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Projective crossed modules of algebras and cyclic homology

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Abstract: In this work, we give a characterization of free crossed modules and also get a relation between projective crossed modules and the cyclic homology of associative algebras by using Hopf-type formulas.

Key words: Crossed module of algebras, free crossed module, projective crossed module, cyclic homology

1. Introduction

Crossed modules were first defined by Whitehead [19]. Areas in which crossed modules have been applied include the theory of group presentations, algebraic K-theory, and homological algebra. The crossed module theory has been deeply analyzed by Brown et al.'s book "Nonabelian Algebraic Topology", [5], and in the book by Porter "The Crossed Menagerie"[16]. As an application of cyclic homology of associative algebras, the Hopf-type formulas for the cyclic homology given in Brown and Ellis [4] is developed by using the way of n-fold Čech derived functors in [7]. In this paper, we consider the work by Ratcliffe [17], involving a homological characterization of free and projective crossed modules of groups. He mainly proved that C is a projective crossed G-module if and only if C/C^2 is projective as a coker∂-module and the second homology morphism $\partial_* : H_2C \longrightarrow H_2N$ is trivial where $N = \partial(C)$. In this paper, we want to describe an analogous philosophy for crossed modules of associative algebras, which will be called crossed modules of algebras hereafter. The results are important for examining the first cyclic homology of associative algebras, in terms of Hopf-type formulas.

It is well known that all free modules are projective modules but the converse is not true. Therefore, we give a counterexample about this situation for crossed modules of algebras.

In order to achieve our goals, we organized the paper as follows.

In Section 2, we recall some necessary definitions and examples about crossed modules of algebras.

In Section 3, we give the construction of free crossed modules and give an algebra version of the result that was proved by Ratcliffe [17] and later by Ellis and Porter [10] for the group version to use in Theorem 3.

In Section 4, we give two results: the first one is that the relation between free crossed modules and first cyclic homology of algebras by Hopf-type formula as analogous of Ratcliffe [17]; the second is the projective version of the first one.

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2. Crossed modules of algebras

Detailed information about crossed modules of (commutative) algebras can be obtained from [14–16]. In this section we recall the crossed modules of algebras and give material needed for the rest of the work. Refer to [8] for details. Computing crossed modules of algebras were recently given by Arvasi and Odabaş [2].

Let k be a fixed commutative ring, R be a k-algebra with identity. All algebras in the rest of the paper will be associative.

Let A, B be algebras. An *action* (i.e. derived action) of B on A is a pair of bilinear maps

$$B \times A \longrightarrow A, \ A \times B \longrightarrow A$$

denoted respectively as $(b, a) \longmapsto b \cdot a$, $(a, b) \longmapsto a \cdot b$, with conditions

$$\begin{aligned}
(b_1 \cdot b_2) \cdot a &= b_1 \cdot (b_2 \cdot a) \\
a \cdot (b_1 \cdot b_2) &= (a \cdot b_1) \cdot b_2 \\
(b_1 \cdot a) \cdot b_2 &= b_1 \cdot (a \cdot b_2) \\
b \cdot (a_1 \cdot a_2) &= (b \cdot a_1) \cdot a_2 \\
(a_1 \cdot a_2) \cdot b &= a_1 \cdot (a_2 \cdot b) \\
a_1 \cdot (b \cdot a_2) &= (a_1 \cdot b) \cdot a_2
\end{aligned}$$

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$.

A *crossed module*, (C, R, ∂), (or shortly crossed R-module) consists of an R-algebra C and a morphism $\partial : C \longrightarrow R$ with actions R on C, written $(r, c) \longmapsto r \cdot c$ and $(c, r) \longmapsto c \cdot r$ for $r \in R$, $c \in C$, satisfying the following conditions:

(CM1) for all $r \in R$ and $c \in C$,

$$\partial(r \cdot c) = r\partial(c) \text{ and } \partial(c \cdot r) = \partial(c)r$$

(CM2) for all $c, c' \in C$,

$$\partial(c) \cdot c' = cc' = c \cdot \partial(c')$$

(CM2 is called the Peiffer identity)

If (C, R, ∂) and (C', R', ∂') are crossed modules, a *morphism*, $(\mu, \eta) : (C, R, \partial) \longrightarrow (C', R', \partial')$ *of crossed modules* consists of morphisms $\mu : C \longrightarrow C'$ and $\eta : R \longrightarrow R'$ such that

$$(i) \quad \eta\partial = \partial'\mu \quad \text{and} \quad (ii) \quad \mu(r \cdot c) = \eta(r) \cdot \mu(c) \ \& \ \mu(c \cdot r) = \mu(c) \cdot \eta(r)$$

for all $c \in C$, $r \in R$.

