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On coprimely structured rings

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Abstract: In this paper, we define coprimely structured rings, which are the generalization of strongly 0-dimensional rings. Furthermore, we investigate coprimely structured rings and give some relations between other rings such as Artinian rings, strongly 0-dimensional rings, and h-local domains.

Key words: Prime ideal, Artinian ring, coprimely structured ring

1. Introduction

Throughout this paper, we assume that R is a commutative ring with identity. A proper ideal P of R is called as a prime ideal if for any $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. Moreover, $Spec(R)$ denotes the set of prime ideals of R and $MaxSpec(R)$ denotes the set of maximal ideals of R. It is known that the nilradical of R, $\mathfrak{N}$, is equal to the intersection of all prime ideals of R. The dimension of R, denoted by $\dim(R)$, is defined to be $\sup\{n \in \mathbb{Z}^+ \mid \text{there exists a strict chain of prime ideals of } R \text{ of length } n\}$ (see also [11], for more information).

Prime ideals are an area of interest in many fields such as algebra and algebraic geometry. The following is the well-known prime avoidance theorem: if an ideal I of a ring R is contained in a union of finitely many prime ideals P_i's, then it is contained in P_i for some i. This property was extended to infinite union by Reis and Viswanathan in [10] and they called these rings compactly packed rings. They also characterized Noetherian compactly packed rings by the property that prime ideals are radicals of principal ideals. Moreover, they showed that if a Noetherian ring R is compactly packed, then $\dim(R) \leq 1$. After this, Erdoğdu generalized the concept of compactly packed rings to coprimely packed rings and investigated the properties of such rings in his paper entitled "Coprimely Packed Rings" [4]. Gilmer formed the dual notion of compactly packed rings in his paper "An Intersection Condition for Prime Ideals" and studied some properties of this notion [6]. Recently, Jayaram et al. called this notion a strongly prime ideal and they also called the ring in which all prime ideals are strongly prime a strongly 0-dimensional ring [7].

In the present paper, based on the definition of a strongly 0-dimensional ring, we define a coprimely structured ideal and coprimely structured ring as follows: a prime ideal P is said to be a coprimely structured ideal if $I_i + P = R$ for $i \in \Lambda$; then $\bigcap_{i\in\Lambda} I_i \nsubseteq P$ for any index set Λ. A ring R is called a coprimely structured

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ring if all its prime ideals are coprimely structured ideals. In this work it will be generalized the notion of strongly 0-dimensional rings to coprimely structured rings and examine the properties of these rings. It will be proved that every strongly 0-dimensional ring is a coprimely structured ring. However, it will be given that the converse is true when R is zero-dimensional in Corollary 2.

In the Figure below, the diagram indicates relations among the rings mentioned above.

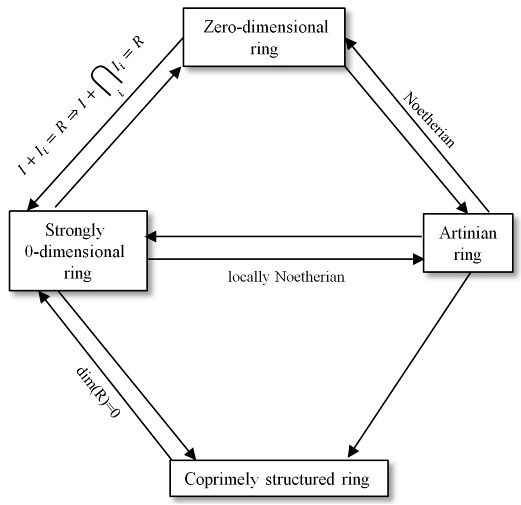

Figure.

2. Coprimely structured rings

In this section we give the definition of coprimely structured rings and examples of such rings. After this we investigate coprimely structured rings and examine some of their properties.

Definition 1 *Let R be a ring, Λ be an index set, and I_i's be ideals of R for $i \in \Lambda$. A prime ideal P of R is said to be a coprimely structured ideal if $I_i + P = R$ for all $i \in \Lambda$; then $\bigcap_{i\in\Lambda} I_i \nsubseteq P$. A ring R is called a coprimely structured ring if all its prime ideals are coprimely structured ideals.*

Note that in an integral domain the ideal (0) is always a coprimely structured ideal. Therefore, every field is a coprimely structured ring. Note also that there exist integral domains that are not coprimely structured rings such as $\mathbb{Z}$.

