

1-1-2016

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Recommended Citation

ZHU, GUANGJUN (2016) "A lower bound for Stanley depth of squarefree monomial ideals," *Turkish Journal of Mathematics*: Vol. 40: No. 4, Article 10. <https://doi.org/10.3906/mat-1505-90>
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A lower bound for Stanley depth of squarefree monomial ideals

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Received: 27.05.2015

Accepted/Published Online: 06.11.2015

Final Version: 16.06.2016

Abstract: Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K in n variables and I a squarefree monomial ideal of S with Schmitt–Vogel number $sv(I)$. In this paper, we show that $\text{sdepth}(I) \geq \max\{1, n - 1 - \lfloor \frac{sv(I)}{2} \rfloor\}$, which improves the lower bound obtained by Herzog, Vladioiu, and Zheng. As some applications, we show that Stanley’s conjecture holds for the edge ideals of some special n -cyclic graphs with a common edge.

Key words: Stanley depth, Stanley conjecture, monomial ideal, Schmitt–Vogel number, n -cyclic graph

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K in n variables and M a finitely generated \mathbb{Z}^n -graded S -module. For a homogeneous element $u \in M$ and a subset $Z \subseteq \{x_1, \dots, x_n\}$, $uK[Z]$ denotes the K -subspace of M generated by all the homogeneous elements of the form uv , where v is a monomial in $K[Z]$. The \mathbb{Z}^n -graded K -subspace $uK[Z]$ is said to be a Stanley space of dimension $|Z|$ if it is a free $K[Z]$ -module, where $|Z|$ denotes the cardinality of Z . A Stanley decomposition of M is a decomposition of M as a finite direct sum of \mathbb{Z}^n -graded K -vector spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^r u_i K[Z_i]$$

where each $u_i K[Z_i]$ is a Stanley space of M . The number $\text{sdepth}_S(\mathcal{D}) = \min\{|Z_i| : i = 1, \dots, r\}$ is called the Stanley depth of decomposition \mathcal{D} and the number

$$\text{sdepth}_S(M) := \max\{\text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

is called the Stanley depth of M .

In [4], Schmitt and Vogel introduced the Schmitt–Vogel number, which is given in the following definition.

Definition 1.1 Let I be a monomial ideal and $G(I)$ the set of its minimal monomial generators. The Schmitt–Vogel number of I , denoted by $sv(I)$, is the smallest integer t for which there exist subsets P_1, \dots, P_t of $G(I)$ such that

$$(i) \bigcup_{i=1}^t P_i = G(I);$$

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2010 AMS Mathematics Subject Classification: 13C15, 13P10, 13F20.

(ii) P_1 has exactly one element;

(iii) if p and p' are different elements of P_i ($2 \leq i \leq t$), then there is an integer i' with $1 \leq i' < i$ and an element in $P_{i'}$ that divides pp' .

They proved that for any monomial ideal I , the Schmitt–Vogel number $sv(I)$ is an upper bound for the arithmetical rank of I . It is clear that $sv(I) \leq |G(I)|$, and this inequality is strict in general. Herzog et al. [1] proved the following result:

Lemma 1.2 ([1, Proposition 3.4]) *Let $I \subset S$ be a monomial ideal with $|G(I)| = m$. Then $sdepth_S(I) \geq \max\{1, n - m + 1\}$.*

Recall that a monomial $v \in S$ is said to be squarefree if the exponent of each x_i in v is less than or equal to 1, and a monomial ideal I is said to be squarefree if it is generated by some squarefree monomials. The main result in this paper is the following: for a squarefree monomial ideal I , we have that

$$sdepth_S(I) \geq \max\{1, n - 1 - \lfloor \frac{sv(I)}{2} \rfloor\}.$$

Our result improves the lower bound obtained by Herzog et al. stated above. As some applications, we show that Stanley’s conjecture holds for the edge ideals of some special n -cyclic graphs with a common edge.

In this paper, we will focus on the case where I is a squarefree monomial ideal in S and let $G(I) = \{v_1, \dots, v_m\}$ be the set of its minimal squarefree monomial generators.

2. Preliminaries

We first recall some definitions and basic facts about the edge ideal of a graph and the lower bounds for Stanley depth of some special monomial ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [2, 3, 6].

