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On $*$ -commuting mappings and derivations in rings with involution

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Abstract: Let R be a ring with involution $*$. A mapping $f : R \rightarrow R$ is said to be $*$ -commuting on R if $[f(x), x^*] = 0$ holds for all $x \in R$. The purpose of this paper is to describe the structure of a pair of additive mappings that are $*$ -commuting on a semiprime ring with involution. Furthermore, we study the commutativity of prime rings with involution satisfying any one of the following conditions: (i) $[d(x), d(x^*)] = 0$, (ii) $d(x) \circ d(x^*) = 0$, (iii) $d([x, x^*]) \pm [x, x^*] = 0$ (iv) $d(x \circ x^*) \pm (x \circ x^*) = 0$, (v) $d([x, x^*]) \pm (x \circ x^*) = 0$, (vi) $d(x \circ x^*) \pm [x, x^*] = 0$, where d is a nonzero derivation of R . Finally, an example is given to demonstrate that the condition of the second kind of involution is not superfluous.

Key words: Prime ring, semiprime ring, involution, additive mapping, $*$ -commuting mapping, skew $*$ -commuting mapping, derivation

1. Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. For $a, b \in R$, $[a, b]$ will be the element $ab - ba$ and $a \circ b$ the element $ab + ba$, respectively. However, given two subsets A and B of R , then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ where $a \in A$ and $b \in B$ and $A \circ B$ is defined similarly. Furthermore, \bar{A} will be the subring of R generated by A . A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$ and is called a semiprime ring in the case that $aRa = (0)$ implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known as a ring with involution or $*$ -ring. An element x in a ring with involution $*$ is said to be Hermitian if $x^* = x$ and skew-Hermitian if $x^* = -x$. The sets of all Hermitian and skew-Hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$; otherwise, it is said to be of the second kind. In the latter case $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$ where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal, i.e. $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [12], where further references can be found.

An additive mapping $d : R \rightarrow R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the

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last 30 years, several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps on R . The first result in this direction was due to Divinsky [11], who proved that a simple Artinian ring is commutative if it has commuting nontrivial automorphisms. Two years later, Posner [16] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined and extended these results in various directions (e.g., [2, 4, 5], where further references can be found).

Let R be a ring with involution $*$ and S be a nonempty subset of R . Following [1], a mapping f from R into itself is called $*$ -centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in S$, and is called $*$ -commuting on S if $[f(x), x^*] = 0$ for all $x \in S$. Similarly, we can define the notions of skew $*$ -centralizing and skew $*$ -commuting mappings as follows: a mapping f from R into R is called skew $*$ -centralizing on S if $f(x) \circ x^* \in Z(R)$ for all $x \in S$, and is called skew $*$ -commuting on S if $f(x) \circ x^* = 0$ for all $x \in S$. In [6], Brešar proved that if the additive mapping $f : R \rightarrow R$ is commuting on a prime ring R , then $f(x) = \lambda x + \mu(x)$, where $\lambda \in C$ the extended centroid of R and $\mu : R \rightarrow C$. Further, he extended this result for a semiprime ring in [8]. In the present paper, we present the $*$ -version of the above mentioned result in the setting of semiprime rings with involution $*$. Moreover, we also extended this result for the pair of additive mappings in a semiprime ring with involution $*$. In fact, we prove the following result: let R be a 2-torsion free semiprime ring with involution $*$ and let f, g be any pair of additive mappings of R such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$; then there exists $\lambda \in C$ and an additive mapping $\mu : R \rightarrow C$ such that $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$. Further, in the last section, we discuss the commutativity of a prime ring R with involution of the second kind involving a nonzero derivation d satisfying any one of the following properties: (i) $[d(x), d(x^*)] = 0$, (ii) $d(x) \circ d(x^*) = 0$, (iii) $d([x, x^*]) \pm [x, x^*] = 0$, (iv) $d(x \circ x^*) \pm (x \circ x^*) = 0$ (v) $d([x, x^*]) \pm (x \circ x^*) = 0$, (vi) $d(x \circ x^*) \pm [x, x^*] = 0$, for all $x \in R$. Finally, an example is provided, which states that the above results does not hold in the case that the involution is of the first kind.

