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Research Article

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Abstract: A spliced sequence is formed by combining all of the terms of two or more convergent sequences, in their original order, into a new spliced sequence. In this paper replacing convergent sequences by bounded sequences, we study the summability of spliced sequences and give some inequalities that provide us with approximation of the core of transformation of these sequences by a summability matrix. We also present some further results via the Lebesgue integral.

Key words: Matrix summability, spliced sequences, A-density, core of a sequence

1. Introduction

In 1930, the core of a sequence that is strongly related to the set of limit points of the sequence was introduced by Knopp [10] and several properties of this concept have been studied since then. Particularly papers that investigate under which conditions the core of the transformation of a sequence by a summability matrix is a subset of the core of the original sequence have been written by many authors [1, 2, 6, 9, 12, 13]. In the present paper we give some inequalities that are closely connected to the core of the transformation of a spliced sequence by a summability matrix.

Recently Osikiewicz [14] studied the summability of spliced sequences. He has shown that A-limits of spliced sequences are closely related to A-densities of the sets in the partition. Furthermore, Unver et al. [16] have investigated the summability of spliced sequences in metric spaces and given the Bochner integral representation of A-limits of the spliced sequences in Banach spaces.

A spliced sequence is formed by combining all of the terms of two or more convergent sequences, in their original order, into a new spliced sequence [14]. In this paper replacing convergent sequences by bounded sequences, we give some inequalities that help us to approximate the core of transformation of a spliced sequence. Our results also reduce to Osikiewicz's equalities in special cases. We also obtain some inequalities via a Lebesgue integral. These inequalities extend a result in [16] for a real case. Throughout the paper we deal with the real valued sequences and recall that the core of a bounded real sequence x is the interval $[\liminf x, \limsup x]$.

Let $A = (a_{nk})$ be a summability matrix and let $x = (x_k)$ be a sequence. If the sequence $(Ax)_n = \sum_k a_{nk}x_k$ exists, i.e. the series $\sum_k a_{nk}x_k$ is convergent for each $n \in \mathbb{N}$ then we say that Ax is the A-transformation of x, where $\mathbb{N}$ is the set of all positive integers. If the sequence Ax converges to a number

L then we say that x is A-summable to L. A summability matrix A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_k x_k = L$. The following theorem characterizes the regular matrices:

Theorem 1 *A summability matrix $A = (a_{nk})$ is regular if and only if*

i) $\displaystyle \sup_n \sum_k |a_{nk}| < \infty,$

ii) $\displaystyle \lim_n \sum_k a_{nk} = 1,$

iii) $\displaystyle a_k := \lim_n a_{nk} = 0$ *for all $k \in \mathbb{N}$ [3].*

Let $A = (a_{nk})$ be a nonnegative regular summability matrix. Then the $A - density$ of $K \subseteq \mathbb{N}$ is given by

$$\delta_A(K) := \lim_n \sum_{k \in K} a_{nk}$$

whenever the limit exists [4, 5, 7, 8, 11].

Let $A = (a_{nk})$ be a summability matrix. Then we let

$$\chi(A) := \lim_n \sum_k a_{nk} - \sum_k a_k$$

whenever the series are convergent and the limit exists. Note that if the matrix A is conservative then $\chi(A)$ exists [3]. Let A be a summability matrix and let $K = \{\nu_j\}$ be an infinite subset of $\mathbb{N}$. Then the matrix $A^{[K]} = (b_{nk})$ is said to be a column submatrix of A, where $b_{nk} = a_{n,\nu_k}$ for all $n, k \in \mathbb{N}$. Now we recall the following theorem of Rhoades [15]:

Theorem 2 *Let $A = (a_{nk})$ be a summability matrix for which $\chi(A)$ is defined. If there exists an integer q such that $a_{nk} \geq 0$ for all $k \geq q$ then*

$$\liminf_n (Ax)_n \geq \sum_{k=1}^{\infty} a_k x_k + \chi(A) \liminf_n x_n$$

and

$$\limsup_n (Ax)_n \leq \sum_{k=1}^{\infty} a_k x_k + \chi(A) \limsup_n x_n$$

whenever the series $\displaystyle \sum_{k=1}^{\infty} a_k x_k$ is convergent [15].

