Turkish Journal of Physics

Volume 24 | Number 3 Article 28

1-1-2000

Skyrmions from Harmonic Maps

WOJTEK ZAKRZEWSKI

Turk J Phy
24 (2000) , 489 – 500.

Skyrmions from Harmonic Maps*

Wojtek ZAKRZEWSKI
Dept. of Mathematical Sciences, Univ. of Durham,
Durham, DH1 3LE, UK

Received 21.04.2000

Abstract

We describe some relations between solitonic solutions of various models in different dimensions. We present some examples and then concentrate on some of our recent work (performed in collaboration with Ioannidou and Piette) [1] [2] which shows how some harmonic maps from S^2 to CP^{N-1} can be used to find nontrivial spherically symmetric static solutions of the $SU(N)$ Skyrme model in 3 dimensions and to generate some of its low energy field configurations.

1. Introduction

Solitons (or, in general, extended structures) arise as classical solutions of many models in (1+1) dimensions. A good and exhaustive review of their properties can be found in the book by Ablowitz and Clarkson [3]. In fact there are two classes of solitons, those that arise in integrable models and those that are topological in nature. Those in the first class have been studied in much detail. Many of their properties can be proved by exploiting the integrability of the models. In physical applications, hoverver, one often encounters the solitons of the second class. Such solitons are based on topological considerations; the basic field takes values in a manifold with topologically nontrivial properties which are responsible for the existence of solitons. In this talk I want to concentrate on the discussion of such topological solitons.

The fields describing them correspond to maps

$$\begin{array}{ccc} S^1 & \rightarrow & S^1 \\ S^2 & \rightarrow & S^2 \\ S^3 & \rightarrow & S^3 \end{array} \qquad (1)$$

*Talk presented in Regional Conference on Mathematical Physics IX held at Feza Gürsey Institute, Istanbul, August 1999.

etc.

1.1. One dimension

In this case our solitons correspond to $S^1 \to S^1$ maps. An example of such a map is a soliton (kink) of the Sine-Gordon model, or of $\lambda\phi^4$ model etc. In each of these cases we have a field

$$\phi(x), \qquad x \in (-\infty, \infty) \tag{2}$$

which, when $x \to \pm\infty$, takes the value corresponding to the vacuum of the model, *ie*

$$\phi(-\infty) = -\phi(+\infty) = \pm A, \tag{3}$$

where $V(\pm A) = 0$ for the $\lambda\phi^4$ model, or

$$\phi(-\infty) = 2k\pi, \quad \phi(+\infty) = \phi(-\infty) + 2n\pi \tag{4}$$

for the Sine-Gordon model.

A typical soliton (kink) solution is then given by $\phi(x)$ as shown in fig.1.

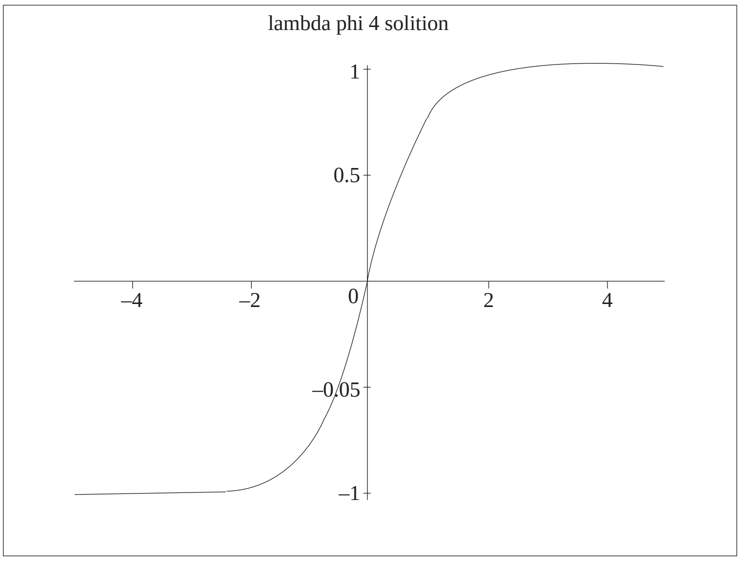

Figure 1. A typical soliton - solution of $\lambda\phi^4$ model.

