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## Factorizations related to the reciprocal Pascal matrix

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**Abstract:** The reciprocal Pascal matrix has entries  $\binom{i+j}{j}^{-1}$ . Explicit formulæ for its LU-decomposition, the LU-decomposition of its inverse, and some related matrices are obtained. For all results,  $q$ -analogues are also presented.

**Key words:** Pascal matrix, LU-decomposition,  $q$ -analogue, Zeilberger's algorithm

### 1. Introduction

Recently, there has been some interest in the *reciprocal Pascal matrix*  $M$ , defined by

$$M_{i,j} = \binom{i+j}{j}^{-1};$$

the indices start here for convenience with  $0, 0$ , and the matrix is either infinite or has  $N$  rows and columns, depending on the context.

Richardson [7] has provided the decomposition  $S = GMG$ , where the diagonal matrix  $G$  has entries  $G_{i,i} = \binom{2i}{i}$ , and  $S$  is the *super Catalan matrix* [2, 4] with entries

$$S_{i,j} = \frac{(2i)!(2j)!}{i!j!(i+j)!}.$$

We want to give an alternative decomposition of  $M$ , provided by the LU-decomposition. We will give explicit expressions for  $L$  and  $U$ , defined by  $LU = M$ , as well as for  $L^{-1}$  and  $U^{-1}$ .

Since there is also interest in  $M^{-1}$ , in particular in the integrality of its coefficients, we also provide the LU-decomposition  $AB = M^{-1}$ , and give expressions for  $A$ ,  $B$ ,  $A^{-1}$ , and  $B^{-1}$ .

In the following section, we provide  $q$ -analogues of these results.

The paper closes with a list of similar results with two additional parameters, but for the matrix with entries  $\binom{i+r+j+s}{j+s}^{-1}$  and  $\binom{i+r+j+s}{j+s}$ .

We would like to mention that results of the type as presented here are useful to find and prove new expansion formulæ for “Fibonomial sums”; see, for instance, [5].

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**2. Identities**

The LU-decomposition  $M = LU$  is given by

$$L_{i,j} = \frac{i!(2j)!}{(i+j)!(i-j)!j!}$$

and

$$U_{i,j} = \frac{(-1)^i j! i!(i-1)!}{(j+i)!(j-i)!(2i-1)!} \quad \text{for } i \geq 1.$$

For  $i = 0$ , the formula is  $U_{0,j} = 1$ .

The formula that needs to be proved is

$$\sum_{0 \leq k \leq \min\{i,j\}} L_{i,k} U_{k,j} = \binom{i+j}{j}^{-1},$$

which is equivalent to

$$1 + \frac{2i!i!j!j!}{(2i)!(2j)!} \sum_{1 \leq k \leq \min\{i,j\}} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k} = \binom{i+j}{j}^{-1}.$$

The von Szily identity [2, 3, 8] is

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \sum_{k \in \mathbb{Z}} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k},$$

and an equivalent form is, by symmetry,

$$\frac{(2i)!(2j)!}{i!j!(i+j)!} = \binom{2i}{i} \binom{2j}{j} + 2 \sum_{k \geq 1} (-1)^k \binom{2i}{i+k} \binom{2j}{j+k}.$$

Thus, the identity to be proven is now

$$\binom{i+j}{j} + \frac{i!j!(i+j)!}{(2i)!(2j)!} \left[ \frac{(2i)!(2j)!}{i!j!(i+j)!} - \binom{2i}{i} \binom{2j}{j} \right] = 1,$$

which is obviously correct.

The formula for  $L^{-1}$  is for  $i \geq j \geq 0$ :

$$L_{i,j}^{-1} = \frac{(-1)^{i-j} i!(i+j-1)!}{(2i-1)!(i-j)!j!}.$$

If necessary ( $i = j = 0$ ), this must be interpreted as a limit.

To check this, we consider

$$\begin{aligned} & \sum_k \frac{i!(2k)!}{(i+k)!(i-k)!k!k!} \frac{(-1)^{k-j} k!(k+j-1)!}{(2k-1)!(k-j)!j!} \\ &= \frac{2i!i!(-1)^j}{j!j!} \sum_{j \leq k \leq i} \frac{k}{(i+k)!(i-k)!} \frac{(-1)^k (k+j-1)!}{(k-j)!} \end{aligned}$$

The sum can be evaluated by computer algebra (or otherwise), and the result is indeed  $\llbracket i = j \rrbracket$ , as desired.