Example 1 *(i) Every ring that has finitely many ideals is a coprimely structured ring.*
(ii) Let $(R, \mathfrak{M})$ be a local ring. Then $\mathfrak{M}$ is a coprimely structured ideal and from this we obtain R is a coprimely structured ring. In particular, every valuation ring is a coprimely structured ring.

Theorem 1 *Every homomorphic image of a coprimely structured ring is a coprimely structured ring.*

Proof Let R be a coprimely structured ring, S be any ring, and without loss of generality $f : R \longrightarrow S$ be an epimorphism. Suppose that $\mathfrak{J}_i + \mathfrak{P} = S$ for some ideals $\mathfrak{J}_i$ of S and $\mathfrak{P} \in Spec(S)$ with the index set Λ and $i \in \Lambda$. Then there exist ideals I_i and $P \in Spec(R)$ containing $\ker f$ such that $f(I_i) = \mathfrak{J}_i$ and $f(P) = \mathfrak{P}$. Then $f(I_i) + f(P) = f(R)$ and so $f(I_i + P) = S = f(R)$. Since I_i and P contains $\ker f$, we get $I_i + P = R$.

Now since R is a coprimely structured ring, we obtain $\bigcap_{i\in\Lambda} I_i \nsubseteq P$. Therefore, $f\left(\bigcap_{i\in\Lambda} I_i\right) \nsubseteq f(P)$, that is, $\bigcap_{i\in\Lambda} \mathfrak{I}_i \nsubseteq \mathfrak{P}$. □

Corollary 1 *Let R be a ring and I be an ideal of R. If R is a coprimely structured ring, then R/I is a coprimely structured ring.*

Proof It follows from Theorem 1. □

Proposition 1 *Every Artinian ring is a coprimely structured ring.*

Proof Let R be an Artinian ring and $\bigcap_{i\in\Lambda} I_i \subseteq P$ for a family $\{I_i\}_{i\in\Lambda}$ of ideals of R and prime ideal P of R, where Λ is an index set. Since R is an Artinian ring, $\bigcap_{i\in\Lambda} I_i = \bigcap_{\alpha=1}^{n} I_{i_\alpha}$ for a finite subset $\{i_\alpha\}_{\alpha=1}^{n}$ of Λ where $n \in \mathbb{Z}^+$. Then $\bigcap_{\alpha=1}^{n} I_{i_\alpha} \subseteq P$, and so $I_{i_\alpha} \subseteq P$ for some i_α. Consequently, $I_{i_\alpha} + P \neq R$ and hence R is a coprimely structured ring. □

The following theorem shows that coprimely structured rings can be also characterized by their maximal ideals instead of prime ideals.

Theorem 2 *Let R be a ring. If every maximal ideal of R is a coprimely structured ideal then R is a coprimely structured ring.*

Proof Let $P \in Spec(R)$, Λ be an index set and I_i's be ideals of R such that $I_i + P = R$ where $i \in \Lambda$. Then there exists a maximal ideal $\mathfrak{M}$ of R that contains P. Therefore, we get $I_i + \mathfrak{M} = R$ for all $i \in \Lambda$. Since $\mathfrak{M}$ is a coprimely structured ideal, we obtain $\bigcap_{i\in\Lambda} I_i \nsubseteq \mathfrak{M}$. Thus $\bigcap_{i\in\Lambda} I_i \nsubseteq P$ and hence R is a coprimely structured ring. □