2. On additive mappings in rings with involution

We begin this section with the following important result:

Proposition 2.1 *Let R be a semiprime ring with involution $*$ such that $\text{char}(R) \neq 2$. If an additive mapping $f : R \rightarrow R$ is $*$ -commuting on R , then there exists $\lambda \in C$ and an additive mapping $\mu : R \rightarrow C$ such that $f(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.*

Proof By hypothesis we have $[f(x), x^*] = 0$ for all $x \in R$. Replacing x by x^* , we obtain $[f(x^*), x] = 0$ for all $x \in R$. Let $g : R \rightarrow R$ be defined by $g(x) = f(x^*)$ for all $x \in R$. Then it is easy to verify that g is additive and therefore we have $[g(x), x] = 0$ for all $x \in R$. In view of [[8], Corollary 4.2], we conclude that $g(x) = \lambda x + \mu(x)$ for all $x \in R$ and hence $f(x^*) = \lambda x + \mu(x)$ where $\mu : R \rightarrow C$ and $\lambda \in C$. Replacing x by x^* in the last expression, we obtain $f(x) = \lambda x^* + \mu(x^*)$ for all $x \in R$. This implies $f(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$. This completes the proof. \square

Using a similar approach as in Proposition 2.1 and making use of Theorem 2 in [7] instead of Corollary 4.2 in [8], we have the following result.

Proposition 2.2 *Let R be a 2-torsion free semiprime ring with involution $*$. If an additive mapping $f : R \rightarrow R$ is skew $*$ -commuting on R , then $f = 0$.*

Theorem 2.3 *Let R be a 2-torsion free semiprime ring with involution $*$ and let f, g be the additive mappings of R commuting with $*$ such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$; then there exists $\lambda \in C$ and an additive mapping $\mu : R \rightarrow C$ such that $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$.*

Proof Suppose $f : R \rightarrow R$ is an additive mapping such that $f(x)x^* - x^*g(x) = 0$ for all $x \in R$. We use the fact that R being a semiprime ring, the intersection of all prime ideals in R is zero. Let P be the prime ideal such that the quotient ring R/P is of characteristic not two. Since P is a prime ideal, the quotient ring R/P is a prime ring. Now by the given hypothesis we have

$$f(x)x^* - x^*g(x) = 0 \tag{2.1}$$

for all $x \in R$. Replacing x by x^* in (2.1), we have

$$f(x^*)x - xg(x^*) = 0 \tag{2.2}$$

for all $x \in R$. Linearizing (2.1), we get

$$f(x)y^* + f(y)x^* - x^*g(y) - y^*g(x) = 0$$

for all $x, y \in R$. Replacing y by p^* , where $p \in P$, in the above equation, we get $f(p^*)x^* - x^*g(p^*) \in P$ for all $p \in P$ and $x \in R$. Replacing x by x^* , we get

$$f(p^*)x - xg(p^*) \in P \tag{2.3}$$

for all $p \in P$ and $x \in R$. In particular, $f(p^*)xy - xyg(p^*) \in P$; that is, $(f(p^*)x - xg(p^*))y - x(yg(p^*) - g(p^*)y) \in P$ for all $p \in P$ and $x \in R$. The first term is contained in P because of (2.3), and hence $x(yg(p^*) - g(p^*)y) \in P$ for all $p \in P$ and $x, y \in R$. P being the prime ideal of R , we have