Now we have the following

Lemma 1 *Let A be a nonnegative regular summability matrix, let $K := \{\nu_j\}$ be an infinite subset of $\mathbb{N}$, and let $x = (x_k)$ be a bounded sequence. If $\delta_A(K)$ exists then*

$$\liminf_n (A^{[K]}x)_n \geq \delta_A(K) \liminf_n x_n \tag{1.1}$$

and

$$\limsup_n (A^{[K]}x)_n \leq \delta_A(K) \limsup_n x_n. \tag{1.2}$$

Proof   Since A is regular then $a_k = 0$ for all $k \in \mathbb{N}$, which implies $b_k := \lim_n b_{nk} = 0$ for all $k \in \mathbb{N}$, where $b_{nk} = a_{n,\nu_k}$ for all $n, k \in \mathbb{N}$. Hence we get from Theorem 2 that

$$\liminf_n (A^{[K]}x)_n \geq \sum_{k=1}^{\infty} b_k x_k + \chi(A^{[K]}) \liminf_n x_n$$

$$= \chi(A^{[K]}) \liminf_n x_n$$

$$= \left(\lim_n \sum_k b_{nk} - \sum_k b_k \right) \liminf_n x_n$$

$$= \left(\lim_n \sum_k a_{n,\nu_k} \right) \liminf_n x_n$$

$$= \left(\lim_n \sum_{k \in K} a_{nk} \right) \liminf_n x_n$$

$$= \delta_A(K) \liminf_n x_n.$$

Taking $-x$ instead of x in (1.1) it is easy to prove (1.2).   □

2. Finite Splices

Definition 1 *Let M be a fixed positive integer. An M-partition of $\mathbb{N}$ consists of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i = 1, 2, ..., M$ such that $\bigcup_{i=1}^{M} K_i = \mathbb{N}$ and for all $i \neq r$ $K_i \cap K_r = \varnothing$.*

Definition 2 *Let $\{K_i : i = 1, 2, ..., M\}$ be a fixed M-partition of $\mathbb{N}$ and let $x^{(i)} = \left(x_j^{(i)} \right)$ be a bounded sequence for $i = 1, 2, ..., M$. If $k \in K_i$, then $k = \vartheta_i(j)$ for some j. Define $x = (x_k)$ as $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an M^*-splice over $\{K_i : i = 1, 2, ..., M\}$.*

Note that in [14] the spliced sequences (M-splice) are obtained from convergent sequences and every M-splice is also an M^*-splice. Note that any M^*-splice is bounded.

The following theorem shows how we can approximate the core of Ax.

Theorem 3 *Let A be a nonnegative regular summability matrix and let $\{K_i = \{\vartheta_i(j)\} : i = 1, 2, ..., M\}$ be an M-partition of $\mathbb{N}$. If $\delta_A(K_i)$ exists for all $i = 1, 2, ..., M$ then for any M^*-splice x over $\{K_i\}$ we have*

$$\liminf_n (Ax)_n \geq \sum_{i=1}^{M} \delta_A(K_i)\alpha_i \tag{2.1}$$

and

$$\limsup_n (Ax)_n \le \sum_{i=1}^M \delta_A(K_i)\beta_i \qquad (2.2)$$

where $\alpha_i = \liminf_j x_j^{(i)}$ *and* $\beta_i = \limsup_j x_j^{(i)}$.

Proof Assume that $\delta_A(K_i)$ exists for all $i = 1, 2, ..., M$ and let x be an M^*-splice over $\{K_i\}$. Then for all $n \in \mathbb{N}$ we have as in [14]

$$
\begin{aligned}
(Ax)_n &= \sum_{i=1}^\infty a_{nk}x_k \\
&= \sum_{i=1}^M \left(\sum_{k \in K_i} a_{nk}x_k \right) \\
&= \sum_{i=1}^M \left(\sum_{j=1}^\infty a_{n,\vartheta_i(j)} x_{\vartheta_i(j)} \right) \\
&= \sum_{i=1}^M \left(\sum_{j=1}^\infty a_{n,\vartheta_i(j)} x_j^{(i)} \right) \\
&= \sum_{i=1}^M \left(A^{[K_i]} x^{(i)} \right)_n. \qquad (2.3)
\end{aligned}
$$

