x, \theta) - y\theta)} a(x, y) \psi(y) dy d\theta.$$

**Theorem 3** *Let  $F_h$  be the integral operator of distribution kernel*

$$K(x, y; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, y) - y\theta)} a(x, y) \widehat{d\theta}, \tag{5.1}$$

where  $\widehat{d\theta} = (2\pi h)^{-n} d\theta$ ,  $a \in \Gamma_k^m(\mathbb{R}_{x, \theta}^{2n})$ ,  $k = 0, 1, h \in ]0, h_0]$  and  $S$  satisfies  $(G_1), (G_2)$ , and  $(G_3)$ . Then  $F_h F_h^*$  and  $F_h^* F_h$  are  $h$ -admissible operators with symbol in  $\Gamma_k^m(\mathbb{R}^{2n})$ ,  $k = 0, 1$ , given by

$$\begin{aligned} \sigma(F_h F_h^*)(x, \partial_x S(x, \theta)) &\equiv |a(x, \theta)|^2 \left| \left( \det \frac{\partial^2 S}{\partial \theta \partial x} \right)^{-1} (x, \theta) \right|, \\ \sigma(F_h^* F_h)(\partial_\theta S(x, \theta), \theta) &\equiv |a(x, \theta)|^2 \left| \left( \det \frac{\partial^2 S}{\partial \theta \partial x} \right)^{-1} (x, \theta) \right|, \end{aligned}$$

we denote here  $a \equiv b$  for  $a, b \in \Gamma_k^{2p}(\mathbb{R}^{2n})$  if  $(a - b) \in \Gamma_k^{2p-2}(\mathbb{R}^{2n})$  and  $\sigma$  stands for the symbol.

**Proof** If  $u \in S(\mathbb{R}^n)$ , then  $F_h u(x)$  is given by

$$\begin{aligned} (F_h u)(x) &= \int_{\mathbb{R}^n} K(x, y) u(y) dy. \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta) - y\theta)} a(x, \theta) u(y) dy d\widehat{\theta}. \\ &= \int_{\mathbb{R}^n} e^{\frac{i}{h}S(x, \theta)} a(x, \theta) \left( \int_{\mathbb{R}^n} e^{-\frac{i}{h}y\theta} u(y) dy \right) d\widehat{\theta}. \\ &= \int_{\mathbb{R}^n} e^{\frac{i}{h}S(x, \theta)} a(x, \theta) \mathcal{F}_h u(\theta) d\widehat{\theta}, \end{aligned} \tag{5.2}$$

where  $\mathcal{F}_h$  is the Fourier transformation.

Here  $F_h$  is a continuous linear mapping from  $S(\mathbb{R}^n)$  to  $S(\mathbb{R}^n)$  (by Theorem 4). Let  $v \in S(\mathbb{R}^n)$ , then

$$\begin{aligned} \langle F_h u, v \rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{\frac{i}{h}S(x, \theta)} a(x, \theta) \mathcal{F}_h u(\theta) d\widehat{\theta} \right) \overline{v(x)} dx. \\ &= \int_{\mathbb{R}^n} \mathcal{F}_h u(\theta) \left( \int_{\mathbb{R}^n} e^{\frac{i}{h}S(x, \theta)} \overline{a(x, \theta)v(x)} dx \right) d\widehat{\theta}. \end{aligned}$$

Thus

$$\langle F_h u(x), v(x) \rangle_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \langle \mathcal{F}_h u(\theta), \mathcal{F}_h((F_h^* v))(\theta) \rangle_{L^2(\mathbb{R}^n)}.$$

where

$$\mathcal{F}_h((F_h^* v))(\theta) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}S(\tilde{x}, \theta)} \bar{a}(\tilde{x}, \theta) v(\tilde{x}) d\tilde{x}. \tag{5.3}$$