1.2. Two dimensions

In this case the topological solitons correspond to

$$S^2 \to S^2 \tag{5}$$

maps.

An example is the S^2 σ model (also called the $O(3)$ model) and its generalisations[4]. Its solitonic solutions are often called baby skyrmions.

In the most common formulation of this model the basic field is $\vec{n}$, satisfying $\vec{n}^2 = 1$ and so the static fields of the model involve maps

$$R^2 \to (n_1, n_2, n_3). \tag{6}$$

However, the condition of finiteness of energy of the soliton imposes a further condition

$$\partial_x n_i = \partial_y n_i \to 0 \qquad \text{as} \quad x^2 + y^2 \to \infty \tag{7}$$

and so R^2 becomes compactified to S^2.

In this case it is more convenient to introduce a complex field W (a stereographic projection of the $\vec{n}^2 = 1$ sphere onto the complex plane) defined by

$$W \;=\; \frac{n_1 + in_2}{1 + n_3} \tag{8}$$

and then consider W as a function of (x, y), (r, θ) or $(z, \bar{z})$, where $z = x + iy$ and (r, θ) are polar coordinates. As $r \to \infty$

$$W \;\to\; \lim_{r\to\infty} R(r, \theta) e^{i\psi(r,\theta)} \tag{9}$$

and so we see that if $\lim_{r\to\infty} \psi(r, \theta) \;=\; \varphi(\theta)$ we have (at $r \to \infty$) a map $S^1 \to S^1$ corresponding to the mapping of the circle at spatial infinity ($r \to \infty$) onto the phase of W.

Can we exploit this observation to relate our $S^2 \to S^2$ maps to the $S^1 \to S^1$ ones?

The answer to this question is affirmative as we will demonstrate in the next section. Moreover, this observation can be generalised further to other $S^k \to S^k$ maps and in the following section we report some recent results exploting these ideas to generate skyrmion fields from $S^2 \to S^2$ maps.

2. $S^1 \to S^1$ and $S^2 \to S^2$ maps

Let us first discuss the reduction, *ie* how to use the $S^2 \to S^2$ maps to obtain topologically nontrivial $S^1 \to S^1$ fields.

In this case, we want to use the W field and to consider its phase at spatial ∞. So we put

$$W \;=\; R\, e^{i\psi} \tag{10}$$

and define

$$A_\mu \;=\; i\,\frac{\partial \psi}{\partial x^\mu}\,\frac{R^2}{1 + R^2}. \tag{11}$$

This A_μ field is really the gauge field of the classically equivalent CP^1 formulation of the S^2 model. ($A_\mu = (z^\dagger \partial_\mu z)$; $\vec{n} = z^\dagger \vec{\sigma} z$, where $\vec{\sigma}$ are Pauli matrices).

Then we can introduce

$$\phi(x) \sim \int_C A_\mu \, dx^\mu, \tag{12}$$

where the curve C is a straight line at x parallel to the y axis[5]. Now it is easy to verify that if we take

$$W = \lambda z, \tag{13}$$

which corresponds to one soliton solution of the S^2 model, we get

$$\phi(x) \sim \frac{\mu x}{\sqrt{1+\lambda^2 x^2}}, \tag{14}$$

whose plot, shown in fig.2, resembles our kink (fig.1).