Definition 2.1 *A finite graph G is an ordered pair $G = (V(G), E(G))$ where $V(G) = \{x_1, \dots, x_n\}$ is the set of vertices of G , and $E(G)$ is a collection of two-element subsets of $V(G)$, usually called the edges of G .*

In this case, we may suppose that x_1, \dots, x_n are indeterminates over the field K . The edge ideal of G in the polynomial ring $S = K[x_1, \dots, x_n]$ is the squarefree monomial ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)).$$

Definition 2.2 *Let $G_i = (V(G_i), E(G_i))$ be some graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, for $i = 1, \dots, k$. The union of the graphs G_1, G_2, \dots, G_k , written $\bigcup_{i=1}^k G_i$, is the graph with vertex set $\bigcup_{i=1}^k V(G_i)$ and edge set $\bigcup_{i=1}^k E(G_i)$.*

Definition 2.3 *Let $G = (V(G), E(G))$ be a graph. A walk of length m in G is an alternating sequence of vertices and edges $w = \{x_1, y_1, x_2, \dots, x_m, y_m, x_{m+1}\}$, where $y_i = \{x_i, x_{i+1}\}$ is the edge joining x_i and x_{i+1} . If $x_1 = x_{m+1}$, we call this walk closed.*

A cycle of length m ($m \geq 3$) is a closed walk in which the vertices x_1, \dots, x_m are distinct. We denote by C_m the graph consisting of a cycle with m vertices. An n -cyclic graph with a common edge is a graph consisting of the union of n cycles $C_{3r_1}, \dots, C_{3r_{k_1}}, C_{3s_1+1}, \dots, C_{3s_{k_2}+1}, C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ connected through a common edge, where $k_1 + k_2 + k_3 = n$, and r_i, s_j, t_l are positive integers for any $1 \leq i \leq k_1, 1 \leq j \leq k_2$ and $1 \leq l \leq k_3$.

The Stanley depth of the complete intersection monomial ideal is completely computed by Shen.

Lemma 2.4 ([5, Theorem 2.4]) *Let $I \subset S$ be a complete intersection monomial ideal with $|G(I)| = m$. Then $sdepth_S(I) = n - \lfloor \frac{m}{2} \rfloor$.*

Keller and Young [2] and Okazaki [3] independently improved this lower bound stated above; they showed that:

Lemma 2.5 *Let $I \subset S$ be a monomial ideal with $|G(I)| = m$. Then $sdepth_S(I) \geq \max\{1, n - \lfloor \frac{m}{2} \rfloor\}$.*

Let $\text{mod}_{\mathbb{Z}}^n(S)$ denote the category whose objects are finitely generated \mathbb{Z}^n -graded S -modules and morphisms are degree-preserving S -homomorphisms, that is, S -homomorphisms $f : M \rightarrow N$ such that $f(M_a) \subseteq N_a$ for $a \in \mathbb{Z}^n$. Clearly, the following lemma holds.

Lemma 2.6 *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence in $\text{mod}_{\mathbb{Z}}^n(S)$. Then $sdepth_S(M) \geq \min\{sdepth_S(L), sdepth_S(N)\}$.*

Let $R = K[x_1, \dots, x_{n-1}]$. We consider the natural map $\varphi : S \rightarrow R$ via $\varphi(x_i) = x_i$ for any $1 \leq i \leq n-1$ and $\varphi(x_n) = 1$. Thus, any \mathbb{Z}^{n-1} graded R -module has a structure of \mathbb{Z}^n -graded S -modules by the map φ . We need the following lemma.

Lemma 2.7 ([3, Lemma 2.5]) *Let v_1, \dots, v_m be monomials in S such that $x_n | v_i$ for $i = 1, \dots, r$ and $x_n \nmid v_i$ for $i = r+1, \dots, m$, where $1 \leq r \leq m-1$. Let $\mathfrak{a} = (v_1, \dots, v_r)$, $\mathfrak{b} = (v_{r+1}, \dots, v_m)$ be monomial ideals generated by v_1, \dots, v_r and v_{r+1}, \dots, v_m , respectively. Let $I = \mathfrak{a} + \mathfrak{b}$ and $I' = \mathfrak{a} + x_n \mathfrak{b}$. Then*

$$I/I' \cong \mathfrak{b} \cap R$$

as \mathbb{Z}^n -graded S -modules, where the structure of \mathbb{Z}^n -graded S -modules $\mathfrak{b} \cap R$ is given as above.

3. Main results

In this section we provide a lower bound for the Stanley depth of squarefree monomial ideals. This lower bound is given in terms of the Schmitt–Vogel number $sv(I)$. In the following three propositions, we consider the behavior of the Schmitt–Vogel number of an arbitrary monomial ideal under the elimination of variables.