The formula for  $U^{-1}$  is for  $j \geq i \geq 1$

$$U_{i,j}^{-1} = \frac{(-1)^i(j+i-1)!(2j)!}{(j-i)!j!(j-1)!i!i!}$$

and for  $i = 0$

$$U_{0,j}^{-1} = \frac{(2j)!}{j!j!}.$$

The fact that  $\sum_k U_{i,k}U_{k,j}^{-1} = \llbracket i = j \rrbracket$  can also be done by computer algebra. Since there are a few cases to be distinguished, it is omitted here.

The LU-decomposition  $AB = M^{-1}$  depends on the dimension  $N$  and is given by

$$A_{i,j} = \frac{(-1)^{i-j}(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B_{i,j} = \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-i)!(N-j-1)!i!}.$$

Since  $M^{-1}$  does not have “nice” entries, we rather provide formulæ for  $A^{-1}$  and  $B^{-1}$  and prove the identity  $B^{-1}A^{-1} = M$  instead. The results are:

$$A_{i,j}^{-1} = \frac{(N-j-1)!j!(N+i-1)!}{i!(N-i-1)!(N+j-1)!(i-j)!},$$

$$B_{i,j}^{-1} = \frac{(-1)^{j+N-1}(N-1-i)!j!i!}{(j-i)!(N+i-1)!}.$$

First we prove that these are indeed the inverses. We consider

$$\begin{aligned} & \sum_k \frac{(-1)^{i-k}(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!} \frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!} \\ &= (-1)^i \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!} \sum_{j \leq k \leq i} \frac{(-1)^k}{(i-k)!(k-j)!} \\ &= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!} \sum_{j \leq k \leq i} (-1)^{i-k} \binom{i-j}{i-k} \\ &= \frac{(N+i-1)!(N-j-1)!j!}{(N-i-1)!(N+j-1)!i!(i-j)!} \llbracket i = j \rrbracket = \llbracket i = j \rrbracket, \end{aligned}$$

which proves  $AA^{-1} = I$ . Similarly

$$\begin{aligned} & \sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!} \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!} \\ &= (-1)^j \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!} \sum_k \frac{(-1)^k}{(k-i)!(j-k)!} \\ &= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!} \sum_k (-1)^{j-k} \binom{j-i}{j-k} \\ &= \frac{(N-1-i)!i!(N+j-1)!}{(N+i-1)!j!(N-j-1)!(j-i)!} \llbracket i=j \rrbracket = \llbracket i=j \rrbracket, \end{aligned}$$

which proves  $B^{-1}B = I$ .

Now we compute an entry in  $B^{-1}A^{-1}$ :

$$\begin{aligned} & \sum_k \frac{(-1)^{k+N-1}(N-1-i)!k!i!}{(k-i)!(N+i-1)!} \frac{(N-j-1)!j!(N+k-1)!}{k!(N-k-1)!(N+j-1)!(k-j)!} \\ &= (-1)^{N-1} \frac{(N-1-i)!i!(N-j-1)!j!}{(N+i-1)!(N+j-1)!} \sum_k \frac{(-1)^k(N+k-1)!}{(k-i)!(N-k-1)!(k-j)!} \\ &= (-1)^{N-1} \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k (-1)^k \binom{N-1-i}{N-1-k} \binom{N+k-1}{N-1+j} \\ &= \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k \binom{i-1-k}{N-1-k} \binom{N+k-1}{N-1+j} \\ &= \frac{i!j!(N-j-1)!}{(N+i-1)!} \sum_k \binom{i-1-k}{i-N} \binom{N+k-1}{N-1+j} \\ &= \frac{i!j!(N-j-1)!}{(N+i-1)!} \binom{i-1+N}{i+j} \\ &= \frac{i!j!}{(i+j)!} = M_{i,j}, \end{aligned}$$

as claimed.