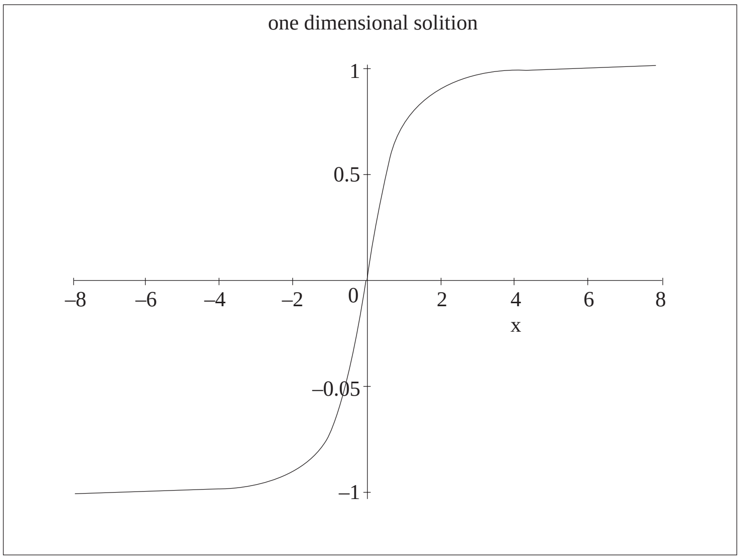

Figure 2. Our soliton solution - given by (14).

Note, however, that $\phi(x)$ is **not** an exact solution of the Sine-Gordon or $\lambda\phi^4$ model but, by conveniently choosing λ in (13), we can make it a very good approximation to the solutions of these models.

In fact, this ϕ field is an exact static soliton-like solution of the $\alpha\phi^6$ model, *ie* model based on the lagrangian density

$$L = \frac{1}{2}(\frac{\partial\phi}{\partial t})^2 - \frac{1}{2}(\frac{\partial\phi}{\partial x})^2 - \frac{|\lambda^2\phi^2 - \mu^2|^3}{\mu^4}, \tag{15}$$

whose potential $V = \frac{|\lambda^2\phi^2-\mu^2|^3}{\mu^4}$ is plotted in fig.3.

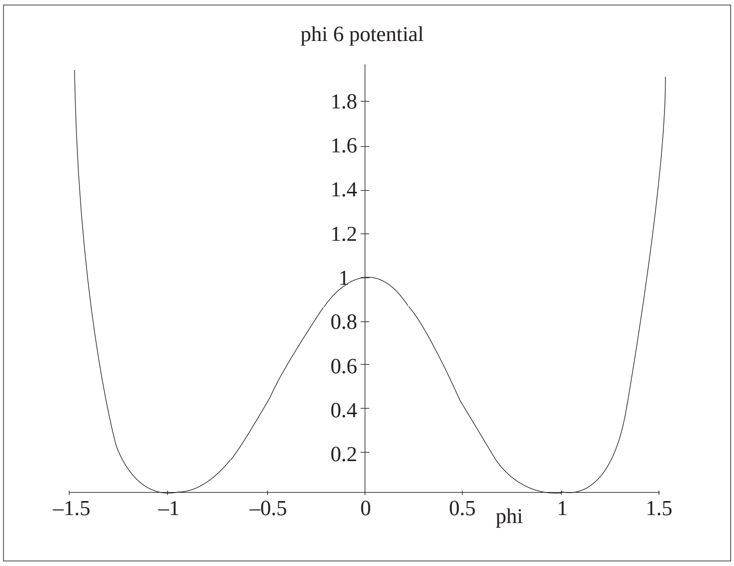

Figure 3. A potential for which our soliton (14) is an exact solution

Can we now go the other way and use $S^1 \to S^1$ maps to generate topologically nontrivial $S^2 \to S^2$ maps?

Yes, we can put

$$W(r,\theta) \;=\; f(r)\, e^{i\phi(\theta)} \tag{16}$$

and impose that $\phi(\theta)$ is a $S^1 \to S^1$ map and that $f(r)$, the "profile function", satisfies

$$f(r) \to \begin{cases} 0 & r \to \infty \\ \infty & r \to 0 \end{cases} \tag{17}$$

or other way round.