Example 2 *Let $R = R_1 \times R_2$ be a quasi-semilocal ring, where $(R_i, \mathfrak{M}_i)$'s are quasilocal rings for $i = 1, 2$. Then the set of ideals of R is of the form $I_\alpha \times J_\beta$, where I_α and J_β are ideals of R_1 and R_2, respectively. Furthermore, $MaxSpec(R) = \{I_\alpha \times J_\beta \mid (I_\alpha = R_1,\ I_\beta = \mathfrak{M}_2)$ or $(I_\alpha = \mathfrak{M}_1,\ I_\beta = R_2)\}$. Now, suppose that $(I_\alpha \times J_\beta) + (\mathfrak{M}_1 \times R_2) = R$ for maximal ideal $\mathfrak{M}_1 \times R_2$ of R. It follows that $I_\alpha + \mathfrak{M}_1 = R_1$, and so $I_\alpha = R_1$. Hence, we get $\bigcap_{(\alpha,\beta)} (I_\alpha \times J_\beta) \nsubseteq (\mathfrak{M}_1 \times R_2)$. Thus, by repeating this argument for $i = 2$, we obtain that R is a coprimely structured ring by Theorem 2. Consequently, it can be verified by induction that $R = R_1 \times R_2 \times ... \times R_n$ is a coprimely structured ring where $(R_i, \mathfrak{M}_i)$'s are quasilocal rings for $i = 1, ..., n$.*

In the following lemma, Arapovic characterizes the embeddability of a ring in a zero-dimensional ring with two properties in his paper entitled "On the embedding of a commutative ring into a 0-dimensional ring" [1].

Lemma 1 *[1, Theorem 7] A ring R is embeddable in a zero-dimensional ring if and only if R has a family of primary ideals $\{Q_\lambda\}_{\lambda\in\Lambda}$, such that:*

A1. $\bigcap_{\lambda\in\Lambda} Q_\lambda = 0$, *and*

A2. For each $a \in R$, *there is* $n \in \mathbb{N}$ *such that for all* $\lambda \in \Lambda$, *if* $a \in \sqrt{Q_\lambda}$, *then* $a^n \in Q_\lambda$.

By the above lemma, the condition (A2) is very useful. Therefore, Brewer and Richman give an equivalent condition in their paper "Subrings of 0-dimensional rings" [2] as follows:

Lemma 2 *[2, Theorem 2] A family* $\{I_\lambda\}_{\lambda\in\Lambda}$ *of ideals in a ring* R *satisfies (A2) if and only if for each (countable) subset* $\Gamma \subset \Lambda$, $\sqrt{\bigcap_{\lambda\in\Gamma} I_\lambda} = \bigcap_{\lambda\in\Gamma}\sqrt{I_\lambda}$.

In the paper "Subrings of Zero-dimensional Rings", Brewer and Richman show that (A2) holds for the family of all ideals in a zero-dimensional ring, and hence also for any family of ideals. Moreover, they prove the following theorem as characterizations of zero-dimensional rings.

Theorem 3 *[2, Theorem 4] The following conditions on a ring* R *are equivalent:*
(i) The ring R *is zero-dimensional.*
(ii) Condition (A2) holds for the family of all ideals of R.
(iii) Condition (A2) holds for the family of all primary ideals of R.

Now, using (A2) condition we give another characterization of a coprimely structured ring.

Theorem 4 *Let* R *be a ring and* Λ *be an index set. If* R *is a coprimely structured ring, then for any family* $\{P_i\}_{i\in\Lambda}$ *of prime ideals and maximal ideal* $\mathfrak{M}$ *of* R, $\bigcap_{i\in\Lambda} P_i \subseteq \mathfrak{M}$ *implies* $P_j \subseteq \mathfrak{M}$ *for some* $j \in \Lambda$. *The converse is true if condition (A2) is satisfied for any family of ideals of* R.

Proof Let R be a coprimely structured ring and $\bigcap_{i\in\Lambda} P_i \subseteq \mathfrak{M}$ for a family $\{P_i\}_{i\in\Lambda}$ of prime ideals of R and maximal ideal $\mathfrak{M}$ of R. Then, by our assumption, we get $P_j + \mathfrak{M} \neq R$ for a prime ideal P_j where $j \in \Lambda$. Thus $P_j \subseteq \mathfrak{M}$.