Proposition 3.1 *Let $I \subset S$ be a squarefree monomial ideal with $G(I) = \{v_1, \dots, v_m\}$ such that $x_n | v_i$ for any $1 \leq i \leq m$. Let $v'_i = v_i/x_n$, and let I' be a monomial ideal of $R = K[x_1, \dots, x_{n-1}]$ generated by $\{v'_1, \dots, v'_m\}$. If $sv(I) \geq 2$, then $sv(I') = sv(I)$.*

Proof Let $sv(I) = t$. Then $v_i \neq x_n$ for any $1 \leq i \leq m$. Otherwise, $m = t = 1$, which contradicts with $t \geq 2$. We can assume that $P_1 = \{v_1\}, P_2 = \{v_2, \dots, v_{s_2}\}, \dots, P_t = \{v_{s_{t-1}+1}, \dots, v_m\}$ are the subsets of $G(I)$. Then $P'_1 = \{v'_1\}, P'_2 = \{v'_2, \dots, v'_{s_2}\}, \dots, P'_t = \{v'_{s_{t-1}+1}, \dots, v'_m\}$ are the subsets of $G(I')$. Since I is a squarefree monomial ideal, it is obvious that P_1, \dots, P_t satisfy the conditions of Definition 1.1 if and only if P'_1, \dots, P'_t satisfy the conditions of Definition 1.1. Therefore, $sv(I') = sv(I)$. This completes the proof. \square

Proposition 3.2 Let $I \subset S$ be a squarefree monomial ideal with $G(I) = \{v_1, \dots, v_m\}$ such that $x_n|v_i$ for $i = 1, \dots, r$ and $x_n \nmid v_i$ for $i = r + 1, \dots, m$, where $2 \leq r \leq m - 1$. Let $v'_i = v_i/x_n$ for any $1 \leq i \leq r$, and let I' be a squarefree monomial ideal of $R = K[x_1, \dots, x_{n-1}]$ generated by $\{v'_1, \dots, v'_r, v_{r+1}, \dots, v_m\}$. If $sv(I) \geq 2$, then $sv(I') \leq sv(I)$.

Proof Note that for any $1 \leq i \leq r$, $v_i \neq x_n$. Otherwise, $r = 1$, and this contradicts with $r \geq 2$. Let π be a permutation of the set $\{1, \dots, m\}$ and $sv(I) = t$, and let $P_1 = \{v_{\pi(1)}\}, P_2 = \{v_{\pi(2)}, \dots, v_{\pi(s_2)}\}, \dots, P_t = \{v_{\pi(s_{t-1}+1)}, \dots, v_{\pi(m)}\}$ be the subsets of $G(I)$. Then $P'_1 = \{v'_{\pi(1)}\}, P'_2 = \{v'_{\pi(2)}, \dots, v'_{\pi(s_2)}\}, \dots, P'_t = \{v'_{\pi(s_{t-1}+1)}, \dots, v'_{\pi(m)}\}$ are the subsets of $G(I')$ such that $\bigcup_{i=1}^t P'_i = G(I')$, where v'_i , for $i = 1, \dots, m$ is the monomial obtained by substitution of 1 to x_n in v_i . Hence, in order to prove the assertion, it is enough to prove that the sets P'_1, \dots, P'_t satisfy conditions (ii) and (iii) of Definition 1.1. It is clear that $P'_i \neq \emptyset$. Assume that $v'_{\pi(i)}$ and $v'_{\pi(j)}$ are different elements of P'_k for some k with $2 \leq k \leq t$. Then $v_{\pi(i)}$ and $v_{\pi(j)}$ are different elements of P_k . Since P_1, \dots, P_t satisfy condition (iii) of Definition 1.1, it follows that there exists an integer s with $1 \leq s < k$ and some monomial $v_{\pi(l)} \in P_s$ such that $v_{\pi(l)}|v_{\pi(i)}v_{\pi(j)}$. Since $v_{\pi(l)}, v_{\pi(i)}$ and $v_{\pi(j)}$ are squarefree, we have that $v'_{\pi(l)}|v'_{\pi(i)}v'_{\pi(j)}$. Thus, $v'_{\pi(l)} \in P'_s$. Therefore, $sv(I') \leq sv(I)$. This completes the proof. \square

Proposition 3.3 Let $I \subset S$ be a squarefree monomial ideal with $G(I) = \{v_1, \dots, v_m\}$ such that $x_n|v_i$ for $i = 1, \dots, r$ and $x_n \nmid v_i$ for $i = r + 1, \dots, m$, where $2 \leq r \leq m - 1$. Let I' be a squarefree monomial ideal of $K[x_1, \dots, x_{n-1}]$ generated by $\{v_{r+1}, \dots, v_m\}$. If $sv(I) \geq 2$, then $sv(I') \leq sv(I)$.