From given definitions above, crossed modules and their morphisms form a category, of course. It will usually be denoted **XMod**. There is, for a fixed algebra R, a subcategory **XMod**$/R$, which has, as objects, those crossed modules with R as the "base", i.e. all (C, R, ∂) for this fixed R, and having morphism from (C, R, ∂) to (C', R, ∂') just those (μ, η) in **XMod** in which $\eta : R \longrightarrow R$ is the identity morphism on R.

Examples:

(i) Any two-sided ideal I in algebra R, (I,R,i) is a crossed module where i is an inclusion map. Conversely, given any crossed module (C,R,∂), the image $I=\partial(C)$ is an ideal in R. (See [15], for commutative version.)

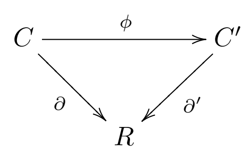
(ii) Any R-module M can be considered an R-algebra with zero multiplication and hence $(M,R,0)$ is a crossed module. Conversely, if (C,R,∂) is a crossed module, $\ker\partial$ is an $R/\partial(C)$-module. (See [15], for commutative version.)

(iii) The kernel, $\ker\partial$, lies in the annihilator of C. (See. [1], for details.)

3. Free crossed modules

Free crossed modules of (commutative) algebras were analyzed in [3, 14].

Let C be a crossed R-module, let Y be a set, and let $v : Y \longrightarrow C$ be a function; then C is said to be a *free crossed R-module with basis v* or, alternatively, on the function $\partial v : Y \longrightarrow R$ if for any crossed R-module C' and function $v' : Y \longrightarrow C'$ such that $\partial' v' = \partial v$, there is a unique morphism of crossed modules



such that $\phi v = v'$.

Theorem 1 *[18] A free crossed R-module exists on any function $f : Y \longrightarrow R$ with codomain R.*

Proof Given a function from a set Y to the R, $f : Y \longrightarrow R$, consider $E = R\langle Y\rangle$, the (free) monoid algebra on Y. Thus each element c of $R\langle Y\rangle$ has the form

$$c = \sum r_i y_i s_i r_j y_j s_j \ldots r_m y_m s_m$$

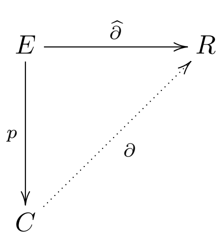
where y'$s \in Y$ and the r's and s'$s \in R$. The function f induces a morphism of R-algebras

$$\widehat{\partial} : E \longrightarrow R$$

given by $\widehat{\partial}(y) = f(y)$. Let W be the ideal of E (sometimes called by analogy with the group theoretic case the Peiffer ideal relative to f) generated by all elements of the form

$$W = \{qq' - \widehat{\partial}(q)\cdot q', qq' - q\widehat{\partial}(q') : q, q' \in E\}.$$

Clearly $\widehat{\partial}(W) = 0$ and so putting $C = E/W$ one obtains an induced morphism (by universal property)



which is the required free crossed R-module on f. $\square$

The following lemma was first proved by Ratcliffe [17]; later Ellis and Porter gave a short proof of that in [10]. We prove an algebra version of that to use in Theorem 3.

Lemma 2 *If E is the (free) monoid R-algebra and C is the quotient algebra defined in proof of Theorem 1, then $P \cap E^2 = EI$, where $I = \ker(\widehat{\partial} : E \longrightarrow R)$ and $P = \ker(p : E \longrightarrow C)$.*

Proof Suppose that $t \in E$ and $i \in I$. Then $p(i) \in \ker\partial$, which is in the annihilator of C; hence

$$p(ti) = p(t)p(i) = 0.$$

That is, $ti \in \ker p = P$. This shows that $EI \subseteq P \cap E^2$.