Now we use the form  $M^{-1} = AB$  and write the  $(i, j)$  entry:

$$\begin{aligned} & \sum_k \frac{(N-k-1)!k!(N+i-1)!}{i!(N-i-1)!(N+k-1)!(i-k)!} \frac{(-1)^{j+N-1}(N+j-1)!}{j!(j-k)!(N-j-1)!k!} \\ &= \frac{(N+i-1)!(N+j-1)!}{i!(N-i-1)!j!(N-j-1)!} \sum_k \frac{(N-k-1)!}{(N+k-1)!(i-k)!} \frac{(-1)^{j+N-1}}{(j-k)!} \\ &= \binom{N-1}{i} \binom{N+j-1}{j} \sum_{0 \leq k \leq \min\{i,j\}} (-1)^{j+N-1} \binom{N+i-1}{i-k} \binom{N-k-1}{j-k}. \end{aligned}$$

From this representation, it is clear that this is an integer. This was a question that was addressed in the affirmative in [7].

**3.  $q$ -analogues**

In this section we present  $q$ -analogues. Define  $(q)_n := (1 - q)(1 - q^2) \dots (1 - q^n)$ , and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_k (q)_{n-k}};$$

these definitions are standard, see [1]. Then we have the following results for the matrix  $M$  with entries  $\begin{bmatrix} i+j \\ j \end{bmatrix}^{-1}$ .

$$L_{i,j} = \frac{(q)_i (q)_i (q)_{2j}}{(q)_{i+j} (q)_{i-j} (q)_j (q)_j},$$

$$U_{i,j} = \frac{(-1)^i q^{i(3i-1)/2} (1 + q^i) (q)_j (q)_j (q)_i (q)_i}{(q)_{i+j} (q)_{j-i} (q)_{2i}} \quad \text{for } i \geq 1, \quad U_{0,j} = 1,$$

$$L_{i,j}^{-1} = \frac{q^{i(i-1)/2 + j(j+1)/2 - ij} (-1)^{i-j} (q)_i (q)_i (q)_{i+j-1}}{(q)_{2i-1} (q)_{i-j}} \quad \text{for } j < i, \quad L_{i,i}^{-1} = 1,$$

$$U_{i,j}^{-1} = \frac{(-1)^i q^{-j^2 - j + i(i+1)/2} (q)_{j+i-1} (q)_{2j} (q)_i (q)_i}{(q)_{j-i} (q)_j (q)_{j-1}} \quad \text{for } j > i,$$

$$U_{i,i}^{-1} = \frac{(-1)^i q^{i(3i+1)/2} (q)_{2i} (q)_{2i}}{(q)_i (q)_i (q)_i (q)_i (1 + q^i)} \quad \text{for } i \geq 1, \quad U_{0,0}^{-1} = 1,$$

$$A_{i,j} = \frac{(-1)^{i-j} q^{(i+j+3)(i-j)/2 + N(j-i)} (q)_{N-j-1} (q)_j (q)_{N+i-1}}{(q)_{N-i-1} (q)_i (q)_{N+j-1} (q)_{i-j}},$$

$$B_{i,j} = \frac{(-1)^{j+N-1} q^{i^2 + j(j+3)/2 - Nj - N(N-1)/2} (q)_{N+j-1}}{(q)_j (q)_{j-i} (q)_{N-j-1} (q)_i},$$

$$A_{i,j}^{-1} = \frac{q^{(i-j)(i-N+1)} (q)_{N-j-1} (q)_{N+i-1} (q)_j}{(q)_{N-i-1} (q)_{N+j-1} (q)_i (q)_{i-j}},$$

$$B_{i,j}^{-1} = \frac{(-1)^{j+1+N} q^{-j(j+1)/2 - ij + N(N-1)/2 + (N-1)i} (q)_{N-1-i} (q)_j (q)_i}{(q)_{j-i} (q)_{N+i-1}}.$$

Note that for  $q \rightarrow 1$ , we get the previous formulæ. We do not display all the proofs here, since Zeilberger’s algorithm (aka WZ-theory) [6] proves all these results (which were obtained by guessing), using a computer algebra system (such as, e.g., Maple). However, as suggested by a referee, in the next section, we provide how a typical proof is obtained with a computer.