Proof Let $R = K[x_1, \dots, x_{n-1}]$; then $G(I') = G(I) \cap R$. Note that for any $1 \leq i \leq r$, $v_i \neq x_n$ from the proof of Proposition 3.2. Let $sv(I) = t$, and P_1, \dots, P_t be the subsets of $G(I)$ that satisfy the conditions of Definition 1.1. Set $P'_i = P_i \cap R$ for any $1 \leq i \leq t$ and $P_1 = \{u\}$. We distinguish two cases:

(1) If $x_n \nmid u$, then $P'_1 \neq \emptyset$ and it is obviously seen that P'_1, \dots, P'_t are the subsets of $G(I')$ that satisfy the conditions of Definition 1.1. Thus, $sv(I') \leq sv(I)$.

(2) If $x_n|u$, then $P'_1 = \emptyset$. Thus, there exist integers $2 \leq i_1 < i_2 < \dots < i_l \leq t$ such that $P'_{i_k} \neq \emptyset$ for any $1 \leq k \leq l$ and $P'_j = \emptyset$ for any $j \notin \{i_1, \dots, i_l\}$. It is clear that $G(I') = \bigcup_{k=1}^l P'_{i_k}$. Since $i_1 \geq 2$, it follows that $l \leq t - 1$. We claim that the sets $P'_{i_1}, \dots, P'_{i_l}$ satisfy conditions (ii) and (iii) of Definition 1.1.

We first verify condition (ii). Assume that $|P'_{i_1}| \geq 2$. This implies that there exist two different monomials μ_1, μ_2 in P_{i_1} that are not divisible by x_n . Thus, by condition (iii) of Definition 1.1, there exists an integer $q < i_1$ and some monomial $\mu_3 \in P_q$ with $\mu_3|\mu_1\mu_2$. However, this is not possible because $P'_q = \emptyset$ and therefore every element of P_q and in particular μ_3 is divisible by x_n . This proves condition (ii).

Now we verify condition (iii). Let ν_1, ν_2 be two different monomials in P'_{i_k} for some k with $2 \leq k \leq l$. Then $\nu_1, \nu_2 \in P_{i_k}$ and since P_1, \dots, P_t satisfy condition (iii) of Definition 1.1, it follows that there exists an integer s with $1 \leq s < i_k$ and some monomial $\nu_3 \in P_s$, such that $\nu_3 | \nu_1 \nu_2$. Since ν_1 and ν_2 are not divisible by x_n , we conclude that $x_n \nmid \nu_3$. Thus, $s \in \{i_1, \dots, i_l\}$ and $\nu_3 \in P'_s$. This verifies condition (iii) of Definition 1.1. Thus, $sv(I') \leq sv(I) - 1$. This completes the proof. \square

Now we state and prove the main result of this section.

Theorem 3.4 *Let I be a squarefree monomial ideal of S with Schmitt–Vogel number $sv(I)$. Then:*

$$sdepth_S(I) \geq \max\{1, n - 1 - \lfloor \frac{sv(I)}{2} \rfloor\}.$$

Proof It suffices to show that $sdepth_S(I) \geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor$ by Lemma 1.2. Let $G(I) = \{v_1, \dots, v_m\}$. We use induction on n . If $n = 1$ or $sv(I) = 1$, then I is a principal ideal, so we have $sdepth_S(I) = n$. Thus, the assertion holds. Now we assume that $n \geq 2$ and the assertion holds for $n - 1$. It suffices to consider only the case $sv(I) \geq 2$. For $i = 1, \dots, n$, we set $t_i(I) = |\{v_j \in G(I) \mid x_i \text{ divides } v_j\}|$. If $t_i(I) \leq 1$ for any $1 \leq j \leq m$, then I is a complete intersection and $sv(I) = |G(I)| = m$, and hence we obtain that the assertion holds by Lemma 2.4. Thus, we may assume that $t_i(I) \geq 2$ for some i , and hence, without loss of generality, that $t_n(I) \geq 2$. We distinguish the following two cases:

(1) If $t_n(I) = m$, then $x_n | v_i$ for any $1 \leq i \leq m$. Set $v'_i = v_i/x_n$, and let I' be a squarefree monomial ideal of S generated by v'_1, \dots, v'_m . It is readily seen that I' is naturally isomorphic to I in $\text{mod } \frac{n}{2}(S)$ up to degree shifting, and it follows that $sdepth_S(I) = sdepth_S(I')$. Note that I' is also a squarefree monomial ideal of $R = K[x_1, \dots, x_{n-1}]$. By inductive hypothesis, Proposition 3.1, and [1, Lemma 3.6], we have

$$sdepth_S(I) = sdepth_S(I') = sdepth_R(I') + 1 \geq (n - 1) - \lfloor \frac{sv(I')}{2} \rfloor + 1 > n - 1 - \lfloor \frac{sv(I)}{2} \rfloor.$$

(2) If $2 \leq t_n(I) \leq m - 1$, we set $r = t_n(I)$. Without loss of generality, we may assume that $x_n | v_i$ for $i = 1, \dots, r$ and $x_n \nmid v_i$ for $i = r + 1, \dots, m$. Let $\mathfrak{a} = (v_1, \dots, v_r)$, $\mathfrak{b} = (v_{r+1}, \dots, v_m)$ be squarefree monomial ideals generated by v_1, \dots, v_r and v_{r+1}, \dots, v_m , respectively. Then $I = \mathfrak{a} + \mathfrak{b}$. Set $I' = \mathfrak{a} + x_n \mathfrak{b}$; thus, each minimal generator of I' can be divided by x_n . Set $v'_i = v_i/x_n$ for $1 \leq i \leq r$, and let I'' be the squarefree monomial ideal generated by $\{v'_1, \dots, v'_r, v_{r+1}, \dots, v_m\}$. By the same argument as in case (1), we have that $sdepth_S(I'') = sdepth_S(I')$. Applying our inductive hypothesis and Proposition 3.2, we have

$$sdepth_S(I') = sdepth_S(I'') \geq n - 1 - \lfloor \frac{sv(I'')}{2} \rfloor \geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor.$$

We consider the exact sequence

$$0 \rightarrow I' \rightarrow I \rightarrow I/I' \rightarrow 0.$$

It follows from Lemma 2.6 that

$$sdepth_S(I) \geq \min\{sdepth_S(I'), sdepth_S(I/I')\}.$$

As for $sdepth_S(I/I')$, we can apply Lemma 2.7, and it follows that

$$sdepth_S(I/I') = sdepth_S(\mathfrak{b} \cap R).$$

Note that $\mathfrak{b} \cap R$ is minimally generated by v_{r+1}, \dots, v_m as an ideal of R . By inductive hypothesis, Proposition 3.3, and [1, Lemma 3.6], we have

$$\begin{aligned} \text{sdepth}_S(\mathfrak{b} \cap R) &= \text{sdepth}_R(\mathfrak{b} \cap R) + 1 \\ &\geq (n - 2) - \lfloor \frac{sv(\mathfrak{b} \cap R)}{2} \rfloor + 1 \\ &= n - 1 - \lfloor \frac{sv(\mathfrak{b} \cap R)}{2} \rfloor \\ &\geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor. \end{aligned}$$

Summing up, we conclude that $\text{sdepth}_S(I) \geq n - 1 - \lfloor \frac{sv(I)}{2} \rfloor$, which completes the proof. □

Lemma 3.5 (Auslander–Buchsbaum). *Let M be a finitely generated graded S -module. Then*

$$pd_S(M) + \text{depth}(M) = \dim(S),$$

where $pd_S(M)$ is the projective dimension of M .

Zhu et al. [6] provided some upper bounds for Schmitt–Vogel number $sv(I(G))$ of the edge ideals $I(G)$ of some special graphs G with a common edge and the lower bounds for the projective dimensions of their quotient ring $S/I(G)$.

Lemma 3.6 (1) *Let G be a graph consisting of the union of k_1 cycles $C_{3r_1}, \dots, C_{3r_{k_1}}$ with a common edge.*

Then $pd_S(S/I(G)) = 1 + \sum_{i=1}^{k_1} (2r_i - 1)$ and $sv(I(G)) \leq 1 + \sum_{i=1}^{k_1} (2r_i - 1)$.

(2) *Let G be a graph consisting of the union of k_2 cycles $C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ with a common edge. Then*

$pd_S(S/I(G)) \geq 2 - k_2 + 2 \sum_{i=1}^{k_2} s_i$ and $sv(I(G)) \leq 1 + 2 \sum_{i=1}^{k_2} s_i$.