$$yg(p^*) - g(p^*)y \in P \tag{2.4}$$

for all $p \in P$ and $y \in R$. Combining (2.3) and (2.4), we obtain

$$f(p^*)x - g(p^*)x \in P$$

for all $p \in P$ and $x \in R$. Let $h = f - g$, and then we have $h(p^*)x \in P$ for all $p \in P$ and $x \in R$. Since P is a prime ideal of R , we get $h(p^*) \in P$ for all $p \in P$. Therefore, h will induce an additive mapping F on R/P , defined by $F(x + P) = h(x^*) + P$ for all $x \in R$. Since both f and g commute with $*$ and making use of equation (2.2), it is easy to prove that F is skew-commuting on R/P . Therefore, in view of [[7], Theorem 2], $F = 0$. This implies that $h(x) \in P$ for all $x \in R$. That is, we have proved that the range of h is contained in any prime ideal P such that R/P is of characteristic different from two. We show that the intersection of all such prime ideal is zero. Now since R is a semiprime ring, there exists a family of prime ideals $\{P_a/a \in A\}$ such that $\cap_a P_a = 0$. Let $B = \{b \in A \mid \text{the char}(R/P_b) \text{ is not } 2\}$ and $C = \{c \in A \mid \text{the char}(R/P_c) \text{ is } 2\}$. Then $2x \in \cap_c P_c$ for every $x \in R$. Therefore, given any $x \in \cap_b P_b$, we have $2x \in (\cap_c P_c) \cap (\cap_b P_b) = \cap_a P_a = 0$, and so $x = 0$, since R is 2-torsion free. Thus, $\cap_b P_b = 0$. This proves that $h(x) = 0$; that is, $f(x) = g(x)$ for all $x \in R$. This gives because of (2.1) $g(x)x^* - x^*g(x) = 0$ for all $x \in R$. Therefore, in view of Theorem 2.1, we obtain $f(x) = g(x) = \lambda x^* + \mu'(x)$ for all $x \in R$, where $\mu'(x) = \mu(x^*)$. This completes the proof of the theorem. \square

3. On derivations in rings with involution

In [13], Herstein proved that a prime ring R of characteristic not two with a nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be commutative. Further, Daif [9] showed that for a 2-torsion free semiprime ring R admitting a derivation d such that $d(x)d(y) = d(y)d(x)$ for all $x, y \in I$, where I is a nonzero ideal of R and d is nonzero on I , then R contains a nonzero central ideal. Further, this result was extended by many authors (e.g., [2, 15], where further references can be found). In view of Herstein's result [13, Theorem 2] mentioned above, it is natural to ask what we can say about the commutativity of a prime ring if we replace y by x^* in the above condition. In this direction, we succeeded in establishing the following result:

Theorem 3.1 *Let R be a prime ring with involution $*$ of the second kind such that $\text{char}(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(x^*)] = 0$ for all $x \in R$. Then R is commutative.*

For developing the proof of the above result, we need the following lemma.

Lemma 3.2 [1, Lemma 2.1] *Let R be a prime ring with involution $*$ such that $\text{char}(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Proof [Proof of Theorem 3.1] By the assumption, we have

$$[d(x), d(x^*)] = 0 \tag{3.1}$$

for all $x \in R$. A linearization of (3.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \tag{3.2}$$

for all $x, y \in R$. Replacing y by xx^* in (3.2), we arrive at

$$\begin{aligned} 0 &= [d(x), d(x)x^* + xd(x^*)] + [d(x)x^* + xd(x^*), d(x^*)] \\ &= d(x)[d(x), x^*] + [d(x), x]d(x^*) + x[d(x), d(x^*)] \\ &+ [d(x), d(x^*)]x^* + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) \\ &= d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) \end{aligned}$$

for all $x \in R$. That is,

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \tag{3.3}$$

for all $x \in R$. Replacing x by $x + h'$, where $h' \in H(R) \cap Z(R)$, we obtain

$$d(h')[d(x), x^*] + [d(x), x]d(h') + d(h')[x^*, d(x^*)] + [x, d(x^*)]d(h') = 0.$$

This can be further written as

$$d(h')([d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]) = 0$$

for all $h' \in H(R) \cap Z(R)$ and $x \in R$. Since the center of a prime ring is free from zero divisors we get either $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Suppose

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{3.4}$$

Replacing h' by $(k')^2$ in (3.4), where $k' \in S(R) \cap Z(R)$, we get

$$0 = d(h') = d((k')^2) = d(k')k' + k'd(k') = 2d(k')k'.$$

Since $\text{char}(R) \neq 2$, we arrive at

$$d(k')k' = 0 \text{ for all } k \in S(R) \cap Z(R).$$

Now since the center of a prime ring is free from zero divisors, we get for each $k' \in S(R) \cap Z(R)$ either $d(k') = 0$ or $k' = 0$. Since $k' = 0$ implies $d(k') = 0$, we may write

$$d(k') = 0 \text{ for all } k' \in S(R) \cap Z(R). \tag{3.5}$$

Let $x \in Z(R)$. Since $\text{char}(R) \neq 2$, every $x \in Z(R)$ can be represented as $2x = h + k$, where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. This implies that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = 0$. Since $\text{char}(R) \neq 2$, we get