Hence, for all  $v \in S(\mathbb{R}^n)$ ,

$$(F_h F_h^* v)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta) - S(\tilde{x}, \theta))} \bar{a}(x, \theta) \bar{a}(\tilde{x}, \theta) v(\tilde{x}) d\tilde{x} d\widehat{\theta}. \tag{5.4}$$

The main idea to show that  $F_h F_h^*$  is a  $h$ -pseudodifferential operator is to use the fact that  $S(x, \theta) - S(\tilde{x}, \theta)$  can be expressed by the scalar  $\langle x - \tilde{x}, \xi(x, \tilde{x}, \theta) \rangle$ , after considering the change of variables  $(x, \tilde{x}, \theta) \rightarrow (x, \tilde{x}, \xi = \xi(x, \tilde{x}, \theta))$ .

The distribution kernel of  $F_h F_h^*$  is

$$K(x, \tilde{x}; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta) - S(\tilde{x}, \theta))} a(\tilde{x}, \theta) \bar{a}(x, \theta) d\widehat{\theta}.$$

we obtain from (4.3) that if

$$|x - \tilde{x}| \geq \frac{\varepsilon}{2} \lambda(x, \tilde{x}, \theta), \text{ where } \varepsilon > 0 \text{ is sufficiently small,}$$

then

$$|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)| \geq \frac{\varepsilon}{2C_5} \lambda(x, \tilde{x}, \theta). \tag{5.5}$$

Choosing  $\omega \in C^\infty(\mathbb{R}^n)$  such that

$$\begin{aligned} \omega(x) &\geq 0, \quad \forall x \in \mathbb{R}, \\ \omega(x) &= 1, \quad \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ \text{supp } \omega &\subset ]-1, 1[, \end{aligned}$$

and setting

$$b(x, \tilde{x}, \theta) := a(x, \theta) \bar{a}(\tilde{x}, \theta) = b_1^\varepsilon(x, \tilde{x}, \theta) + b_2^\varepsilon(x, \tilde{x}, \theta).$$

$$b_1^\varepsilon(x, \tilde{x}, \theta) = \omega\left(\frac{|x - \tilde{x}|}{\varepsilon \lambda(x, \tilde{x}, \theta)}\right) b(x, \tilde{x}, \theta).$$

$$b_2^\varepsilon(x, \tilde{x}, \theta) = \left[1 - \omega\left(\frac{|x - \tilde{x}|}{\varepsilon \lambda(x, \tilde{x}, \theta)}\right)\right] b(x, \tilde{x}, \theta).$$

We have  $K(x, \tilde{x}; h) = K_1^\varepsilon(x, \tilde{x}; h) + K_2^\varepsilon(x, \tilde{x}; h)$ , where

$$K_j^\varepsilon(x, \tilde{x}; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x, \theta) - S(\tilde{x}, \theta))} b_j^\varepsilon(x, \tilde{x}, \theta) \widehat{d\theta}, \quad j = 1, 2.$$

We will study separately the kernels  $K_1^\varepsilon$  and  $K_2^\varepsilon$ .

On the support of  $b_2^\varepsilon$ , inequality (5.5) is satisfied and we have

$$K_2^\varepsilon(x, \tilde{x}; h) \in S(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, using the oscillatory integral method, there is a linear partial differential operator  $L$  of order 1 such that

$$L\left(e^{\frac{i}{h}(S(x, \theta) - S(\tilde{x}, \theta))}\right) = e^{\frac{i}{h}(S(x, \theta) - S(\tilde{x}, \theta))},$$

where

$$L = \frac{h \sum_l [(\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta)] \partial_{\theta_l}}{i |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^2}.$$