REMARK. Richardson’s decomposition  $S = GMG$  still holds when all binomial coefficients are replaced by the corresponding Gaussian  $q$ -binomial coefficients.

**4. A sample proof**

We deal here with

$$\begin{aligned} \sum_{j \leq k \leq i} A_{i,k} A_{k,j}^{-1} &= \sum_{j \leq k \leq i} \frac{(-1)^{i-k} q^{(i+k+3)(i-k)/2+N(k-i)} (q)_{N-k-1} (q)_k (q)_{N+i-1}}{(q)_{N-i-1} (q)_i (q)_{N+k-1} (q)_{i-k}} \\ &\quad \times \frac{q^{(k-j)(k-N+1)} (q)_{N-j-1} (q)_{N+k-1} (q)_j}{(q)_{N-k-1} (q)_{N+j-1} (q)_k (q)_{k-j}} \\ &= \frac{q^{i(i+3)/2-j+N(j-i)} (q)_{N+i-1} (q)_{N-j-1} (q)_j}{(q)_{N-i-1} (q)_i (q)_{N+j-1} (q)_{i-j}} \sum_{j \leq k \leq i} \frac{(-1)^{i-k} q^{k(k-1)/2-kj} (q)_{i-j}}{(q)_{i-k} (q)_{k-j}}. \end{aligned}$$

Now Zeilberger’s algorithm provides the formula

$$\frac{(-1)^{i-k} q^{k(k-1)/2-kj} (q)_{i-j}}{(q)_{i-k} (q)_{k-j}} = \frac{(-1)^{i-k} q^{k(k+1)/2-j-kj} (q)_{i-j-1}}{(q)_{i-1-k}} - \frac{(-1)^{i-(k-1)} q^{k(k-1)/2-j-(k-1)j} (q)_{i-j-1}}{(q)_{i-k}},$$

so the sum over  $k$  is telescoping, with the result

$$\sum_{j \leq k \leq \ell} \frac{(-1)^{i-k} q^{k(k-1)/2-kj} (q)_{i-j}}{(q)_{i-k} (q)_{k-j}} = \frac{(-1)^{i-\ell} q^{\ell(\ell+1)/2-j-\ell j} (q)_{i-j-1} (1 - q^{i-\ell})}{(q)_{i-\ell}}.$$

For  $j < i$  and  $\ell = i$ , this evaluates to 0. For  $j = i$ , we have directly

$$\sum_{i \leq k \leq i} \frac{(-1)^{i-k} q^{k(k-1)/2-ki}}{(q)_{i-k} (q)_{k-i}} = q^{i(i-1)/2-i^2} = q^{-i(i+1)/2}.$$

Therefore

$$\sum_{i \leq k \leq i} A_{i,k} A_{k,j}^{-1} = \frac{q^{i(i+3)/2-i} (q)_{N+i-1} (q)_{N-i-1} (q)_i}{(q)_{N-i-1} (q)_i (q)_{N+i-1}} q^{-i(i+1)/2} = 1,$$

as desired.

**5. A two parameter extension**

It is even possible to extend the results by replacing  $i \rightarrow i + r$  and  $j \rightarrow j + s$ , for  $r, s \geq 0$ . In other words, the matrix now has entries  $\binom{i+r+j+s}{j+s}^{-1}$ .

We only give the formulæ in a list:

$$\begin{aligned}
 L_{i,j} &= \frac{(i+r)!i!(2j+r+s)!}{(i+j+r+s)!(i-j)!(j+r)!j!}, \\
 U_{i,j} &= \frac{(-1)^i(j+s)!j!(i+r)!(i+r+s-1)!}{(j+i+r+s)!(j-i)!(2i+r+s-1)!}, \\
 L_{i,j}^{-1} &= \frac{(-1)^{i-j}(i+r)!i!(i+j+r+s-1)!}{(2i+r+s-1)!(i-j)!(j+r)!j!}, \\
 U_{i,j}^{-1} &= \frac{(-1)^i(i+j+r+s-1)!(2j+r+s)!}{(j-i)!(j+r)!(j+i)!(j+r+s-1)!(i+s)!i!}, \\
 A_{i,j} &= \frac{(-1)^{i-j}(N-j-1)!(j+s)!(N+i+r+s-1)!}{(i+s)!(N-i-1)!(N+j+r+s-1)!(i-j)!}, \\
 B_{i,j} &= \frac{(-1)^{j+N-1}(N+j+r+s-1)!}{(j+r)!(j-i)!(N-j-1)!(i+s)!}, \\
 A_{i,j}^{-1} &= \frac{(N-j-1)!(j+s)!(N+i+r+s-1)!}{(i+s)!(N-i-1)!(N+j+r+s-1)!(i-j)!}, \\
 B_{i,j}^{-1} &= \frac{(-1)^{j+1+N}(N-1-i)!(j+s)!(i+r)!}{(j-i)!(N+i+r+s-1)!}.
 \end{aligned}$$