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{3.6}$$

Replacing y by $k'y$ in (3.2), where $k' \in S(R) \cap Z(R)$ and using (3.6), we arrive at

$$k'(-[d(x), d(y^*)] + [d(y), d(x^*)]) = 0$$

for all $k' \in S(R) \cap Z(R)$ and $x, y \in R$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-[d(x), d(y^*)] + [d(y), d(x^*)] = 0 \tag{3.7}$$

for all $x, y \in R$. On comparing (3.2) and (3.7), we obtain $2[d(x), d(y^*)] = 0$. Replacing y by y^* and using the fact that $\text{char}(R) \neq 2$, we conclude that $[d(x), d(y)] = 0$ for all $x, y \in R$. Therefore, in view of [13], we get that R is commutative. Now we consider the case

$$[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$$

for all $x \in R$. Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$, we get $4[d(k), h] = 0$. Since $\text{char}(R) \neq 2$, we obtain

$$[d(k), h] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{3.8}$$

Replacing h by k_0k' , where $k_0 \in S(R)$ and $k' \in S(R) \cap Z(R)$, we arrive at $([d(k), k_0])k' = 0$. Using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we get

$$[d(k), k_0] = 0 \text{ for all } k, k_0 \in S(R). \tag{3.9}$$

Now since $\text{char}(R) \neq 2$, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, so in view of equations (3.8) and (3.9), we are forced to conclude that

$$[d(k), x] = 0 \text{ for all } k \in S(R) \text{ and } x \in R. \tag{3.10}$$

That is, $d(k) \in Z(R)$ for all $k \in S(R)$. First we assume that $d(S(R)) = (0)$. Then we have $d(x - x^*) = 0$ for all $x \in R$. That is, $d(x) = d(x^*)$ for all $x \in R$. Now for $k \in S(R)$ and $x \in R$, we have

$0 = d(kx + x^*k) = kd(x) + d(x^*)k = kd(x) + d(x)k$ for all $x \in R$. This further implies that $k^2d(x) = d(x)k^2$ for all $x \in R$. Thus, by the theorem of [14], we conclude that $k^2 \in Z(R)$ for all $k \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, let $0 \neq k_0 \in S(R) \cap Z(R)$ and let k be an arbitrary element of $S(R)$. Then $(k + k_0)^2 = k^2 + k_0^2 + 2kk_0 \in Z(R)$ and hence $2kk_0 \in Z(R)$. Since $char(R) \neq 2$, we get $kk_0 \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. This further implies that $k \in Z(R)$ for all $k \in S(R)$ and hence R is normal. Thus, R is commutative in view of Lemma 3.2. Now suppose $d(S(R)) \neq (0)$. For $k_o \in S(R)$ with $d(k_o) \neq 0$ and $k \in [S(R), S(R)]$, we have $d(k_o k k_o) \in Z(R)$. The last expression can be written as $d(k_o) k k_o + k_o k d(k_o) \in Z(R)$, since $d([S(R), S(R)]) = 0$. Thus, $d(k_o)(k_o k + k k_o) \in Z(R)$ and hence $k_o k + k k_o \in Z(R)$ for all $k \in [S(R), S(R)]$. This implies that $d(k_o k + k k_o) \in Z(R)$ and hence $2d(k_o)k \in Z(R)$. Since $char(R) \neq 2$ and R is prime, the above relation yields that $k \in Z(R)$. That is, $[S(R), S(R)] \subseteq Z(R)$. Suppose $[S(R), S(R)] \neq (0)$ and let $k, k_o \in S(R)$ such that $[k, k_o] \neq 0$. Since $kk_o k \in S(R)$, we have $[k, k k_o k] = k[k, k_o]k = k^2[k, k_o] \in Z(R)$. This implies that $k^2 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$ as proved earlier. Therefore, R is commutative in view of Lemma 3.2. Now suppose $[S(R), S(R)] = (0)$. Since $\overline{S(R)^2}$ is both a Lie ideal and a commutative subring of R , by [12, Theorem 2.1.2], $k^2 \in Z(R)$ for all $k \in S(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. Thus, R is normal and hence R is commutative by Lemma 3.2. This completes the proof of the theorem. \square

If we replace the commutator by anticommutator in Theorem 3.1, the corresponding result also holds.