Conversely, suppose that $q \in P \cap E^2$. Then $q \in W$. It is clear since P is the kernel of the quotient map $E \longrightarrow C$. Since $\widehat{\partial}(W) = 0$, $W \subseteq \ker\widehat{\partial}$ and $q \in \ker\widehat{\partial} = I \subseteq EI$. Consequently, we obtain that $q \in EI$. Thus, $P \cap E^2 \subseteq EI$. $\square$

4. Free (projective) crossed modules and cyclic homology

In this section we give the relation between free crossed modules and first cyclic homology of algebras by using a Hopf-type formula as analogous of Ratcliffe [17]. Then we prove the projective version of this relation. Before that we recall the cyclic homology of algebras from [12] in the following.

Cyclic homology, $HC(A)$ of an algebra A, is Hochschild homology on cyclically invariant chains.

In [9], Donadze et al. obtain the generalized Hopf-type formulas for the cyclic homology of algebras, using the method of n-fold Čech derived functors [7, 11]. They get the exact sequence

$$HC_1(C) \longrightarrow HC_1(M) \longrightarrow A = (J/JC) \longrightarrow C/C^2 \longrightarrow M/M^2 \longrightarrow 0 \tag{1}$$

where $C \xrightarrow{\partial} M$ is surjective and $J = \ker\partial$. If ∂ is free presentation of the algebra M, then (1) induces the Hopf formula

$$HC_1(M) \cong (J \cap C^2)/JC. \tag{2}$$

They also generalized formula (2) to any dimension.

4.1. A characterization of free crossed modules

Theorem 3 *: Let C be a crossed R-module, $M = \partial(C)$ and $Y = \{y_\alpha\}$ be an indexed subset of C. Then C is a free crossed R-module on $f : Y \longrightarrow R$ if and only if*

(i) C/C^2 is a free $\operatorname{coker}\partial$-module with basis $\overline{Y} = \{\overline{y_\alpha}\}$, which is an indexed subset of C/C^2;

(ii) M is an ideal of R, which is the smallest ideal that contains $\{\partial y_\alpha\}$;

(iii) $\partial_* : HC_1(C) \longrightarrow HC_1(M)$ is trivial.

Proof Let C be a free crossed R-module on $f : Y \longrightarrow R$. As properties of free crossed modules and modules, one can easily check that the conditions (i) and (ii) are satisfied.

It is clear that there exist free presentations for M and C

$$\begin{aligned} 0 &\longrightarrow I \longrightarrow E \xrightarrow{\widehat{\partial}} M \longrightarrow 0 \\ 0 &\longrightarrow P \longrightarrow E \xrightarrow{p} C \longrightarrow 0. \end{aligned}$$

Since

$$HC_1(M) \cong (I \cap E^2)/IE$$

and

$$HC_1(C) \cong (P \cap E^2)/PE$$

by [9] the exact sequence (1) turns into the sequence

$$(P \cap E^2)/EP \longrightarrow (I \cap E^2)/EI \longrightarrow I/P \longrightarrow C/C^2 \longrightarrow M/M^2 \longrightarrow 0.$$

Moreover, the first two homomorphisms are induced by inclusion. By Lemma 2, we obtain that $\partial_* : HC_1(C) \longrightarrow HC_1(M)$ is trivial.

Conversely, let the conditions (i)–(iii) be satisfied and C' be a free crossed R-module on $f' : Y' \longrightarrow R$, where $Y' = \{y'_\alpha\}$ is an indexed subset of C' such that $\partial'(y'_\alpha) = \partial(y_\alpha)$ for each α. Since M is an ideal of R, which is the smallest ideal that contains $\{\partial y_\alpha\} = \{\partial' y'_\alpha\}$ by (ii), $\operatorname{Im} \partial' = M$. On the other hand, since C' is a free, there is a unique morphism $\phi : C' \longrightarrow C$ such that $\phi(y'_\alpha) = y_\alpha$. Therefore, ϕ induces a homomorphism $\phi_0 : A' \longrightarrow A$ by $\partial' = \partial \circ \phi$.

