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AMIR MAFI

DLER NADERI

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A note on reduction numbers and Hilbert–Samuel functions of ideals over Cohen–Macaulay rings

Amir MAFI*, Dler NADERI

Department of Mathematics, University of Kurdistan, Sanandaj, Iran

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Abstract: Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension $d \geq 2$ with infinite residue field and I an \mathfrak{m} -primary ideal of R . Let I be integrally closed and J be a minimal reduction of I . In this paper, we show that the following are equivalent: (i) $P_I(n) = H_I(n)$ for $n = 1, 2$; (ii) $P_I(n) = H_I(n)$ for all $n \geq 1$; (iii) $I^3 = JI^2$. Moreover, if $\dim R = 3$, $n(I) \leq 1$ and $\text{grade } gr_I(R)_+ > 0$, then the reduction number $r(I)$ is independent.

Key words: Cohen–Macaulay rings, Hilbert–Samuel functions

1. Introduction

Throughout this note, we assume that (R, \mathfrak{m}) is a Cohen–Macaulay (abbreviated CM) local ring of dimension $d > 0$ with infinite residue field and I is an \mathfrak{m} -primary ideal of R . Let $\ell(R/I^n)$ denote the length of the R -module R/I^n . The Hilbert–Samuel function of I is the function $H_I(n) = \ell(R/I^n)$. It is well known that this function coincides with a polynomial $P_I(n)$ of degree d for all sufficiently large integers n and we set $n(I) = \max\{n \in \mathbb{Z} : P_I(n) \neq H_I(n)\}$; this number first introduced by Ooishi [11]. $P_I(n)$ is called the Hilbert–Samuel polynomial of I . Northcott and Rees [10] defined a minimal reduction of I (abbreviated $MR(I)$) to be a d -generated ideal $J \subseteq I$ of R such that $JI^n = I^{n+1}$ for some nonnegative integers n . We denote $r_J(I) = \min\{n \in \mathbb{Z} : JI^n = I^{n+1}\}$. The reduction number $r(I)$ is defined as $r(I) = \min\{r_J(I) : J \in MR(I)\}$. The reduction number $r(I)$ is said to be independent if $r(I) = r_J(I)$ for all $J \in MR(I)$. We denote the associated graded ring of I by $gr_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and $gr_I(R)_+$ denotes the ideal $\bigoplus_{n \geq 1} I^n/I^{n+1}$. Huckaba [3] and Trung [14] showed that if $\text{grade } gr_I(R)_+ \geq d - 1$, then $r(I)$ is independent (see also [8]).

Hoa [2] proved that if $d = 2$ and I agrees with its Ratliff–Rush closure \tilde{I} , then the following are equivalent: (i) $H_I(n) = P_I(n)$ for $n = 1, 2$; (ii) $H_I(n) = P_I(n)$ for all $n \geq 1$; (iii) $r(I) \leq 2$ and $\text{grade } gr_I(R)_+ \geq 1$ (see also [4]). For basic definitions, we refer the reader to [1, 9].

The main aim of this paper is to prove the following theorems.

Theorem 1.1 *Let $d \geq 2$ and J be a minimal reduction of I . If $\tilde{I} = I$, then the following are equivalent:*

- (i) $P_I(n) = H_I(n)$ for $n = 1, 2$.
- (ii) $P_I(n) = H_I(n)$ for all $n \geq 1$.

*Correspondence: a.mafi@ipm.ir

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(iii) $r_J(I) \leq 2$.

Theorem 1.2 (Compare with Corollary 5 of [16].) *Let $d = 3$ and $\bar{I} = I$. If $n(I) \leq 1$ and $\text{grade } gr_I(R)_+ > 0$, then $r(I)$ is independent.*