The transpose operator of  $L$  is

$${}^tL = \sum_{l=1}^n F_{h,l}(x, \tilde{x}, \theta) \partial_{\theta_l} + G_h(x, \tilde{x}, \theta),$$

where  $F_l(x, \tilde{x}, \theta) \in \Gamma_0^{-1}(\Omega_\varepsilon)$ ,  $G(x, \tilde{x}, \theta) \in \Gamma_0^{-2}(\Omega_\varepsilon)$ ,

$$F_{h,l}(x, \tilde{x}, \theta) = \frac{h [(\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta)]}{i |(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^2},$$

$$G_h(x, \tilde{x}, \theta) = \frac{h}{i} \sum_{l=1}^n \partial_{\theta_l} \frac{(\partial_{\theta_l} S)(x, \theta) - (\partial_{\theta_l} S)(\tilde{x}, \theta)}{|(\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta)|^2},$$

$$\Omega_\varepsilon = \left\{ (x, \tilde{x}, \theta) \in \mathbb{R}^{3n} : (\partial_\theta S)(x, \theta) - (\partial_\theta S)(\tilde{x}, \theta) > \frac{\varepsilon}{2C_5} \lambda(x, \tilde{x}, \theta) \right\}.$$

On the other hand, we prove by inducting on  $q$  that

$$({}^tL)^q b_2^\varepsilon(x, \tilde{x}, \theta) = \sum_{|\gamma| \leq q, \gamma \in \mathbb{N}} g_{\gamma,q}(x, \tilde{x}, \theta) \partial_\theta^\gamma b_2^\varepsilon(x, \tilde{x}, \theta), \quad g_\gamma^{(q)} \in \Gamma_0^{-q}(\Omega_\varepsilon),$$

and so

$$K_2^\varepsilon(x, \tilde{x}; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta) - S(\tilde{x},\theta))} ({}^tL)^q b_2^\varepsilon(x, \tilde{x}, \theta) \widehat{d\theta}.$$

Using Leibnitz's formula,  $(G_2)$ , and the form  $({}^tL)^q$ , we can choose  $q$  large enough such that for all  $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n, \exists C_{\alpha, \alpha', \beta, \beta'} > 0$ ,

$$\sup_{x, \tilde{x} \in \mathbb{R}^n} \left| x^\alpha \tilde{x}^{\alpha'} \partial_x^\beta \partial_{\tilde{x}}^{\beta'} K_2^\varepsilon(x, \tilde{x}) \right| \leq C_{\alpha, \alpha', \beta, \beta'}.$$

Next, we study  $K_1^\varepsilon$ : this is more difficult and depends on the choice of the parameter  $\varepsilon$ . It follows from Taylor's formula that

$$\begin{aligned} S(x, \theta) - S(\tilde{x}, \theta) &= \langle x - \tilde{x}, \xi(x, \tilde{x}, \theta) \rangle_{\mathbb{R}^n} \\ \xi(x, \tilde{x}, \theta) &= \int_0^1 (\partial_x S)(\tilde{x} + t(x - \tilde{x}), \theta) dt \end{aligned}$$

we define the vectorial function

$$\tilde{\xi}^\varepsilon(x, \tilde{x}, \theta) = \omega\left(\frac{|x - \tilde{x}|}{2\varepsilon\lambda(x, \tilde{x}, \theta)}\right) \xi(x, \tilde{x}, \theta) + \left(1 - \omega\left(\frac{|x - \tilde{x}|}{2\varepsilon\lambda(x, \tilde{x}, \theta)}\right)\right) (\partial_x S)(\tilde{x}, \theta).$$

We have

$$\tilde{\xi}^\varepsilon(x, \tilde{x}, \theta) = \xi(x, \tilde{x}, \theta), \quad \text{on the supp } b_1^\varepsilon.$$

Moreover, for  $\varepsilon$  sufficiently small,

$$\lambda(x, \theta) \simeq \lambda(\tilde{x}, \theta) \simeq \lambda(x, \tilde{x}, \theta), \quad \text{on the supp } b_1^\varepsilon. \tag{5.6}$$

Let us consider the mappings

$$\mathbb{R}^{3n} \ni (x, \tilde{x}, \theta) \rightarrow (x, \tilde{x}, \tilde{\xi}^\varepsilon(x, \tilde{x}, \theta)). \tag{5.7}$$

for which the Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \partial_x \tilde{\xi}^\varepsilon & \partial_{\tilde{x}} \tilde{\xi}^\varepsilon & \partial_\theta \tilde{\xi}^\varepsilon \end{pmatrix}.$$