For the converse, suppose that $\bigcap_{i\in\Lambda} I_i \subseteq P$ for ideals I_i and a prime ideal P of R. Then $\sqrt{\bigcap_{i\in\Lambda} I_i} \subseteq \sqrt{P} = P$ and there exists a maximal ideal $\mathfrak{M}$ of R such that $P \subseteq \mathfrak{M}$. Since $\sqrt{\bigcap_{i\in\Lambda} I_i} = \bigcap_{i\in\Lambda}\sqrt{I_i}$ by Lemma 2, we get $\bigcap_{i\in\Lambda}\sqrt{I_i} \subseteq P$. For all I_i, there exist prime ideals $\{P_{\alpha,i}\}_{\alpha\in\aleph}$ such that $\sqrt{I_i} = \bigcap_{\alpha\in\aleph} P_{i,\alpha}$. Thus we have $\mathfrak{M} \supseteq \sqrt{\bigcap_{i\in\Lambda} I_i} = \bigcap_{i\in\Lambda}\sqrt{I_i} = \bigcap_{i\in\Lambda}\left(\bigcap_{\alpha\in\aleph} P_{\alpha,i}\right) = \bigcap_{i\in\Lambda,\ \alpha\in\aleph} P_{i,\alpha}$. By our assumption we get $P_{j,\beta} \subseteq \mathfrak{M}$ for some $j \in \Lambda$, $\beta \in \aleph$. Therefore, $P_{j,\beta} + \mathfrak{M} \neq R$ and from this $P_{j,\beta} + P \neq R$. Hence $I_j + P \neq R$. □

Theorem 5 *Let* R *be a ring satisfying the condition (A2) for any family of ideals of* R *and* I *be an ideal of* R *contained in the nilradical of* R. *Then* R/I *is a coprimely structured ring if and only if* R *is a coprimely structured ring.*

Proof For the sufficient condition, let I be an ideal of R contained in the nilradical of R and R/I be a coprimely structured ring. Let $P_i + \mathfrak{M} = R$ for some $\{P_i\}_{i\in\Delta} \subseteq Spec(R)$ and $\mathfrak{M} \in MaxSpec(R)$. Then $R/I = (P_i + \mathfrak{M})/I = P_i/I + \mathfrak{M}/I$. Since R/I is a coprimely structured ring, we get $\bigcap_{i\in\Delta} P_i/I \nsubseteq \mathfrak{M}/I$. Therefore, $\left(\bigcap_{i\in\Delta} P_i\right)/I \nsubseteq \mathfrak{M}/I$ and then we obtain $\bigcap_{i\in\Delta} P_i \nsubseteq \mathfrak{M}$. Thus R is a coprimely structured ring by Theorem 4. The necessary condition follows from Corollary 1. □

3. Localization of coprimely structured rings

In this section we will give some relations between coprimely structured rings and their localizations. First of all we give the localization. Note that R does not have to be a coprimely structured ring even if $S^{-1}R$ is coprimely structured. As an example of this, $\mathbb{Q}$ is a coprimely structured ring but $\mathbb{Z}$ is not.

Proposition 2 *Let R be a coprimely structured ring and P be a prime ideal of R. Then the localization of R at P, R_P, is a coprimely structured ring.*

Proof Let P be a prime ideal of R. Then R_P is a local ring and so it is a coprimely structured ring by Theorem 2. □

Theorem 6 *Let R be a coprimely structured ring and S be the complement of the union of a family of maximal ideals of R. Then $S^{-1}R$ is a coprimely structured ring.*

Proof Let Δ be a subset of $MaxSpec(R)$ and let S be the complement of the union of elements of Δ. Then it is sufficient to show that the maximal ideal $S^{-1}\mathfrak{M}$ of $S^{-1}R$ is coprimely structured ideal by Theorem 2, where $\mathfrak{M} \in \Delta$. For this, suppose that $S^{-1}I_i + S^{-1}\mathfrak{M} = S^{-1}R$ where $S^{-1}I_i$ are ideals of $S^{-1}R$ for $i \in \Lambda$. Then we get $I_i + \mathfrak{M} = R$. Since R is a coprimely structured ring, we get $\bigcap_{i\in\Lambda} I_i \nsubseteq \mathfrak{M}$ and hence $\bigcap_{i\in\Lambda} S^{-1}I_i \nsubseteq S^{-1}\mathfrak{M}$. □

Before giving Theorem 7, for another quotient ring, we give the following well-known lemma.