An example here is

$$W \;=\; r\, e^{i\theta}. \tag{18}$$

Note that this works well if $\phi(\theta)$ describes the simplest case, *eg* $\phi(\theta) = \theta$. Note also that the energy density of the resultant $S^2 \to S^2$ soliton is radially symmetric. Finally, observe that if $\phi(\theta) = \theta$ our new field is an exact solution of the S^2 model if $f(r)$ satisfies the appropriate ordinary differential equation; otherwise it is a good approximation to a solution of this model.

We can go further and look at

$$S^3 \to S^3, \qquad S^4 \to S^4, \ldots$$

maps. Such fields arise in many physical applications; *ie* they describe magnetic monopoles and bubbles, cosmic textures etc, skyrmions, instantons and many other topologically nontrivial field configurations.

Many of these, physically relevant structures, have their topology related to the $S^3 \to S^3$ maps, *eg* magnetic monopoles or skyrmions.

Again, ona can relate these topological structures to each other. In fact, already in 1978 Manton[6] showed that by assuming no dependence on one variable in the $SU(2)$ self dual Yang Mills equation one can obtain an exact solution describing one $SU(2)$ monopole. Later, Atiyah and Manton[7] showed also how to derive approximate solutions of $SU(2)$ skyrmions from the $SU(2)$ instantons. In both these cases one performs a reduction. Last year Houghton, Manton and Sutcliffe[8] went the other way and showed how to use some $S^2 \to S^2$ maps to generate good approximations to $SU(2)$ multiskyrmion fields. Recently this construction was generalised further[1, 2] to $SU(N)$ Skyrme fields. So in the rest of this talk I will discuss this generalisation. To do this, I have to say a few words about the Skyrme model.

3. Skyrme model

This model was initially proposed as a theory of strong interactions of hadrons [9], but later, it was shown to be the low energy limit of QCD in the large N_c limit [10]. As such it presents an opportunity to understand nuclear physics as a low energy limit of quantum chromodynamics (QCD). Here we make the assumption that topologically nontrivial solutions of the Skyrme model, known as skyrmions, can be identified with classical ground states of light nuclei. However a better understanding of the structure and dynamics of multi-skyrmion configurations is required and quantum corrections reliably computed, before we can be sure as to the validity of this claim.

The $SU(N)$ Skyrme model involves fields which take value in $SU(N)$; *ie* are described by $SU(N)$ valued functions of $\vec{x}$ and t. Its static solutions provide us with multiskyrmions and in the remaining part of this talk we will discuss some of our recent work [1][2].

Static fields describing multiskyrmions are stationary points (either maxima or saddle points) of the static energy functional, which is given by

$$E = \int_{R^3} \left\{ -\frac{1}{2}\mathrm{tr}\left(\partial_i U\, U^{-1}\right)^2 - \frac{1}{16}\mathrm{tr}\left[\partial_i U\, U^{-1},\ \partial_j U\, U^{-1}\right]^2 \right\} d^3\vec{x}. \tag{19}$$

where $U(\vec{x}) \in SU(N)$. Here, for simplicity, all the mass terms have been set to zero. All the discussion can be generalised to include various mass terms as indicated in [1].

The equations for static multiskyrmions are therefore

$$\partial_i \left(\partial_i U\, U^{-1} - \frac{1}{4}\left[\partial_j U\, U^{-1}, \left[\partial_j U\, U^{-1}, \partial_i U\, U^{-1}\right]\right] \right) = 0. \tag{20}$$

Finiteness of the energy functional requires that $U(\vec{x})$ approaches a constant matrix at spatial infinity, which without any loss of generality, can be chosen to be the identity matrix.

Since $U \to I$ as $|\vec{x}| \to \infty$ $U(\vec{x})$ is a mapping from $S^3 \to SU(N)$ which can be classified by the integer valued winding number

$$B(U) = \frac{1}{24\pi^2} \int_{R^3} \varepsilon_{ijk} \, \mathrm{tr} \left(\partial_i U \, U^{-1} \partial_j U \, U^{-1} \partial_k U \, U^{-1} \right) d^3\vec{x}, \tag{21}$$

which is a topological invariant. This winding number classifies the solitonic sectors in the model [9] and $B(U)$ is to be identified with the baryon number of the field configuration.