We have

$$\begin{aligned} \frac{\partial \tilde{\xi}_j^\varepsilon}{\partial \theta_i}(x, \tilde{x}, \theta) &= \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) + \omega\left(\frac{|x - \tilde{x}|}{2\varepsilon\lambda(x, \tilde{x}, \theta)}\right) \left( \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right) \\ &\quad - \frac{|x - \tilde{x}|}{2\varepsilon\lambda(x, \tilde{x}, \theta)} \frac{\partial \lambda}{\partial \theta_i}(x, \tilde{x}, \theta) \lambda^{-1}(x, \tilde{x}, \theta) \omega' \left( \frac{|x - \tilde{x}|}{2\varepsilon\lambda(x, \tilde{x}, \theta)} \right) \left( \xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \left| \frac{\partial \tilde{\xi}_j^\varepsilon}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| &\leq \left| \omega \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda(x, \tilde{x}, \theta)} \right) \right| \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \\ &\quad + \lambda^{-1}(x, \tilde{x}, \theta) \left| \omega' \left( \frac{|x - \tilde{x}|}{2\varepsilon \lambda(x, \tilde{x}, \theta)} \right) \right| \left| \xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right|. \end{aligned}$$

Now it follows from  $(G_2)$ , (5.6), and Taylor's formula that

$$\begin{aligned} \left| \frac{\partial \xi_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| &\leq \int_0^1 \left| \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| dt \\ &\leq C_{10} |x - \tilde{x}| \lambda^{-1}(x, \tilde{x}, \theta), \quad C_{10} > 0. \end{aligned} \tag{5.8}$$

$$\begin{aligned} \left| \xi_j(x, \tilde{x}, \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right| &\leq \int_0^1 \left| \frac{\partial S}{\partial x_j}(\tilde{x} + t(x - \tilde{x}), \theta) - \frac{\partial S}{\partial x_j}(\tilde{x}, \theta) \right| dt \\ &\leq C_{11} |x - \tilde{x}|, \quad C_{10} > 0. \end{aligned} \tag{5.9}$$

From (5.8) and (5.9), there exists a positive constant  $C_{12} > 0$ , such that

$$\left| \frac{\partial \tilde{\xi}_j}{\partial \theta_i}(x, \tilde{x}, \theta) - \frac{\partial^2 S}{\partial \theta_i \partial x_j}(\tilde{x}, \theta) \right| \leq C_{12} \varepsilon, \quad \forall i, j \in \{1, \dots, n\}. \tag{5.10}$$

If  $\varepsilon < \frac{\delta_0}{2\tilde{C}}$ , then (5.10) and  $(G_3)$  yield the estimate

$$\frac{\delta_0}{2} \leq -\tilde{C}\varepsilon + \delta_0 \leq -\tilde{C}\varepsilon + \det \frac{\partial^2 S}{\partial \theta \partial x} \leq \det \partial_\theta \tilde{\xi}^\varepsilon(x, \tilde{x}, \theta). \tag{5.11}$$

with  $\tilde{C} > 0$ . If  $\varepsilon$  is such that (5.6) and (5.11) hold, then the mapping given in (5.7) is global diffeomorphism of  $\mathbb{R}^{3n}$ . Hence there exists a mapping

$$\theta : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (x, \tilde{x}, \xi) \rightarrow \theta(x, \tilde{x}, \xi) \in \mathbb{R}^n,$$

such that

$$\begin{aligned} \tilde{\xi}^\varepsilon(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) &= \xi, \\ \theta(x, \tilde{x}, \tilde{\xi}^\varepsilon(x, \tilde{x}, \theta)) &= x, \\ \partial^\alpha \theta(x, \tilde{x}, \xi) &= o(1), \quad \forall \alpha \in \mathbb{N}^{3n} \setminus \{0\}. \end{aligned} \tag{5.12}$$