For  $\left[ \begin{smallmatrix} i+r+j+s \\ j+s \end{smallmatrix} \right]^{-1}$  we get  $q$ -analogues:

$$\begin{aligned}
 L_{i,j} &= \frac{(q)_{i+r}(q)_i(q)_{2j+r+s}}{(q)_{i+j+r+s}(q)_{i-j}(q)_{j+r}(q)_j}, \\
 U_{i,j} &= \frac{(-1)^i q^{i(3i-1)/2} (q)_{j+s}(q)_j (q)_{i+r}(q)_{i+r+s-1}}{(q)_{j+i+r+s}(q)_{j-i}(q)_{2i+r+s-1}}, \\
 L_{i,j}^{-1} &= \frac{(-1)^{i-j} q^{i(i-1)/2+j(j+1)/2-ij} (q)_{i+r}(q)_i (q)_{i+j+r+s-1}}{(q)_{2i+r+s-1}(q)_{i-j}(q)_{j+r}(q)_j}, \\
 U_{i,j}^{-1} &= \frac{(-1)^i q^{i(i+1)/2-j^2-ij-(r+s)j} (q)_{i+j+r+s-1}(q)_{2j+r+s}}{(q)_{j-i}(q)_{j+r}(q)_{j+i}(q)_{j+r+s-1}(q)_{i+s}(q)_i}, \\
 A_{i,j} &= \frac{(-1)^{i-j} q^{i(i+3)/2-j(j+3)/2+N(j-i)} (q)_{N-j-1}(q)_{j+s}(q)_{N+i+r+s-1}}{(q)_{i+s}(q)_{N-i-1}(q)_{N+j+r+s-1}(q)_{i-j}}, \\
 B_{i,j} &= \frac{(-1)^{j+N-1} q^{(r+s)(i+1)+i^2+j(j+3)/2-(r+s+j)N-N(N-1)/2} (q)_{N+j+r+s-1}}{(q)_{j+r}(q)_{j-i}(q)_{N-j-1}(q)_{i+s}}, \\
 A_{i,j}^{-1} &= \frac{q^{i(i+1)-j-ij+N(j-i)} (q)_{N-j-1}(q)_{j+s}(q)_{N+i+r+s-1}}{(q)_{i+s}(q)_{N-i-1}(q)_{N+j+r+s-1}(q)_{i-j}}, \\
 B_{i,j}^{-1} &= \frac{(-1)^{j+1+N} q^{-j(j+1)/2-ij+N(N-1)/2+(N-1)(r+s+i)-(r+s)j} (q)_{N-1-i}(q)_{j+s}(q)_{i+r}}{(q)_{j-i}(q)_{N+i+r+s-1}}.
 \end{aligned}$$

The previous results follow from these by plugging in  $r = s = 0$  or taking appropriate limits.