(3) *Let G be a graph consisting of the union of k_3 cycles $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ with a common edge. Then*

$pd_S(S/I(G)) = 1 + 2 \sum_{i=1}^{k_3} t_i$ and $sv(I(G)) \leq 1 + 2 \sum_{i=1}^{k_3} t_i$.

As a consequence of Theorem 3.4 and Lemma 3.6, we have the following results.

Theorem 3.7 (1) *Let G be a graph consisting of the union of k_1 cycles $C_{3r_1}, \dots, C_{3r_{k_1}}$ with a common edge. Then Stanley’s conjecture holds for $I(G)$.*

(2) *Let G be a graph consisting of the union of k_2 cycles $C_{3s_1+1}, \dots, C_{3s_{k_2}+1}$ with a common edge. Then Stanley’s conjecture holds for $I(G)$.*

(3) *Let G be a graph consisting of the union of k_3 cycles $C_{3t_1+2}, \dots, C_{3t_{k_3}+2}$ with a common edge. Then Stanley’s conjecture holds for $I(G)$.*

Proof Cases (1) and (3) can be shown by similar arguments, so we only prove case (1). Note that the number of vertices of the graph G is $n = \sum_{i=1}^{k_1} 3r_i - 2(k_1 - 1)$. Thus, by Lemma 3.6 (1), we have

$$\begin{aligned} n - 1 - \lfloor \frac{sv(I)}{2} \rfloor &\geq \sum_{i=1}^{k_1} 3r_i - 2(k_1 - 1) - 1 - \lfloor \frac{1 + \sum_{i=1}^{k_1} (2r_i - 1)}{2} \rfloor \\ &= 1 + \sum_{i=1}^{k_1} r_i - k_1 + \lceil \frac{1 + \sum_{i=1}^{k_1} (2r_i - 1)}{2} \rceil, \end{aligned}$$

and

$$\begin{aligned} \text{depth}(I(G)) &= \text{depth}(S/I(G)) + 1 = n - \text{pd}_S(S/I(G)) + 1 \\ &\leq \sum_{i=1}^{k_1} 3r_i - 2(k_1 - 1) - (1 + \sum_{i=1}^{k_1} (2r_i - 1)) + 1 \\ &= 1 + \sum_{i=1}^{k_1} r_i - k_1 + 1. \end{aligned}$$

Since $k_1 \geq 2$ and $r_i \geq 1$ for any $1 \leq i \leq k_1$, we have that $\lceil \frac{1 + \sum_{i=1}^{k_1} (2r_i - 1)}{2} \rceil \geq 1$. Therefore, by Theorem 3.4, we have that

$$\text{sdepth}_S(I(G)) \geq n - 1 - \lfloor \frac{sv(I(G))}{2} \rfloor \geq \text{depth}(I(G)).$$

(2) Note that the number of vertices of the graph G is $n = \sum_{i=1}^{k_2} (3s_i + 1) - 2(k_2 - 1)$. Thus, by Lemma 3.6 (2), we have

$$\begin{aligned} n - 1 - \lfloor \frac{sv(I)}{2} \rfloor &\geq \sum_{i=1}^{k_2} (3s_i + 1) - 2(k_2 - 1) - 1 - \lfloor \frac{1 + 2 \sum_{i=1}^{k_2} s_i}{2} \rfloor \\ &= 1 + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_2} (s_i - 1), \end{aligned}$$

and

$$\begin{aligned} \text{depth}(I(G)) &= \text{depth}(S/I(G)) + 1 = n - \text{pd}_S(S/I(G)) + 1 \\ &\leq \sum_{i=1}^{k_2} (3s_i + 1) - 2(k_2 - 1) - (2 - k_2 + 2 \sum_{i=1}^{k_2} s_i) + 1 \\ &= 1 + \sum_{i=1}^{k_2} s_i. \end{aligned}$$

Therefore, by Theorem 3.4, we have that

$$\text{sdepth}_S(I(G)) \geq n - 1 - \lfloor \frac{sv(I(G))}{2} \rfloor = 1 + \sum_{i=1}^{k_2} s_i + \sum_{i=1}^{k_2} (s_i - 1) \geq \text{depth}(I(G)).$$

□

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11271275) and by the Foundation of the Jiangsu Overseas Research & Training Program for University Prominent Young & Middle-Aged Teachers and Presidents and by foundation of the Priority Academic Program Development of Jiangsu Higher Education Institutions. The author would like to express her sincere thanks to the editor for help and encouragement. Special thanks are due to the referee for a careful reading and pertinent comments.

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