Lemma 3 *[5, Theorem 22.1] Let R be an integral domain. The following conditions are equivalent:*
(i) R_P is a valuation ring for each proper prime P of R.
(ii) R_M is a valuation ring for each maximal ideal M of R.
(iii) R is a Prüfer domain.

Theorem 7 *Let R be a Prüfer domain and let S be any multiplicatively closed subset of R. If R is a coprimely structured ring, then $S^{-1}R$ is a coprimely structured ring.*

Proof Let $S^{-1}P$ be any prime ideal of $S^{-1}R$ and $S^{-1}I_i$ ($i \in \Lambda$, Λ be an index set) be ideals of $S^{-1}R$ such that $S^{-1}I_i + S^{-1}P = S^{-1}R$. Hence we get $S^{-1}(I_i + P) = S^{-1}R$. Assume that $I_i + P \neq R$. Then there exists a maximal ideal $\mathfrak{M}$ of R such that $I_i + P \subseteq \mathfrak{M}$. Since $I_i \subseteq \mathfrak{M}$, there exists a minimal prime ideal P_i of I_i such that $I_i \subseteq P_i \subseteq \mathfrak{M}$ and $P_i + P \subseteq \mathfrak{M}$. Since R is a Prüfer domain, P_i and P are comparable. Thus $S^{-1}I_i + S^{-1}P = S^{-1}P$ or $S^{-1}I_i + S^{-1}P \subseteq S^{-1}P_i$. Thus, in either case, we have a contradiction. Hence $I_i + P = R$ and so we obtain $\bigcap_{i\in\Lambda} I_i \nsubseteq P$. Therefore, $\bigcap_{i\in\Lambda} \left(S^{-1}I_i\right) \nsubseteq S^{-1}P$.

□

4. Rings related to coprimely structured rings

In this section, we will examine the correspondence between coprimely structured rings, strongly 0-dimensional rings, and h-local domains. Before this, we will give some definitions.

Definition 2 *Let S be an index set and $\{I_i \mid i \in \Lambda\}$ be any family of ideals of R. A prime ideal P is said to be a strongly prime ideal if $\bigcap_{i\in\Lambda} I_i \subseteq P$; then $I_j \subseteq P$ for some $j \in \Lambda$. A ring R is said to be a strongly 0-dimensional ring if all prime ideals of R are strongly prime ideals.*

The following two theorems give some equivalences of strongly 0-dimensional rings in [7].

Theorem 8 *[7, Theorem 2.9] The following statements on a ring R are equivalent:*
(i) R is a strongly 0-dimensional ring.
(ii) R is a zero-dimensional ring satisfying the following property: if for any family $\{I, I_i\}_{i\in\Delta}$ of ideals of R, $I + I_i = R$ for all $i \in \Delta$ implies $I + \left(\bigcap_{i\in\Delta} I_i\right) = R$.

Theorem 9 *[7, Theorem 2.18] The following statements on a ring R are equivalent:*
(i) R is an Artinian ring.
(ii) R is strongly 0-dimensional and locally Noetherian.
(iii) R is a Noetherian ring and $R_{\mathfrak{M}}$ is a strongly 0-dimensional ring for all maximal ideals $\mathfrak{M}$ of R.

The following theorem gives the correspondence between strongly 0-dimensional rings and coprimely structured rings.

Theorem 10 *Every strongly 0-dimensional ring is a coprimely structured ring.*

Proof Let R be a ring and P be a prime ideal of R. Assume that for $i \in \Lambda$, I_i's are ideals of R such that $I_i + P = R$. Suppose that $\bigcap_{i\in\Lambda} I_i \subseteq P$. Since R is a strongly 0-dimensional ring, we get $I_j \subseteq P$ for some $j \in \Lambda$. Then $I_j + P = P$ and this gives a contradiction. Hence $\bigcap_{i\in\Lambda} I_i \nsubseteq P$ and so R is a coprimely structured ring. □

As mentioned in the introduction, the converse of the above theorem is not true in general. Indeed, we know that the ring $R = k[[x, y]]$ is a coprimely structured ring for the field k by Example 1. However, this ring is not a strongly 0-dimensional ring [7, Example 2.4].