Until very recently most of the studies of the Skyrme model have concentrated on the $SU(2)$ version of the model and its embeddings into $SU(N)$. The simplest nontrivial classical solution involves a single skyrmion ($B = 1$) and has already been discussed by Skyrme [9]. Its energy density is radially symmetric and, as a result, using the so-called hedgehog ansatz one can reduce (20) to an ordinary differential equation, which then has to be solved numerically.

Many solutions with $B > 1$ of the $SU(2)$ model have also been computed numerically and, in all cases, the solutions have turned out to be very symmetrical (cf. Battye et al. [11] and references therein). These studies have shown that the energy density of the two skyrmion solution forms a torus, while the energy density of the $B = 3$ solution has the symmetry of a tetrahedron. For larger B the solutions describe semi-radially symmetric structures in which skyrmions are split into connected parts which are all located on a spherical hollow shell. These observations have led Houghton et al. [8] to their ansatz (of exploiting $S^2 \to S^2$ to generate $SU(2)$ skyrmions).

Can one go beyond $SU(2)$ skyrmions and consider solutions of other $SU(N)$ Skyrme models? Moreover, are there any finite energy solutions of the $SU(N)$ ($N > 2$) model which are not embeddings of the $SU(2)$ model and, if they exist, whether they have lower energies than their $SU(2)$ counterparts.

Some such *non-embedding* solutions have already been found and studied [12][13] but it is generally believed that the $SU(2)$ embeddings provide the lowest energy states. Recently a generalisation of the ansatz of Houghton et al[8] has been found [1][2]. It gives some new solutions of $SU(N)$ Skyrme models (only radially symmetric) and also presents a general procedure for finding low energy field configurations of a given $SU(N)$ Skyrme model. In the remainder of this talk we will discuss the main ideas of this procedure.

4. Skyrmions from Harmonic Maps

The idea of Houghton et al. [8] was to exploit the $S^2 \to S^2$ maps and so they put U, the $SU(2)$ field in the form:

$$U(r, \theta, \phi) = \exp(ig(r) \, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}), \tag{22}$$

where (r, θ, ϕ) are the usual polar coordinates on $\mathbf{R^3}$, and

$$\hat{\boldsymbol{n}} = \frac{1}{1 + |R|^2} (2\Re(R), 2\Im(R), 1 - |R|^2), \tag{23}$$

where R are some rational functions of $\xi = \tan(\theta/2)e^{i\phi}$ *ie* the variables on the r =const S^2 sphere and where $g(r)$ is a real function satisfying the boundary conditions: $g(0) = \pi$

and $g(\infty) = 0$. In other words, the configurations (22) factorise into products of a radial profile function $g(r)$ and a harmonic map from the two dimensional sphere, which can be identified with concentric spheres, centered at the origin, in $\mathbf{R^3}$, into an S^2 submanifold of $SU(2) \equiv S^3$. Moreover, it is easy to check that the baryon number B is given by the degree of the harmonic map $\hat{\boldsymbol{n}}$.

The idea of the Houghton et al. was generalised in [1] to $SU(N)$ by modifying their ansatz to

$$U(r,\theta,\phi) = e^{2ig(r)(P-I/N)} = e^{-2ig(r)/N}\left(I + (e^{2ig}-1)P\right), \tag{24}$$

where P is a $N \times N$ Hermitian projector which depends only on the angular variables (θ,ϕ). When $N = 2$ this expression coincides with (22). Note that, the matrix P is a harmonic map from S^2 into CP^{N-1}. Hence it is convenient, instead of using the polar coordinates, to map the sphere onto the complex plane via a stereographic projection and so use the complex coordinate ξ and its conjugate. Thus, P can be written as

$$P(V) = \frac{V \otimes V^\dagger}{|V|^2}, \tag{25}$$

where V is a N-component complex vector (dependent on ξ and $\bar{\xi}$).