Theorem 3.3 *Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(x^*) = 0$ for all $x \in R$. Then R is commutative.*

Proof By the assumption, we have $d(x) \circ d(x^*) = 0$ for all $x \in R$. This can be further written as

$$d(x)d(x^*) + d(x^*)d(x) = 0 \tag{3.11}$$

for all $x \in R$. Replacing x by $x + y$ in (3.11), we get

$$d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) + d(y^*)d(x) = 0 \tag{3.12}$$

for all $x, y \in R$. Taking $x = h'$, where $h' \in H(R) \cap Z(R)$ in (3.11), we have $2d(h')^2 = 0$ for all $h' \in H(R) \cap Z(R)$. Since $char(R) \neq 2$, using the primeness of R we obtain

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{3.13}$$

Using the same technique that we used after (3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{3.14}$$

Replacing y by $k'y$, where $k' \in S(R) \cap Z(R)$ in (3.12) and using (3.14), we get

$$k'(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = 0$$

for all $k' \in S(R) \cap Z(R)$ and $x, y \in R$. Using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we have

$$-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x) = 0 \tag{3.15}$$

for all $x, y \in R$. On comparing (3.12) and (3.15), we obtain $2(d(x)d(y^*) + d(y^*)d(x)) = 0$. Since $\text{char}(R) \neq 2$ and on replacing y by y^* , we finally arrive at $d(x) \circ d(y) = 0$ for all $x, y \in R$. Hence, R is commutative in view of Theorem 4.3 in [2]. \square

In [10], Daif and Bell proved that if we let R be a prime ring admitting a derivation d and I be a nonzero ideal of R such that $d([x, y]) - [x, y] = 0$ for all $x, y \in I$ or $d([x, y]) + [x, y] = 0$ for all $x, y \in I$, then R is commutative. In the same paper they extended this result for the semiprime R . In the present section, we generalize the above-mentioned result in the setting of a prime ring with involution $*$ by replacing y by x^* .

Theorem 3.4 *Let R be a prime ring with involution $*$ of the second kind such that $\text{char}(R) \neq 2$. Let d be a nonzero derivation of R such that $d([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$. Then R is commutative.*

Proof We first consider the case

$$d([x, x^*]) - [x, x^*] = 0 \tag{3.16}$$

for all $x \in R$. A linearization of (3.16) yields that

$$d([x, y^*] + [y, x^*]) - ([x, y^*] + [y, x^*]) = 0 \tag{3.17}$$

for all $x, y \in R$. Replacing y by xx^* in (3.17), we get

$$\begin{aligned} 0 &= d(x[x, x^*]) + [x, x^*]x^* - x[x, x^*] - [x, x^*]x^* \\ &= d(x)[x, x^*] + xd([x, x^*]) + d([x, x^*])x^* \\ &+ [x, x^*]d(x^*) - x[x, x^*] - [x, x^*]x^* \\ &= d(x)[x, x^*] + [x, x^*]d(x^*) \end{aligned}$$

for all $x \in R$. That is,

$$d(x)[x, x^*] + [x, x^*]d(x^*) = 0 \tag{3.18}$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$, $k \in S(R)$, we obtain

$$d(h)[h, k] + d(k)[h, k] + [h, k]d(h) - [h, k]d(k) = 0 \tag{3.19}$$

for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $-k$ in (3.19), we obtain

$$-d(h)[h, k] + d(k)[h, k] - [h, k]d(h) - [h, k]d(k) = 0 \tag{3.20}$$

for all $h \in H(R)$ and $k \in S(R)$. Adding (3.19) and (3.20) and using the fact $\text{char}(R) \neq 2$, we get

$$d(k)[h, k] - [h, k]d(k) = 0 \tag{3.21}$$

for all $h \in H(R)$ and $k \in S(R)$. Replacing h by $k_o k'$ in (3.21), where $k_o \in S(R)$ and $k' \in S(R) \cap Z(R)$, we arrive at