**6. Additional results**

For completeness, we also deal with the binomial matrix (no reciprocals)

$$\mathcal{M}_{i,j} = \left( \binom{i+r+j+s}{j+s} \right)_{i,j \geq 0}.$$

We get the same type of factorizations and use calligraphic letters to mark the difference. We only cite the results; justifications are in the same style as in the previous instances.

$$\begin{aligned} \mathcal{L}_{i,j} &= \frac{i!(i+r+s)!(j+r)!}{(i-j)!j!(i+r)!(j+r+s)!}, \\ \mathcal{U}_{i,j} &= \frac{(j+r+s)!j!}{(j-i)!(i+r)!(j+s)!}, \\ \mathcal{L}_{i,j}^{-1} &= \frac{(-1)^{i-j}(i+r+s)!i!(j+r)!}{(i-j)!(j+r+s)!(i+r)!j!}, \\ \mathcal{U}_{i,j}^{-1} &= \frac{(-1)^{i-j}(j+r)!(i+s)!}{(j-i)!(i+r+s)!i!}, \\ \mathcal{A}_{i,j} &= \frac{(-1)^{i-j}(N-j-1)!(i+s)!(2j+r+s+1)!}{(i-j)!(N-i-1)!(i+j+r+s+1)!(j+s)!}, \\ \mathcal{B}_{i,j} &= \frac{(-1)^{i-j}(N+i+r+s)!(j+r)!(i+s)!}{(j-i)!(N-j-1)!(2i+r+s)!(i+j+r+s+1)!}, \\ \mathcal{A}_{i,j}^{-1} &= \frac{(N-j-1)!(i+j+r+s)!(i+s)!}{(i-j)!(N-i-1)!(2i+r+s)!(j+s)!}, \\ \mathcal{B}_{i,j}^{-1} &= \frac{(N-i-1)!(2j+r+s+1)!(i+j+r+s)!}{(j-i)!(N+j+r+s)!(i+r)!(j+s)!}. \end{aligned}$$

There are also  $q$ -analogues for the matrix

$$\begin{aligned} \mathcal{M}_{i,j} &= \left( \begin{bmatrix} i+r+j+s \\ j+s \end{bmatrix} \right)_{i,j \geq 0}. \\ \mathcal{L}_{i,j} &= \frac{(q)_i(q)_{i+r+s}(q)_{j+r}}{(q)_{i-j}(q)_j(q)_{i+r}(q)_{j+r+s}}, \\ \mathcal{U}_{i,j} &= \frac{q^{i^2+(r+s)i}(q)_{j+r+s}(q)_j}{(q)_{j-i}(q)_{i+r}(q)_{j+s}}, \\ \mathcal{L}_{i,j}^{-1} &= \frac{(-1)^{i-j}q^{i(i-1)/2-ij+j(j+1)/2}(q)_{i+r+s}(q)_i(q)_{j+r}}{(q)_{i-j}(q)_{j+r+s}(q)_{i+r}(q)_j}, \\ \mathcal{U}_{i,j}^{-1} &= \frac{(-1)^{i-j}q^{i(i+1)/2-ij-j(j+1)/2-(r+s)j}(q)_{j+r}(q)_{i+s}}{(q)_{j-i}(q)_{i+r+s}(q)_i}. \end{aligned}$$

$$\begin{aligned} \mathcal{A}_{i,j} &= \frac{(-1)^{i-j} q^{i(i+3)/2-j(j+3)/2+N(j-i)} (q)_{N-j-1} (q)_{i+s} (q)_{2j+r+s+1}}{(q)_{i-j} (q)_{N-i-1} (q)_{i+j+r+s+1} (q)_{j+s}}, \\ \mathcal{B}_{i,j} &= \frac{(-1)^{i-j} q^{(j+1)(j+2)/2+3i(i+1)/2+(r+s)(i+1)-N(j+1+i+r+s)} (q)_{N+i+r+s} (q)_{j+r} (q)_{i+s}}{(q)_{j-i} (q)_{N-j-1} (q)_{2i+r+s} (q)_{i+j+r+s+1}}, \\ \mathcal{A}_{i,j}^{-1} &= \frac{q^{i^2-(N-1)(i-j)-ij} (q)_{N-j-1} (q)_{i+j+r+s} (q)_{i+s}}{(q)_{i-j} (q)_{N-i-1} (q)_{2i+r+s} (q)_{j+s}}, \\ \mathcal{B}_{i,j}^{-1} &= \frac{q^{(i+j+1+r+s)(N-j-1)} (q)_{N-i-1} (q)_{2j+r+s+1} (q)_{i+j+r+s}}{(q)_{j-i} (q)_{N+j+r+s} (q)_{i+r} (q)_{j+s}}. \end{aligned}$$

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