Since the 5-term first cyclic homology sequence of algebras exists, the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & HC_1(M) & \longrightarrow & A' & \longrightarrow & C'/(C')^2 & \xrightarrow{\overline{\partial'}} & M/M^2 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & HC_1(M) & \longrightarrow & A & \longrightarrow & C/C^2 & \xrightarrow{\overline{\partial}} & M/M^2 & \longrightarrow & 0 \end{array}$$

is commutative.

By (iii), the bottom row of the diagram is exact. Furthermore, by (i), $\overline{\phi}(\overline{y'_\alpha}) = \overline{y_\alpha}$ implies that $\overline{\phi} : C'/(C')^2 \longrightarrow C/C^2$ is an isomorphism. Hence, ϕ_0 is an isomorphism by the 5-lemma in [13].

Similarly, the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & C' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

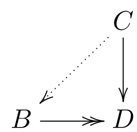
is commutative.

By the short 5-lemma, ϕ is an isomorphism. Moreover, C' is R-isomorphic to C via ϕ. Since $\phi(y'_\alpha) = y_\alpha$ for each α, C is a free crossed R-module on f. □

4.2. Projective crossed modules

For an algebra R, the notion of a free crossed R-module may be generalized:

Let C be a crossed R-module. C is called *projective* if it is a projective object in the category $\mathbf{XMod}/R$. That is, the diagram



is commutative, where B and D are crossed R-modules.

Every free crossed module is clearly projective. Thus, the examples of free crossed modules can be thought of as projective ones.

Examples:

(i) Let C be a free crossed R-module on $\partial v = 0$, where $v : X \longrightarrow C$ is a function; then C is a free R-module on X. Conversely, if C is a free R-module on X, then C is a free crossed R-module on $\partial v = 0v = 0$, where $v : X \longrightarrow C$ is a function.

(ii) Let C be a projective crossed R-module, let B be an arbitrary crossed R-module, and let $\eta : B \longrightarrow C$ be a surjective morphism of crossed R-modules. Then η has a section $s : C \longrightarrow B$.

Before giving the relation between projective crossed modules and first cyclic homology, we give the definition of crossed extension, which will be used in Theorem 4.

Let C be a crossed R-module and A be a coker∂-module. One can regard A as a R-module with trivial action of M.

Then a *crossed extension* of A by C is an extension

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where B is a crossed R-module. $A \times C$ is a crossed R-module with acting diagonally and boundary $\widehat{\partial} : A \times C \longrightarrow R$ given by $\widehat{\partial}(a, c) = \partial(c)$. Moreover, we have the *trivial crossed extension* of A by C :

$$0 \longrightarrow A \longrightarrow A \times C \longrightarrow C \longrightarrow 0.$$

Theorem 4 *C is a projective crossed R-module if and only if C/C^2 is a projective coker∂-module and*

$$\partial_* : HC_1(C) \longrightarrow HC_1(M)$$

is trivial.

Proof Let C be a projective crossed R-module. By (ii) in the above examples, every crossed extension

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\eta} C \longrightarrow 0$$

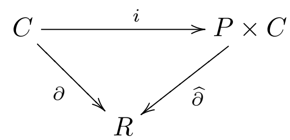
splits.

On the other hand, let $Y = \{y_\alpha\}$ be a set of generators of C, and let B be a free crossed R-module on $f : Y' \longrightarrow B$ where $Y' = \{y'_\alpha\}$ is an indexed subset of B such that $\partial'(y'_\alpha) = \partial(y_\alpha)$ for each α. Since B is free, there is an epimorphism $\eta : B \longrightarrow C$ such that $\eta(y'_\alpha) = y_\alpha$ for each α. Let $P = \ker\eta$. Then there is a crossed extension

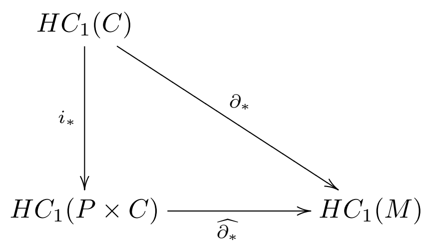
$$0 \longrightarrow P \longrightarrow B \xrightarrow{\eta} C \longrightarrow 0.$$

Hence, this extension splits and $B \cong P \times C$ from properties of split extensions. We know that $P \oplus C/C^2 \cong B/B^2$ as coker∂-modules. Here B/B^2 is a free coker∂-module from Theorem 3. Therefore, P is a projective coker∂-module.