Hence it follows from (2.3) and Lemma 1 that

$$
\begin{aligned}
\liminf_n (Ax)_n &= \liminf_n \sum_{i=1}^M \left(A^{[K_i]} x^{(i)} \right)_n \\
&\ge \sum_{i=1}^M \liminf_n \left(A^{[K_i]} x^{(i)} \right)_n \\
&\ge \sum_{i=1}^M \delta_A(K_i)\alpha_i
\end{aligned}
$$

which proves (2.1). Taking $-x$ instead of x in (2.1) one can prove (2.2). □

If $x^{(i)}$ is convergent for any $i = 1, 2, ..., M$ then $\gamma_i := \alpha_i = \beta_i$ for any $i = 1, 2, ..., M$. Therefore, not only does Theorem 3 prove that the core of Ax does not exceed the interval $\left[\sum_{i=1}^M \delta_A(K_i)\alpha_i, \sum_{i=1}^M \delta_A(K_i)\beta_i \right]$ but also it generalizes the Theorem 2.5 of [14].

3. Infinite Splices

Definition 3 *An ∞-partition on $\mathbb{N}$ consists of an infinite number of infinite sets $K_i = \{\vartheta_i(j)\}$ for $i \in \mathbb{N}$, such that $\bigcup_{i=1}^\infty K_i = \mathbb{N}$ and for all $i \ne r$, $K_i \cap K_r = \varnothing$.*

Definition 4 *Let $\{K_i : i \in \mathbb{N}\}$ be a fixed ∞-partition of $\mathbb{N}$ and let $x^{(i)} = \left(x_j^{(i)} \right)$ be a bounded sequence for $i \in \mathbb{N}$. If $k \in K_i$, then $k = \vartheta_i(j)$ for some j. Define $x = (x_k)$ as $x_k = x_{\vartheta_i(j)} = x_j^{(i)}$. Then x is called an ∞^*-splice over $\{K_i : i \in \mathbb{N}\}$.*

Note that in [14] the spliced sequences (∞-splice) are obtained from convergent sequences and it is obvious that any $\infty-$splice is also an ∞^*-splice. Note that an ∞^*-splice does not need to be bounded.

Similarly the following theorem provides us with estimate of the core of Ax.

Theorem 4 *Let A be a nonnegative regular summability matrix and let $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}\}$ be an ∞-partition of $\mathbb{N}$. If $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \delta_A(K_i) = 1$ then for any bounded ∞^*-splice x over $\{K_i\}$ we get*

$$\liminf_n (Ax)_n \geq \sum_{i=1}^{\infty} \delta_A(K_i)\alpha_i \tag{3.1}$$

and

$$\limsup_n (Ax)_n \leq \sum_{i=1}^{\infty} \delta_A(K_i)\beta_i \tag{3.2}$$

where $\alpha_i = \liminf_j x_j^{(i)}$ and $\beta_i = \limsup_j x_j^{(i)}$.

Proof Assume that $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$ with $\sum_{k=1}^{\infty} \delta_A(K_i) = 1$ and let x be an ∞^*-splice x over $\{K_i\}$. Then for all $n \in \mathbb{N}$ we have as in [14]

$$
\begin{aligned}
(Ax)_n &= \sum_{i=1}^{\infty} a_{nk} x_k \\
&= \sum_{i=1}^{\infty} \left(\sum_{k \in K_i} a_{nk} x_k \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{n,\vartheta_i(j)} x_{\vartheta_i(j)} \right) \\
&= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{n,\vartheta_i(j)} x_j^{(i)} \right) \\
&= \sum_{i=1}^{\infty} \left(A^{[K_i]} x^{(i)} \right)_n.
\end{aligned}
\tag{3.3}
$$

For all n define $f_n : \mathbb{N} \to \mathbb{C}$ and $g_n : \mathbb{N} \to \mathbb{C}$ by

$$f_n(i) := \left(A^{[K_i]} x^{(i)} \right)_n \text{ and } g_n(i) := H \left(A^{[K_i]} e \right)_n$$

where $H := \sup_k |x_k|$ and $e = (1, 1, ...)$. Now let μ be the counting measure. From Theorem 1.2 in [14] we know that

$$\lim_n g_n(i) = H\delta_A(K_i)$$

and

$$\lim_n \int_{\mathbb{N}} g_n(i)d\mu = \int_{\mathbb{N}} \left(\lim_n g_n(i)\right) d\mu = H > 0. \tag{3.4}$$