2. The results

Following Marley [8], let $S = \bigoplus_{n \geq 0} S_n$ be a Noetherian graded ring where S_0 is an Artinian local ring, S is generated by 1-forms over S_0 , and $S_+ = \bigoplus_{n \geq 1} S_n$. If $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a finitely generated graded S -module, then for $i \geq 0$ we let $H_{S_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{S_+}^i(M)_n$ be the i th local cohomology modules of M with support in S_+ . It was shown that these modules are Artinian and that each $H_{S_+}^i(M)_n$ is finitely generated. Thus, $H_{S_+}^i(M)_n = 0$ for n sufficiently large. If $H_{S_+}^i(M) \neq 0$, we let $a_i(M) = \max\{n \in \mathbb{Z} : H_{S_+}^i(M)_n \neq 0\}$. For convenience, we define $a_i(M) = \infty$ for $i < \text{depth}_{S_+} M$ and $a_i(M) = \infty$ for $i > \dim M$. We will use $a_i(I)$ to denote $a_i(gr_I(R))$.

The following lemma extends Theorem 2.1 of [13].

Lemma 2.1 *Let $\bar{I} = I$ and $r(I) \leq 2$. Then $gr_I(R)$ is Cohen–Macaulay.*

Proof Suppose J is a minimal reduction of I . Then $\bar{I}^2 = \bar{J}^2$ and by Theorem 1 of [6] we have $\bar{I}^2 \cap J = \bar{I}J = IJ$. Therefore, $I^2 \cap J = IJ$, and since $J I^n = I^{n+1}$ for all $n \geq 2$, we have $I^n \cap J = I^{n-1}J$ for all $n \geq 1$. Hence, by Corollary 2.7 of [15], $gr_I(R)$ is Cohen–Macaulay. □

The following result extends Remark 3.2 of [8].

Lemma 2.2 *Let $d = 2$ and $P_I(n) = H_I(n)$ for all $n \geq 3$. If $\bar{I} = I$, then $r(I)$ is independent.*

Proof If $r(I) \leq 2$, then by Lemma 2.1 $gr_I(R)$ is CM and so $r(I)$ is independent. If $r(I) \geq 3$, then since $n(I) \leq 2$, we have $r(I) \geq n(I) + 1$. Hence, by Theorem 3.1 of [16] $r(I)$ is independent. □

Proposition 2.3 *Let $d = 2$ and J be a minimal reduction of I . If $r_J(I) \leq n(I) + 1$, then $a_1(I) = n(I)$.*

Proof By Theorem 2.1 of [8] and our hypothesis, we have $\text{grade } gr_I(R)_+ = 0$. Therefore, $a_0(I) < a_1(I)$. If $a_1(I) < a_2(I)$, then by Lemma 2.1 and Remark 1.4 of [8] $n(I) + 2 \leq r_J(I)$ and this is a contradiction with our hypothesis. Hence, $a_2(I) \leq a_1(I)$. If $a_2(I) = a_1(I)$, then $r_J(I) = a_2(I) + 2$ and so $a_2(I) + 2 \leq n(I) + 1 \leq a_2(I) + 1$ and this is also a contradiction. Thus, $a_2(I) < a_1(I)$ and so $a_1(I) = n(I)$, as required. □

Proposition 2.4 *Let $d = 2$ and $n(I) + 2 \leq r(I)$. Then $a_2(I) = n(I)$ or $a_2(I) = r(I) - 2$.*

Proof If $\text{grade } gr_I(R)_+ \geq 1$, then $r(I) = n(I) + 2 = a_2(I) + 2$. Hence, $a_2(I) = n(I)$. Suppose $\text{grade } gr_I(R)_+ = 0$ and then $a_0(I) < a_1(I)$. If $a_1(I) < a_2(I)$, then $a_2(I) = n(I)$. If $a_2(I) < a_1(I)$, then $n(I) = a_1(I)$ and so $a_1(I) + 2 = n(I) + 2 \leq r(I) \leq a_1(I) + 1$ and this is a contradiction. If $a_1(I) = a_2(I)$, then $a_2(I) + 2 \leq r(I) \leq a_2(I) + 2$. Therefore, $a_2(I) = r(I) - 2$. □

Ratliff and Rush [12] defined \tilde{I} as follows: $\tilde{I} = \cup_{n \geq 1} I^{n+1} : I^n$. They proved, for all $n > 0$, $\tilde{I}^n = \cup_{k > 0} I^{n+k} : I^k$. It is well known that $\text{grade } gr_I(R)_+ > 0$ if and only if $\tilde{I}^n = I^n$ for all $n > 0$. In the following results we denote the leading form of an element $x \in I \setminus I^2$ with $x^* \in I/I^2 \subseteq gr_I(R)$.