These ideas were generalised further in [2] by involving more projectors. To do this it was recalled [14] that in the N dimensional complex space there is a “natural” set of projectors. They are constructed from the $S^2 \to CP^{N-1}$ maps in the following way:

Write each projector P in the form (25). Now consider a series of such projectors P by changing V. Then for the first projector take $V = f(\xi)$, *ie* an analytic vector of ξ. For the other projectors we take V's which are obtained from the original V by differentiation and Gramm Schmidt orthogonalisation. This construction corresponds to the introduction of an operator P_+ (defined by its action on any vector $v \in \mathrm{C}^N$[14]) as

$$P_+ v = \partial_\xi v - \frac{v\,(v^\dagger\,\partial_\xi v)}{|v|^2}, \tag{26}$$

and then define further vectors $P_+^k v$ by induction: $P_+^k v = P_+(P_+^{k-1} v)$.

Then as our new projectors we take projectors given by (25) with V given by $P_+^k f$, where $f = f(\xi)$, *ie*

$$P_0 = P(f), \qquad P_1 = P(P_+ f), \qquad P_2 = P(P_+^2 f), \qquad \dots, \qquad P_k = P(P_+^k f). \tag{27}$$

We note, that due to the orthogonality of P_i we have $\sum_{k=0}^{N-1} P_k = 1$. This orthogonality follows from the following properties of $P_+^k f$ when f is holomorphic:

$$(P_+^k f)^\dagger\, P_+^l f = 0, \qquad k \neq l, \tag{28}$$

$$\partial_{\bar{\xi}}\left(P_+^k f\right) = -P_+^{k-1} f \frac{|P_+^k f|^2}{|P_+^{k-1} f|^2}, \qquad \partial_\xi \left(\frac{P_+^{k-1} f}{|P_+^{k-1} f|^2}\right) = \frac{P_+^k f}{|P_+^{k-1} f|^2}. \tag{29}$$

Note that for $SU(N)$ the last projector P_{N-1} in this sequence corresponds to an anti-analytic vector; (*ie* , up to an overall constant, the components of $V = P_+^{N-2} f$ are functions of only $\bar{\xi}$).

As we have $\sum_{k=0}^{N-1} P_k = 1$ we see that for $SU(N)$ only $N-1$ projectors are independent so we can choose, say, the first $N-1$ of them *ie* : P_0, P_1 ... P_{N-2}.

Then the multiprojector ansatz of [2] involves the introduction of $N-1$ profile functions and the field of the form:

$$
\begin{aligned}
U &= \exp\left[-ig_0\left(\frac{I}{N}-P_0\right)-ig_1\left(\frac{I}{N}-P_1\right)-...-ig_{N-2}\left(\frac{I}{N}-P_{N-2}\right)\right] \\
&= e^{-ig_0/N}(I+A_0P_0)\, e^{-ig_1/N}(I+A_1P_1)\,\ldots\, e^{-ig_{N-2}/N}(I+A_{N-1}P_{N-2}), \quad (30)
\end{aligned}
$$

where $g_k = g_k(r)$, for $k = 0, \ldots, N-2$, are the profile functions and $A_k = e^{ig_k} - 1$. The vector V in the projector P_i is given by $V = P_+^i f$. The original generalisation[1] corresponds to putting all the profile functions, but the first one, equal to zero.