Furthermore, it is easy to show for all $n, i \in \mathbb{N}$ that

$$|f_n(i)| \le g_n(i).$$

Since f_n and g_n are measurable with respect to μ and $f_n + g_n \ge 0$ for all n, then it follows from (3.4) and Fatou's Lemma that

$$\int_{\mathbb{N}} \liminf_n (f_n + g_n)(i)d\mu \le \liminf_n \int_{\mathbb{N}} (f_n + g_n)(i)d\mu$$

$$= \liminf_n \left(\int_{\mathbb{N}} f_n(i)d\mu + \int_{\mathbb{N}} g_n(i)d\mu\right)$$

$$= \liminf_n \int_{\mathbb{N}} f_n(i)d\mu + \lim_n \int_{\mathbb{N}} g_n(i)d\mu$$

$$= \liminf_n \int_{\mathbb{N}} f_n(i)d\mu + H \sum_{i=1}^{\infty} \delta_A(K_i)$$

$$= \liminf_n \int_{\mathbb{N}} f_n(i)d\mu + H. \tag{3.5}$$

On the other hand, since (g_n) is convergent for all i

$$\int_{\mathbb{N}} \liminf_n (f_n + g_n)(i)d\mu = \int_{\mathbb{N}} \left(\liminf_n f_n(i) + \lim_n g_n(i)\right) d\mu$$

$$= \int_{\mathbb{N}} \liminf_n f_n(i)d\mu + \int_{\mathbb{N}} \lim_n g_n(i)d\mu$$

$$= \int_{\mathbb{N}} \liminf_n f_n(i)d\mu + H \sum_{i=1}^{\infty} \delta_A(K_i)$$

$$= \int_{\mathbb{N}} \liminf_n f_n(i)d\mu + H. \tag{3.6}$$

Hence from (3.5) and (3.6) one can get

$$\int_{\mathbb{N}} \liminf_n f_n(i) d\mu \leq \liminf_n \int_{\mathbb{N}} f_n(i) d\mu$$

$$= \liminf_n \sum_{i=1}^{\infty} \left(A^{[K_i]} x^{(i)} \right)_n$$

$$= \liminf_n (Ax)_n. \tag{3.7}$$

Now using Lemma 1

$$\int_{\mathbb{N}} \liminf_n f_n(i) d\mu = \int_{\mathbb{N}} \liminf_n \left(A^{[K_i]} x^{(i)} \right) d\mu$$

$$\geq \int_{\mathbb{N}} \delta_A(K^{(i)}) \alpha_i d\mu$$

$$= \sum_{i=1}^{\infty} \delta_A(K^{(i)}) \alpha_i. \tag{3.8}$$

Hence by (3.7) and (3.8) we get

$$\sum_{i=1}^{\infty} \delta_A(K^{(i)}) \alpha_i \leq \liminf_n (Ax)_n$$

which concludes the proof of (3.1).

Taking $-x$ instead of x in (3.1), one can prove (3.2) immediately. □

If $x^{(i)}$ is convergent for any $i \in \mathbb{N}$ then $\gamma_i := \alpha_i = \beta_i$ for any $i \in \mathbb{N}$. Hence our Theorem 4 yields Theorem 3.4 in [14]. Moreover, this theorem shows that the core of Ax does not exceed the interval $\left[\sum_{i=1}^{\infty} \delta_A(K_i) \alpha_i, \sum_{i=1}^{\infty} \delta_A(K_i) \beta_i \right]$.

4. Inequalities via Lebesgue Integral

Recently Unver et al. [16] gave the Bochner integral representation of the A-limits of ∞-splices in Banach spaces. In this section we give some inequalities for the limit inferior and the limit superior of A-transformations of ∞^*-splices via a Lebesgue integral. This result extends Proposition 2 in [16] for a real case. First we recall the following definition:

Consider a set function $F : \mathcal{B}(\mathbb{R}) \to [0,1]$ such that $F(\mathbb{R}) = 1$ and if $U_1, U_2, \ldots$ are disjoint sets in $\mathcal{B}(\mathbb{R})$; then

$$F \left(\bigcup_{j=1}^{\infty} U_j \right) = \sum_{j=1}^{\infty} F(U_j)$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel sigma field on $\mathbb{R}$ that is generated by open intervals. Such a function is called a probability measure or a distribution on $\mathbb{R}$.