Theorem 2.5 Let $d \geq 2$ and J be a minimal reduction of I . If $\bar{I} = I$, then the following are equivalent:

- (i) $P_I(n) = H_I(n)$ for $n = 1, 2$.
- (ii) $P_I(n) = H_I(n)$ for all $n \geq 1$.
- (iii) $r_J(I) \leq 2$.

In particular, if one of the above cases holds, then $gr_I(R)$ is CM.

Proof (iii) \implies (ii). $r_J(I) \leq 2$ and by Lemma 2.1 $gr_I(R)$ is CM. Thus, $r(I) = n(I) + d$ and so $n(I) \leq 2 - d$. Therefore, $n(I) \leq 0$ and hence $P_I(n) = H_I(n)$ for all $n \geq 1$.

(ii) \implies (i). It is clear.

(i) \implies (iii). We use induction on d . If $d = 2$, then by Corollary 4 of [7] $P_I(n) = \ell(R/\widetilde{I}^n)$ for all $n \geq 1$. Therefore, $\widetilde{I}^{n+1} = J\widetilde{I}^n$ for all $n \geq 2$ and so $I^3 = JI^2$. We may assume that $d \geq 3$, and that the assertion holds for CM local rings of dimension less than d . By Lemma 8.4.2 of [5] replacing R with $R(X)$, if necessary, and we may assume that there exists $x \in J$ such that x is a part of a system of minimal generators of J and, if we put $A = R/(x)$, by Lemma 11 of [7] $\mathfrak{a} = IA$ is integrally closed. By Lemma 2.3 of [16] $P_{\mathfrak{a}}(n) = H_{\mathfrak{a}}(n)$ for $n = 1, 2$. Therefore, by our induction hypothesis, $\mathfrak{a}^3 = J\mathfrak{a}^2$ and $gr_{\mathfrak{a}}(A)$ is CM; in particular, $\widetilde{\mathfrak{a}}^n = \mathfrak{a}^n$ for all n . Since $\widetilde{I}^n A \subseteq \widetilde{\mathfrak{a}}^n = \mathfrak{a}^n = I^n A$, we have $\widetilde{I}^n + (x) = I^n + (x)$. Therefore, $\widetilde{I}^n = I^n + (x)\widetilde{I}^{n-1}$ for all $n \geq 1$ and so by induction on n , we have $\widetilde{I}^n = I^n$ for all n . Hence, $gr_I(R)$ is CM and x^* is a regular element of $gr_I(R)$. By Lemma 2.2 of [16] $r_J(I) \leq 2$, as required. \square

Theorem 2.6 Let $d = 3$ and $n(I) \leq 1$. If $\text{grade } gr_I(R)_+ > 0$ and $\bar{I} = I$, then $r(I)$ is independent.

Proof Let $J_1 = (x_1, x_2, x_3)$ and $J_2 = (y_1, y_2, y_3)$ be minimal reductions of I . Since $\text{grade } gr_I(R)_+ > 0$, by Lemma 2.7 of [16] we may assume that x_1^* and y_1^* are regular elements in $gr_I(R)$ and $J = (x_1, x_2, y_1)$ is a minimal reduction of I . By Lemma 2.1 of [16] $n(I/(x_1)) = n(I) + 1 \leq 2$. Replacing R with $R(T)$, by Lemma 11 of [7] $I/(x_1)$ is integrally closed and by Lemma 2.1 $r(I/(x_1))$ is independent. Hence, $r(I/(x_1)) = r_{J_1/(x_1)}(I/(x_1)) = r_{J/(x_1)}(I/(x_1))$. Since x_1^* is regular element in $gr_I(R)$, we have $r_{J_1}(I) = r_J(I)$. By the same argument $r_{J_2}(I) = r_J(I)$ and so $r_{J_1}(I) = r_{J_2}(I)$. Thus, $r(I)$ is independent, as required. \square

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