However, the vector f cannot be arbitrary. In fact, in [2] it is shown that if we take

$$N = 2, \qquad f = (1,\, \xi)^t, \tag{31}$$

$$N = 3, \qquad f = (1,\, \sqrt{2}\,\xi,\, \xi^2)^t, \tag{32}$$

$$N = 4, \qquad f = (1,\, \sqrt{3}\xi,\, \sqrt{3}\,\xi^2,\, \xi^3)^t, \tag{33}$$

$$N = n, \qquad f = (f_0,\, f_1, \ldots,\, f_{r-1})^t : \qquad f_k = \xi^k \sqrt{C_{k+1}^{n-1}}, \tag{34}$$

where C_{k+1}^{n-1} denotes the binomial coefficients then the moduli of the N-dimensional vectors $P_+^k f$ $(k = 0, 1, .N-1)$ are all powers of $(1 + |\xi|^2)$. In fact, we have

$$|P_+^k f|^2 = k!(N-1)(N-2)\cdots(N-k)(1+|\xi|^2)^{N-2k-1}. \tag{35}$$

5. Some Results

5.1. Exact Solutions of the $SU(N)$ models

These generalisations have led to several interesting results. All details can be found in [1] and [2]; here we briefly mention only some of them.

First of all, in [2] it was shown that when $N-1$ projectors (for the $SU(N)$ model) are used the procedure reduces the full equations to $N-1$ coupled and nonlinear ordinary equations for the profile functions $g_k(r)$. As for $SU(N)$ $\sum_{k=0}^{N-1} P_k = 1$, we can, say, use the first $N-1$ projectors Then, after the equations for the profile functions have been solved the procedure gives us exact solutions of $SU(N)$ Skyrme models. The equations for the profile functions are nonlinear and coupled and they have to be solved numerically.

It turns out that the key ingredient in this construction is the form of the initial analytic vectors f (34) and the property (35). Detailed calculations reported in [2] show that when (30) is substituted into (20) all the derivative terms of the projectors cancel and all the novanishing terms in the equation of motion are proportional to $\frac{1}{N} - P_i$ (moreover, all the residual dependence on ξ and $\bar{\xi}$ cancels). As there are $N-1$ such independent terms their coefficients provide us with the equations for the profile functions $g_i(r)$. Note that had we taken fewer projectors or different functions f this would not have worked.

The equations for the profile functions are rather complicated and can be found in [2].

5.2. Approximate Solutions of the $SU(N)$ model

When one takes a more general analytic vector f (for a general $SU(N)$) the cancellation does not take place and the procedure does not give a solution of the model. Of course, (24) gives then only a field configuration which, by choosing the parameters of the vector f can be made to be of low energy. Then, one can hope that these field configurations are close to the true solutions of the model.

The energy of the field (24) is given by the expression

$$E = \frac{1}{3\pi}\int dr\left(A_N\, g_r^2\, r^2 + 2\mathcal{N}\,\sin^2 g\,(1+g_r^2) + \mathcal{I}\,\frac{\sin^4 g}{r^2}\right), \tag{36}$$

where $A_N = \frac{2}{N}(N-1)$ and

$$\mathcal{N} = \frac{i}{2\pi}\int d\xi\, d\bar{\xi}\, \mathrm{tr}\left(|\partial_\xi P|^2\right), \tag{37}$$

$$\mathcal{I} = \frac{i}{4\pi}\int d\xi\, d\bar{\xi}\,(1+|\xi|^2)^2\, \mathrm{tr}\left([\partial_\xi P,\, \partial_{\bar{\xi}} P]^2\right). \tag{38}$$

As the integrals $\mathcal{N}$ and $\mathcal{I}$ in (36) are independent of r, to have a field of low energy (36) can be minimised by first minimising $\mathcal{N}$ and $\mathcal{I}$ as functions of the parameters of P and then with respect to the profile function g.

Note that the baryon number of (24) is given by

$$\begin{aligned} B &= \frac{i}{\pi^2}\int d\xi\, d\bar{\xi}\, \mathrm{tr}\left(P\,[\partial_\xi P,\, \partial_{\bar{\xi}} P]\right)\int_0^\infty dr\, \sin^2 g\, g_r \\ &= \frac{i}{2\pi}\int d\xi\, d\bar{\xi}\, \mathrm{tr}\left(P\,[\partial_{\bar{\xi}} P,\, \partial_\xi P]\right), \end{aligned} \tag{39}$$

which is the topological charge of the two-dimensional $CP^{(N-1)}$ sigma model.