Theorem 11 *Let R be a ring, Λ be an index set, and I_i's $(i \in \Lambda)$ be ideals of R. Then the following are equivalent:*
(i) R is a coprimely structured ring.
(ii) Every maximal ideal $\mathfrak{M}$ of R is a strongly prime ideal.
(iii) For any maximal ideal $\mathfrak{M}$ of R, $\mathfrak{M} + I_i = R$ implies $\mathfrak{M} + \left(\bigcap_{i\in\Lambda} I_i\right) = R$.

Proof $(i) \Rightarrow (ii)$ Let R be a coprimely structured ring and suppose that $\bigcap_{i\in\Lambda} I_i \subseteq \mathfrak{M}$ for a family of ideals $\{I_i\}_{i\in\Lambda}$ of R and maximal ideal $\mathfrak{M}$ of R. Assume that $I_i \nsubseteq \mathfrak{M}$ for all i. Then we get $I_i + \mathfrak{M} = R$. Since R is a coprimely structured ring we get $\bigcap_{i\in\Lambda} I_i \nsubseteq \mathfrak{M}$, which is a contradiction. $(ii) \Rightarrow (iii)$ Suppose that the condition (ii) holds and $I_i + \mathfrak{M} = R$ for the family of ideals $\{I_i\}$ and maximal ideal $\mathfrak{M}$. Assume that $\mathfrak{M} + \left(\bigcap_{i\in\Lambda} I_i\right) \neq R$. Then $\bigcap_{i\in\Lambda} I_i \subseteq \mathfrak{M}$, and by our assumption we get $I_j \subseteq \mathfrak{M}$ for some $j \in \Lambda$. This gives a contradiction.
$(iii) \Rightarrow (i)$ Suppose that R satisfies the condition (iii) and $I_i + P = R$ for $P \in Spec(R)$ and ideals I_i of R. Assume that $\bigcap_{i\in\Lambda} I_i \subseteq P$. Then there exists a maximal ideal $\mathfrak{M}$ of R such that $P \subseteq \mathfrak{M}$. Therefore, we get $I_i + \mathfrak{M} = R$ and $\bigcap_{i\in\Lambda} I_i \subseteq \mathfrak{M}$. Thus $\mathfrak{M} + \left(\bigcap_{i\in\Lambda} I_i\right) \neq R$ is a contradiction. □

Corollary 2 *Let R be a ring with Krull dimension 0. Then R is a coprimely structured ring if and only if R is a strongly 0-dimensional ring.*

Proof It follows from Theorem 10 and Theorem 11. □

Theorem 12 *Let R be a ring satisfying the property that for any family of ideals $\{I_\alpha\}_{\alpha\in S}$ of R and any index set S there exists a finite subset S' of S such that $\sqrt{\bigcap_{\alpha\in S} I_\alpha} = \bigcap_{\alpha\in S'} \sqrt{I_\alpha}$, then R is a strongly 0-dimensional ring. The converse is true when R is Noetherian.*

Proof Let $\bigcap_{\alpha\in S} I_\alpha \subseteq P$ for a family of ideals $\{I_\alpha\}_{\alpha\in S}$ and a prime ideal P of R. By our assumption, we obtain $\sqrt{\bigcap_{\alpha\in S} I_\alpha} \subseteq \bigcap_{\alpha\in S'} \sqrt{I_\alpha} \subseteq P$ for all finite subset S' of S, that is, $I_\beta \subseteq \sqrt{I_\beta} \subseteq P$ for some $\beta \in S'$. For the converse, let $\{I_\alpha\}_{\alpha\in S}$ be a family of ideals of R. Then we get $\sqrt{\bigcap_{\alpha\in S} I_\alpha} \subseteq \bigcap_{\alpha\in S'} \sqrt{I_\alpha}$ for all finite subset S' of S. Since R is a Noetherian ring we have

$$\sqrt{\bigcap_{\alpha\in S} I_\alpha} = \bigcap_{P_\gamma \in V(\bigcap_{\alpha\in S} I_\alpha)} P_\gamma = \bigcap_{P_\gamma \in Min(\bigcap_{\alpha\in S} I_\alpha)} P_\gamma$$