If we change the variable  $\xi$  by  $\theta(x, \tilde{x}, \xi)$  in  $K_1^\varepsilon(x, \tilde{x}; h)$ , we obtain

$$K_1^\varepsilon(x, \tilde{x}; h) = \int e^{\frac{i}{h} \langle x - \tilde{x}, \xi \rangle} b_1^\varepsilon(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \right| \widehat{d\xi}. \tag{5.13}$$

From (5.12) we have, for  $k = 0, 1$ , that  $b_1^\varepsilon(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \left| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \right|$  belongs to  $\Gamma_k^{2m}(\mathbb{R}^{2n})$ .

Applying the stationary phase theorem to (5.13), we obtain the expression of the symbol of the  $h$ -admissible operator  $F_h F_h^*$ ,

$$\sigma(F_h F_h^*) = b_1^\varepsilon(x, \tilde{x}, \theta(x, \tilde{x}, \xi)) \Big| \det \frac{\partial \theta}{\partial \xi}(x, \tilde{x}, \xi) \Big|_{x=\tilde{x}} + R(x, \xi),$$

where  $R(x, \xi)$  belongs to  $\Gamma_k^{2m-2}(\mathbb{R}^{2n})$  if  $a \in \Gamma_k^m(\mathbb{R}^{2n})$ ,  $k = 0, 1$ .

For  $x = \tilde{x}$ , we have

$$b_1^\varepsilon(x, x, \theta(x, x, \xi)) = |a(x, \theta(x, x, \xi))|^2,$$

where  $\theta(x, x, \xi)$  is the inverse of the mapping

$$\theta \rightarrow \partial_x S(x, \theta) = \xi.$$

Thus

$$\sigma(F_h F_h^*)(x, \partial_x S(x, \theta)) \equiv |a(x, \theta)|^2 \Big| \det \frac{\partial^2 S}{\partial \theta \partial x}(x, \theta) \Big|^{-1}.$$

From (5.2) and (5.3), we obtain the expression of  $F_h^* F_h : \forall v \in S(\mathbb{R}^n)$  :

$$\begin{aligned} (\mathcal{F}_h(F_h^* F_h) \mathcal{F}_h^{-1})v(\theta) &= \int_{\mathbb{R}^n} e^{-\frac{i}{h}S(x,\theta)} \bar{a}(x, \theta) (F_h(\mathcal{F}_h^{-1}v))(x) dx. \\ &= \int_{\mathbb{R}^n} e^{-\frac{i}{h}S(x,\theta)} \bar{a}(x, \theta) \left( \int_{\mathbb{R}^n} e^{-\frac{i}{h}S(x,\tilde{\theta})} a(x, \tilde{\theta}) (F_h(\mathcal{F}_h^{-1}v))(\tilde{\theta}) \widehat{d\tilde{\theta}} \right) dx. \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{i}{h}(S(x,\theta)-S(x,\tilde{\theta}))} \bar{a}(x, \theta) a(x, \tilde{\theta}) v(\tilde{\theta}) \widehat{d\tilde{\theta}} dx. \end{aligned}$$

Hence the distribution kernel of the integral operator  $\mathcal{F}_h(F_h^* F_h) \mathcal{F}_h^{-1}$  is

$$\tilde{K}(\theta, \tilde{\theta}; h) = \int e^{-\frac{i}{h}(S(x,\theta)-S(x,\tilde{\theta}))} \bar{a}(x, \theta) a(x, \tilde{\theta}) \widehat{dx}.$$