$$(d(k)[k_o, k] - [k_o, k]d(k))k' = 0.$$

Since the center of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we get

$$d(k)[k_o, k] - [k_o, k]d(k) = 0 \tag{3.22}$$

for all $k, k_o \in S(R)$. Now since $char(R) \neq 2$ and every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.21) and (3.22), we are forced to conclude that

$$d(k)[x, k] - [x, k]d(k) = 0 \tag{3.23}$$

for all $k \in S(R)$ and $x \in R$. If $k \in Z(R)$, then $d(k) \in Z(R)$. Now if $k \notin Z(R)$, then the map defined by $D(x) = [x, k]$ for all $x \in R$ is a nonzero derivation and by (3.23) $d(k)D(x) = D(x)d(k)$. Thus, by the theorem of [14], we have $d(k) \in Z(R)$. We now have $d(k) \in Z(R)$ for all $k \in S(R)$. Hence, using the same technique as used in the proof of Theorem 3.1, we get the required result. This completes the proof of the theorem.

By the same arguments, we obtain the same conclusion in the case of $d([x, x^*]) + [x, x^*] = 0$ for all $x \in R$. This proves the theorem. □

If we replace the commutator by the anticommutator in Theorem 3.2, the corresponding result also holds.

Theorem 3.5 *Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x \circ x^*) \pm (x \circ x^*) = 0$ for all $x \in R$. Then R is commutative.*

Proof First we consider the case $d(x \circ x^*) - (x \circ x^*) = 0$ for all $x \in R$. This can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) - xx^* - x^*x = 0 \tag{3.24}$$

for all $x \in R$. Linearization of (3.24) yields that

$$\begin{aligned} d(x)y^* + xd(y^*) + d(y^*)x + y^*d(x) + d(y)x^* + yd(x^*) \\ + d(x^*)y + x^*d(y) - xy^* - y^*x - yx^* - x^*y = 0 \end{aligned} \tag{3.25}$$

for all $x, y \in R$. Replacing y by $h'x$ in (3.25), where $h' \in H(R) \cap Z(R)$ and making use of (3.24), we get $2(x \circ x^*)d(h') = 0$. Since $char(R) \neq 2$, we obtain

$$(x \circ x^*)d(h') = 0 \tag{3.26}$$

for all $x \in R$ and $h' \in H(R) \cap Z(R)$. Since the center of a prime ring is free from zero divisors we get either $d(h') = 0$ for all $h' \in H(R) \cap Z(R)$ or $x \circ x^* = 0$ for all $x \in R$. Suppose

$$d(h') = 0 \text{ for all } h' \in H(R) \cap Z(R). \tag{3.27}$$

Using the same technique as used after (3.4), we finally arrive at

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{3.28}$$

Replacing y by $y_o \in Z(R)$ in (3.25) and using (3.29), we arrive at

$$d(x)y_o^* + y_o^*d(x) + y_o d(x^*) + d(x^*)y_o - xy_o^* - y_o^*x - y_o x^* - x^*y_o = 0 \tag{3.29}$$

for all $x \in R$ and $y_o \in Z(R)$. In particular, taking $y_o = h_o \in H(R) \cap Z(R)$ in (3.29), we obtain

$$(d(x + x^*) - (x + x^*))h_o = 0$$

for all $x \in R$ and $h_o \in H(R) \cap Z(R)$. Using the primeness of R we get either $d(x + x^*) - (x + x^*) = 0$ for all $x \in R$ or $H(R) \cap Z(R) = (0)$, but $H(R) \cap Z(R) = (0)$ implies that $S(R) \cap Z(R) = (0)$, which gives a contradiction since we have assumed $S(R) \cap Z(R) \neq (0)$. Therefore, we are left with the case

$$d(x + x^*) - (x + x^*) = 0 \tag{3.30}$$

for all $x \in R$. Replacing x by $h + k$ in (3.30) where $h \in H(R)$, $k \in S(R)$, we obtain

$$d(h) = h \text{ for all } h \in H(R). \tag{3.31}$$

Taking $y_o = k_o \in S(R) \cap Z(R)$ in (3.29), we obtain

$$(d(x - x^*) - (x - x^*))k_o = 0$$

for all $x \in R$ and $y_o \in Z(R)$. Again using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$d(x - x^*) - (x - x^*) = 0 \tag{3.32}$$