It is clear that $P \oplus C/C^2 \cong B/B^2$ implies that C/C^2 is a projective coker∂-module. There exists a natural injection $i : C \longrightarrow P \times C$. Thus, we can write $\widehat{\partial} \circ i = \partial$ from the following diagram commutativity where $\widehat{\partial}(p,c) = \partial(c)$.



Moreover, the following diagram



is commutative.

Here $P \times C$ is a free crossed R-module. By Theorem 3 (iii), $\widehat{\partial_*}$ is trivial. Since $\widehat{\partial_*} \circ i_* = \partial_*$, ∂_* is trivial.

Conversely, we show that C is a projective crossed R-module. Let $\phi : C' \longrightarrow C$ be an epimorphism. We try to show that $\phi_* : HC_1(C') \longrightarrow HC_1(C)$ is an epimorphism. Let $A = \ker\partial$ and $A' = \ker\partial'$. Since $\partial \circ \phi = \partial'$, ϕ induces a homomorphism $\phi_0 : A' \longrightarrow A$.

It is easily checked that, by using the way given in [6], there is a 6-term exact first cyclic homology sequence for algebras

$$A \otimes C/C^2 \longrightarrow HC_1(C) \xrightarrow{\partial_*} HC_1(M) \longrightarrow A \longrightarrow C/C^2 \xrightarrow{\overline{\partial}} M/M^2 \longrightarrow 0.$$

Similarly, the following diagram

$$\begin{array}{ccccccccccccc}
A' \otimes C'/(C')^2 & \longrightarrow & HC_1(C') & \xrightarrow{\partial'_*} & HC_1(M) & \longrightarrow & A' & \longrightarrow & C'/(C')^2 & \xrightarrow{\overline{\partial'}} & M/M^2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
A \otimes C/C^2 & \longrightarrow & HC_1(C) & \underset{\partial_*}{\longrightarrow} & HC_1(M) & \longrightarrow & A & \longrightarrow & C/C^2 & \underset{\overline{\partial}}{\longrightarrow} & M/M^2 & \longrightarrow & 0.
\end{array}$$

is commutative.

Since ϕ is an epimorphism, $\overline{\phi} : C'/(C')^2 \longrightarrow C/C^2$ is an epimorphism. In [13], the 5-lemma implies that ϕ_0 is an epimorphism. Hence, $\phi_0 \otimes \overline{\phi} : A' \otimes C'/(C')^2 \longrightarrow A \otimes C/C^2$ is an epimorphism. By hypothesis, ∂_* is trivial. Thus, the homomorphism $A \otimes C/C^2 \longrightarrow HC_1(C)$ is an epimorphism. It follows that

$$\phi_* : HC_1(C') \longrightarrow HC_1(C)$$

is an epimorphism.

Consider the 5-term exact first cyclic homology sequence

$$HC_1(C') \xrightarrow{\phi_*} HC_1(C) \longrightarrow \ker\phi \longrightarrow C'/(C')^2 \xrightarrow{\overline{\phi}} C/C^2 \longrightarrow 0.$$

Since ϕ_* is an epimorphism, kerϕ is into $C'/(C')^2$. Hence, the sequence

$$0 \longrightarrow \ker\phi \longrightarrow C'/(C')^2 \longrightarrow C/C^2 \longrightarrow 0$$

is short exact. The sequence splits, since C/C^2 is projective. Therefore,

$$0 \longrightarrow \ker\phi \longrightarrow C' \xrightarrow{\phi} C \longrightarrow 0$$

splits. Hence, we obtain that C is a projective crossed R-module by using the converse of (ii) in the above examples. □

Counterexample: Projective R-modules form examples of projective crossed R-modules. In addition, if I is an ideal of R, then I is a projective crossed R-module if and only if $HC_1(I) = 0$ and I/I^2 is a projective R/I-module An algebra I is said to be superperfect if both $I/I^2 = HC_0(I)$ and $HC_1(I)$ are trivial. Thus, any superperfect ideal I of an algebra R is a projective crossed module, which is not free as a crossed module.

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