We remark that we can deduce  $\tilde{K}(\theta, \tilde{\theta}; h) = \tilde{K}(x, \tilde{x}; h)$  by replacing  $x$  by  $\theta$ . On the other hand, all assumptions used here are symmetrical on  $x$  and  $\theta$ ; therefore,  $\mathcal{F}_h(F_h^* F_h) \mathcal{F}_h^{-1}$  is a nice  $h$ -admissible operator with symbol

$$\sigma(F_h^* F_h)(\partial_\theta S(x, \theta), \theta) = |a(x, \theta)|^2 \Big| \det \frac{\partial^2 S}{\partial \theta \partial x}(x, \theta) \Big|^{-1}.$$

□

**Corollary 1** Let  $F_h$  be the integral operator with the distribution kernel

$$K(x, y; h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,y)-y\theta)} a(x, y) \widehat{d\theta},$$

where  $a \in \Gamma_0^m(\mathbb{R}_{x,\theta}^{2n})$  and  $S$  satisfies  $(G_1)$ ,  $(G_2)$ , and  $(G_3)$ . Then we have

- (1) For any  $m$  such that  $m \leq 0$ ,  $F_h$  can be extended as a bounded linear mapping on  $L^2(\mathbb{R}^n)$ .
- (2) For any  $m$  such that  $m < 0$ ,  $F_h$  can be extended as a compact operator on  $L^2(\mathbb{R}^n)$ .

**Proof** It follows from theorem (3) that  $F_h^*F_h$  is a  $h$ -admissible operator with symbol in  $\Gamma_0^{2m}(\mathbb{R}^{2n})$ .

(1) If  $m \leq 0$ , the weight  $\lambda^{2m}(x, \theta)$  is bounded, and so we can apply the Calderón–Vaillancourt theorem for  $F_h^*F_h$  and obtain the existence of a positive constant  $\gamma(n)$  and a integer  $k(n)$  such that

$$\|(F_h^*F_h)u\|_{L^2(\mathbb{R}^n)} \leq \gamma(n) Q_{k(n)}(\sigma(F_hF_h^*)) \|u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in S(\mathbb{R}^n),$$

where

$$Q_{k(n)}(\sigma(F_hF_h^*)) = \sum_{|\alpha|+|\beta| \leq k(n)} \sup_{(x,\theta) \in \mathbb{R}^n} \left| \partial_x^\alpha \partial_\theta^\beta \sigma(F_hF_h^*)(\partial_\theta S(x, \theta), \theta) \right|.$$

Hence, for all  $v \in S(\mathbb{R}^n)$ ,

$$\|F_h u\|_{L^2(\mathbb{R}^n)} \leq \|F_h^*F_h\|_{\mathcal{L}(L^2(\mathbb{R}^n))}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \leq (\gamma(n) Q_{k(n)}(\sigma(F_hF_h^*)))^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}.$$

Thus,  $F_h$  is also a bounded linear operator on  $L^2(\mathbb{R}^n)$ .

(2) If  $m < 0$ ,  $\lim_{|x|+|\theta| \rightarrow \infty} \lambda^m(x, \theta) = 0$ , and the compactness theorem shows that the operator  $F_h^*F_h$  can be extended as a compact operator on  $L^2(\mathbb{R}^n)$ . Thus, the Fourier integral operator  $F_h$  is compact on  $L^2(\mathbb{R}^n)$ . Indeed, let  $(\varphi_j)_{j \in \mathbb{N}}$  be an orthonormal basis of  $L^2(\mathbb{R}^n)$ , then

$$\left\| F_h^*F_h - \sum_{j=1}^n \langle \varphi_j, \cdot \rangle F_h^*F_h \varphi_j \right\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $F_h$  is bounded, for all  $\psi \in L^2(\mathbb{R}^n)$ ,

$$\left\| F_h \psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle F_h \varphi_j \right\|^2 \leq \left\| F_h^*F_h \psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle F_h^*F_h \varphi_j \right\| \left\| \psi - \sum_{j=1}^n \langle \varphi_j, \psi \rangle \varphi_j \right\|,$$

it follows that

$$\left\| F_h - \sum_{j=1}^n \langle \varphi_j, \cdot \rangle F_h \varphi_j \right\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

□

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