for all $x \in R$. Replacing x by $h + k$ in (3.32) where $h \in H(R)$, $k \in S(R)$, we obtain

$$d(k) = k \text{ for all } k \in S(R). \tag{3.33}$$

Since every $x \in R$ can be represented as $2x = h + k$, $h \in H(R)$, $k \in S(R)$, it follows from (3.31) and (3.33) that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = h + k = 2x$. Since $\text{char}(R) \neq 2$, we obtain $d(x) = x$ for all $x \in R$. Therefore, in view of the relation (3.28), we get $Z(R) = (0)$, which gives a contradiction. Therefore, we are left with the case $x \circ x^* = 0$ for all $x \in R$. Linearization of the last relation yields that $xoy^* + yox^* = 0$ for all $x, y \in R$. Replacing y by x^2 and using the fact that $x \circ x^* = 0$, we obtain

$$\begin{aligned} 0 &= xo(x^*)^2 + x^*ox^2 \\ &= (xox^*)x^* - x^*[x, x^*] + (x^*ox)x - x[x^*, x] \\ &= x[x, x^*] - x^*[x, x^*] \\ &= (x - x^*)[x, x^*] \end{aligned}$$

for all $x \in R$. Substituting $h + k$ for x , where $h \in H(R)$, $k \in S(R)$, we get $2k([k, h] - [h, k]) = 0$ and hence $4k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Since $\text{char}(R) \neq 2$, the above relation forces that $k[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $k + k_1$ where $k_1 \in S(R) \cap Z(R)$, we obtain $k_1[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we conclude that $[h, k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. That is, R is normal. Hence, application of Lemma 3.2 yields the required result. That is, R is commutative.

By the same arguments, we obtain the same conclusion in the case of $d(x \circ x^*) + (x \circ x^*) = 0$ for all $x \in R$. This proves the theorem. □

Theorem 3.6 *Let R be a prime ring with involution $*$ of the second kind such that $\text{char}(R) \neq 2$. Let d be a nonzero derivation of R such that $d([x, x^*]) \pm (x \circ x^*) = 0$ for all $x \in R$. Then R is commutative.*

Proof By the given hypothesis, we have

$$d([x, x^*]) - (x \circ x^*) = 0 \tag{3.34}$$

for all $x \in R$. Linearizing (3.34), we get

$$d([x, y^*]) - (x \circ y^*) + d([y, x^*]) - (y \circ x^*) = 0 \tag{3.35}$$

for all $x, y \in R$. Replacing y by xx^* in (3.35) and making use of (3.34), we obtain

$$d(x)[x, x^*] + [x, x^*]d(x^*) + xd([x, x^*]) - 2x^2x^* + x[x, x^*] = 0$$

for all $x \in R$. This can be further written as $d(x)[x, x^*] + [x, x^*]d(x^*) + xd([x, x^*]) - x(x \circ x^*) = 0$. Using (3.34) again we finally get

$$d(x)[x, x^*] + [x, x^*]d(x^*) = 0 \tag{3.36}$$

for all $x \in R$. The last expression is the same as the equation (3.18) and hence, by using a similar approach as we have used after (3.18) in the proof of Theorem 3.5, we get the required result.

By the same arguments, we obtain the same conclusion in the case of $d[x, x^*] + (x \circ x^*) = 0$ for all $x \in R$. This proves the theorem. □

Using a similar approach as in Theorem 3.6, we have the following result.

Theorem 3.7 *Let R be a prime ring with involution $*$ of the second kind such that $\text{char}(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x \circ x^*) \pm [x, x^*] = 0$ for all $x \in R$. Then R is commutative.*

Finally, let us write an example that shows that the restriction of the second kind of involution in Theorem 3.1 and Theorem 3.4 is not superfluous.

Example 3.1 *Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$. Of course R with matrix addition and matrix multiplication is a prime ring. Define mappings $d : R \rightarrow R$ and $*$: $R \rightarrow R$ such that $d \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Obviously, $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in Z \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d is nonzero and the following conditions are satisfied: (i) $[d(x), d(x^*)] = 0$, (ii) $d([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$. However, R is not commutative. Hence, the hypothesis of the second kind of involution is crucial in Theorem 3.1 and Theorem 3.4.*

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