where $Min(\bigcap_{\alpha\in S} I_\alpha)$ is a finite set, say $Min(\bigcap_{\alpha\in S} I_\alpha) = \{P_{\gamma_1}, P_{\gamma_2}, ..., P_{\gamma_n}\}$. Therefore, we get $\bigcap_{\alpha\in S} I_\alpha \subseteq P_{\gamma_j}$ for all $\gamma_j \in S' = \{\gamma_1, \gamma_2, ..., \gamma_n\}$. Since R is a strongly 0-dimensional ring there exists an $\alpha_j \in S$ such that $I_{\alpha_j} \subseteq P_{\gamma_j}$ and so $\sqrt{I_{\alpha_j}} \subseteq P_{\gamma_j}$. Thus

$$\sqrt{I_{\alpha_1}} \cap ... \cap \sqrt{I_{\alpha_n}} \subseteq P_{\gamma_1} \cap ... \cap P_{\gamma_n}$$

and hence

$$\bigcap_{\alpha_j \in S'} \sqrt{I_{\alpha_j}} \subseteq \bigcap_{\gamma_j \in S'} P_{\gamma_j} = \sqrt{\bigcap_{\alpha\in S} I_\alpha}.$$

□

Theorem 13 *Let R be a ring satisfying the property that for any family of ideals $\{I_\alpha\}_{\alpha\in S}$ of R and any index set S there exists a finite subset S' of S such that $\sqrt{\bigcap_{\alpha\in S} I_\alpha} = \bigcap_{\alpha\in S'} \sqrt{I_\alpha}$. Then R is a coprimely structured ring.*

Proof Let $I_\alpha + P = R$ for any family of ideals $\{I_\alpha\}_{\alpha\in S}$ and prime ideal P of R. Suppose that $\bigcap_{\alpha\in S} I_\alpha \subseteq P$. Then $\sqrt{\bigcap_{\alpha\in S} I_\alpha} \subseteq P$. By our hypothesis, $\bigcap_{\alpha\in S'} \sqrt{I_\alpha} \subseteq P$ for some finite subset S' of S. Therefore, $I_\alpha \subseteq \sqrt{I_\alpha} \subseteq P$ for some $\alpha \in S'$, which is a contradiction. □

Definition 3 *(1) A domain R is said to be h-local provided that every nonzero ideal of R is contained in at most finitely many maximal ideals of R and each nonzero prime ideal of R is contained in a unique maximal ideal of R [9].*
(2) A subdirectly irreducible ring is one in which the intersection of all the nonzero ideals is a nonzero ideal [3,8].

Note that in [9, Theorem 2.1 (8)], Olberding compiles some characterizations of h-local domains from the literature. One of them is given for a domain R as follows: R is an h-local domain if and only if for $\bigcap_{i\in\Delta} I_i \subseteq \mathfrak{M}$ implies $I_i \subseteq \mathfrak{M}$ for some $i \in \Delta$, where $\mathfrak{M}$ is a maximal ideal of R, $\{I_i\}$ is a collection of ideals of R having nontrivial intersection, and Δ is an index set. In the following theorem, we give the relation between a coprimely structured ring and an h-local domain.

Theorem 14 *Let R be a coprimely structured domain. Then R is an h-local domain.*

Proof Let $0 \neq \bigcap_{i\in\Lambda} I_i \subseteq \mathfrak{M}$ for an index set Λ and suppose that $I_i \nsubseteq \mathfrak{M}$ for all i and $\mathfrak{M} \in MaxSpec(R)$. Then $I_i + \mathfrak{M} = R$ for all i. It follows that $\bigcap_{i\in\Lambda} I_i \nsubseteq \mathfrak{M}$, since R is coprimely structured, which is a contradiction.

□

The converse of the above theorem is not true in general. In [9, Example 3.1] Olberding proved that if R is a Noetherian domain of dimension 1, then R is an h-local domain. For example, $\mathbb{Z}$ is an h-local domain that is not a coprimely structured ring.

Proposition 3 *Let R be a subdirectly irreducible ring. Then R is a coprimely structured domain if and only if R is an h-local domain.*

Proof It follows from Theorem 14 and [9, Theorem 2.1 (8)]. □

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