Theorem 5 *Let* $A = (a_{nk})$ *be a nonnegative regular summability matrix such that each row adds up to one and let* $\{K_i = \{\vartheta_i(j)\} : i \in \mathbb{N}\}$ *be an* ∞ *-partition of* $\mathbb{N}$. *If* $\delta_A(K_i)$ *exists for all* $i \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} \delta_A(K_i) = 1$ *then for any bounded* ∞^* *-splice sequence* x *over* $\{K_i\}$ *we have*

$$\liminf_n (Ax)_n \geq \int_{\mathbb{R}} t dF$$

and

$$\limsup_n (Ax)_n \leq \int_{\mathbb{R}} t dG$$

where

$$F(U) = \sum_{\alpha_i \in U} \delta_A(K_i),$$

$$G(U) = \sum_{\beta_i \in U} \delta_A(K_i)$$

and $\alpha_i = \liminf_j x_j^{(i)}$, $\beta_i = \limsup_j x_j^{(i)}$.

Proof Assume that $\delta_A(K_i)$ exists for all $i \in \mathbb{N}$. Now as in Proposition 2 in [16], define the function $s : X \to X$ by

$$s(t) = \begin{cases} \alpha_i, & t = \alpha_i, \ i \in \mathbb{N} \\ \theta, & otherwise \end{cases}$$

and the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = t.$$

Observe that $f = s$ almost everywhere with respect to F. Thus we have

$$\int_{\mathbb{R}} t dF = \int_{\mathbb{R}} s(t) dF. \tag{4.1}$$

Now define a sequence of simple functions (s_m) by

$$s_m(t) = \begin{cases} \alpha_i, & t = \alpha_i, \ i = 1, 2, ..., m \\ \theta, & otherwise. \end{cases}$$

It is easy to see that for all m

$$|s_m(t) - s(t)| = \begin{cases} |\alpha_i|, & t = \alpha_i, \ i > m \\ 0, & otherwise. \end{cases}$$

Thus for all $t \in X$, $\lim_{m \to \infty} |s_m(t) - s(t)| = 0$. On the other hand, since the spliced sequence is bounded there exists an $H > 0$ such that

$$\sup_{t \in X} |s_m(t) - s(t)| \leq \sup_{i > m} |\alpha_i| < H.$$

Then from the Bounded Convergence Theorem we have

$$\lim_{m\to\infty}\int_{\mathbb{R}}|s_m(t)-s(t)|\,dF=\int_{\mathbb{R}}\lim_{m\to\infty}|s_m(t)-s(t)|\,dF=0$$

which implies

$$\int_{\mathbb{R}}s(t)dF=\lim_{m\to\infty}\int_{\mathbb{R}}s_m(t)dF$$

$$=\lim_{m\to\infty}\int_{\mathbb{R}}\left(\sum_{i=1}^{m}I_{\{\alpha_i\}}(t)\alpha_i\right)dF$$

$$=\lim_{m\to\infty}\sum_{i=1}^{m}F\left(\{\alpha_i\}\right)\alpha_i$$

$$=\lim_{m\to\infty}\sum_{i=1}^{m}\delta_A(K_i)\alpha_i$$

$$=\sum_{i=1}^{\infty}\delta_A(K_i)\alpha_i. \tag{4.2}$$

Now from (4.2) and Theorem 4 we get

$$\liminf_{n}(Ax)_n\geq\int_{\mathbb{R}}tdF.$$

Similarly, it is easy to show that

$$\limsup_{n}(Ax)_n\leq\int_{\mathbb{R}}tdG$$

which concludes the proof. □

Note that if $x^{(i)}$ is convergent for any $i\in\mathbb{N}$ then $\alpha_i=\beta_i$ for any $i\in\mathbb{N}$, which implies $H:=F=G$. Hence, we get from Theorem 5 that

$$\int_{\mathbb{R}}tdH\leq\liminf_{n}(Ax)_n\leq\limsup_{n}(Ax)_n\leq\int_{\mathbb{R}}tdH$$

i.e.

$$\lim_{n}(Ax)_n=\int_{\mathbb{R}}tdH$$

which proves Proposition 2 in [16] in a real case.

This theorem may be extended to the Banach Lattices via a Bochner integral.

Acknowledgement

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