In [1] various field configurations of $SU(3)$ model were studied in detail. In all cases the lowest energies were quite close to the energies of $SU(2)$ embeddings but in each case slightly higher. The field configurations had, in most cases, very different symmetries (as seen from their energy of baryon charge densities). So both, the embeddings and the new configurations, could be used as starting points of possible minimisation programs for $SU(3)$ and higher Skyrme models. Only when such work has been performed it will be possible to say whether the new configurations are close to real solutions, how many such solutions there are and which of them are global maxima. Clearly, these are some of the important questions which should be answered in the near future.

6. Conclusions

In this talk we have discussed various relations between topological solitons in different dimensions. Apart from their intrinsic mathematical interest these relations can be used to produce interesting ansatzae for seeking solitonic solutions of some models or for obtaining good approximations to such solutions (which could then be used as starting points for numerical work).

Most of our discussion involved the review of some of our recent work[1] and [2]. We have shown how to construct radially symmetric solutions of the $SU(N)$ Skyrme model. In the general case, such solutions depend on $N-1$ profile functions which have to be determined numerically. In some cases symmetries of the resultant expressions can be exploited to reduce the number of such functions.

We have also shown that when we restrict our attention to a given $SU(N)$ model, our construction gives us nontrivial low energy field configurations. These configurations are not solutions of the full equations of motion but we expect them to be close to true solutions of the model. As such, they should be good starting points for further numerical work designed to find true solutions (based on some relaxation techniques). Finally, the projector and multiprojector ideas has also recently been used in the the context of finding expressions for the Higgs field in monopole systems [15].

Acknowledgements

I would like to thank Y. Nutku for inviting me to the IX Regional Conference on Mathematical Physics and for the hospitality in Istanbul. I would also like to thank T. Ioannidou and B. Piette for collaborating with me on the construction of skyrmions from harmonic maps.

References

[1] T. Ioannidou, B. Piette and W.J. Zakrzewski, *Low Energy States in the $SU(N)$ Skyrme Model*, hep-th 9811071 to be published in Journal of Mathematical Physics.

[2] T. Ioannidou, B. Piette and W.J. Zakrzewski, *Spherically Symmetric Solutions of the $SU(N)$ Skyrme Models*, hep-th 9904160. to be published in Journal of Mathematical Physics

[3] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, CUP (1991).

[4] see *eg* W.J. Zakrzewski, Low dimensional sigma models (IOP, 1989) and references therein.

[5] P.M. Sutcliffe, Phys. Lett. **283B**, 85 (1992); G.N. Stratopoulos and W.J. Zakrzewski, Zeit. Phys. **C59**, 307 (1993).

[6] N.S. Manton, Nucl. Phys. **B135**, 319 (1978).

[7] M.F. Atiyah and N.S. Manton, Phys. Lett. **222B**, 438 (1989).

[8] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, Nucl. Phys. B **510**, 507 (1998).

[9] T.H.R. Skyrme, Nucl. Phys. **31**, 556 (1962).

[10] E. Witten, Nucl. Phys. B **223**, 422 (1983).

[11] R.A. Battye and P.M Sutcliffe, Phys. Rev. Lett. **79**, 363 (1997).

[12] A.P. Balachandran, A. Barducci, F. Lizzi, V.G.J. Rodgers and A. Stern, Phys. Rev. Lett. **52**, 887 (1984).

[13] V.B. Kopeliovich, B.E. Schwesinger and B.E. Stern, JETP Lett. **62**, 185-90 (1995).

[14] A. Din and W.J. Zakrzewski, Nucl. Phys. **B 174**, 397 (1980).

[15] T. Ioannidou and P.M. Sutcliffe, UKC/IMS/99/07 preprint, hep-th/99/3183.