Turkish Journal of Mathematics

Volume 40 | Number 5 Article 1

1-1-2016

Lagrangian description, symplectization, and Eulerian dynamics of incompressible fluids

HASAN GÜMRAL

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2016) 40: 925 – 940

doi:10.3906/mat-1410-38

Lagrangian description, symplectization, and Eulerian dynamics of incompressible fluids

Hasan GÜMRAL*
Department of Mathematics, School of Engineering, Australian College of Kuwait, Kuwait

Received: 16.10.2014 • **Accepted/Published Online:** 18.12.2015 • **Final Version:** 21.10.2016

Abstract: Eulerian dynamical equations in a three-dimensional domain are used to construct a formal symplectic structure on time-extended space. Symmetries, invariants, and conservation laws are related to this geometric structure. The symplectic structure incorporates dynamics of helicities as identities. The generator of the infinitesimal dilation for symplectic two-form can be interpreted as a current vector for helicity. Symplectic dilation implies the existence of contact hypersurfaces. In particular, these include contact structures on the space of streamlines and on the Bernoulli surfaces.

Key words: Incompressible fluid, symplectic and contact structures, symplectic dilation, helicity conservation, Lagrangian description

1. Introduction

1.1. Motivations

The Euler equation for steady flow of incompressible fluid makes the construction of a contact structure on three-dimensional space of Lagrangian trajectories of velocity vector field possible provided the (time-independent) helicity density is nonvanishing. One natural question is to ask to what extent these Eulerian equations characterize the space of integral curves of the velocity field? In this work, we shall be concerned with the relations between Lagrangian description and Eulerian equations of incompressible fluid and exploit the Eulerian evolution equations to obtain geometric structures relevant to a qualitative study of the Lagrangian description of motion.

Our aim is, first, to show that a construction of symplectic structure on the time extended space $\mathbb{R} \times M$ of trajectories that makes use of the (time-dependent) Eulerian equations, as in the case of contact structure for steady flows, is possible and, second, to reduce the dynamics of the Lagrangian description to various three-dimensional contact hypersurfaces of symplectic space at our disposal. The first part has already been studied in our previous works [7] and [8], which we will summarize more compactly in sections two to four. The main contribution of the present work is to analyze the symplectic structure for the existence of contact hypersurfaces and to construct contact structures therein. Our basic motivation is the observation, stated in [9], that volume preserving vector fields (on compact three manifolds) can be characterized as Hamiltonian vector fields, reduced on an energy surface, of an appropriate symplectic manifold in four dimensions.

*Correspondence: h.gumral@ack.edu.kw
2010 *AMS Mathematics Subject Classification:* 70G45, 53D05, 76B99.

1.2. Content

In the next section, we will show how Eulerian dynamical equations can be used to obtain symmetries and invariants of Lagrangian dynamics. For the Euler equation infinitesimal symmetry condition is the vorticity form of equations and invariant function is described by the Bernoulli equation. We point out that the Bernoulli function remains to be an invariant of viscous Lagrangian flow provided the (explicit) time dependence of pressure is proportional to component of viscous stress parallel to the motion.

In section three, we will show that Eulerian equations for velocity field lead us to construct an exact symplectic two-form on space of trajectories of suspended velocity field. Nondegeneracy of this two-form is provided by nonvanishing of vortical helicity or of potential vorticity. We will present Hamiltonian vector fields, which, in particular, include suspended velocity and normalized vorticity fields. Differential identities involving symplectic two-form and canonical one-form are indeed conservation laws, both in Eulerian and in Lagrangian form, for helicity and potential vorticity. We will then introduce symplectic dilation and give a characterization of the helicity evolution equation in terms of this vector field.

In section four, we will show that the suspended velocity field for the Lagrangian description is Hamiltonian with a Bernoulli function. Under certain conditions, helicity and potential vorticity become invariants of Lagrangian motion.

In section five, we will identify hypersurfaces transversal to the symplectic dilation and construct contact structures on spatial hypersurfaces and on those described by level sets of a Bernoulli function. A comparison of a contact structure on spatial hypersurfaces with one associated to the steady Euler equations will also be presented.

In section six, we will first summarize the present construction relating Eulerian equations to a geometric setting for Lagrangian motion and contact dynamics. We will then discuss the physical implications of symplectization, which is the converse of the construction in the previous section. We will conclude by presenting pictorial relations between streamlines and trajectories as well as between solutions of homogeneous Euler equations and the Euler equations.

For more on fluid dynamical content of this work we refer to Refs. [5, 6, 12] and the necessary mathematical background can be found in Refs. [1, 2, 13, 16, 17].

2. Eulerian dynamics

2.1. Euler equations

We shall begin with the Euler equations of ideal incompressible fluids

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p \tag{1}$$

for the divergence-free velocity field $\mathbf{v}$ tangent to the boundary of a connected region $M \subset \mathbb{R}^3$ with coordinates $\mathbf{x}$ and the pressure function p. The identity $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla|\mathbf{v}|^2/2 - \mathbf{v} \times (\nabla \times \mathbf{v})$ can be used to bring the Euler equation (1) into the Bernoulli form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \alpha \tag{2}$$

where the function $\alpha \equiv p + v^2/2$ is the Bernoulli function [4], also called the total or stagnation pressure [12]. In terms of the divergence-free vorticity field $\mathbf{w} \equiv \nabla \times \mathbf{v}$ Eq. (2) gives

$$\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{w}) = 0 \, . \tag{3}$$

It follows from the identity $\nabla\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{w}+(\nabla\cdot\mathbf{w})\mathbf{v}-(\nabla\cdot\mathbf{v})\mathbf{w}$ together with $\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{w} = 0$ that Eq. (3) is equivalent to

$$\frac{\partial \mathbf{w}}{\partial t} + [\mathbf{v}, \mathbf{w}] = 0 \, , \quad [\mathbf{v}, \mathbf{w}] \equiv (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} \tag{4}$$

which means that $\mathbf{w}$ is an infinitesimal time-dependent symmetry of the velocity field $\mathbf{v}$. Solutions $\mathbf{x} = \mathbf{x}(t)$ of the ordinary differential equations $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}, t)$ are trajectories of the velocity field. The time-dependent transformations generated by $\mathbf{w}$ on M leave these trajectories invariant. A time-dependent conserved function for the velocity field can be found again from the Euler equation. We recall that an energy consequence of the Euler equation follows by taking the dot product of its Bernoulli form with the velocity field. The result is known as the Bernoulli equation [6, 12]

$$\frac{\partial}{\partial t}(\frac{1}{2}v^2) + \mathbf{v} \cdot \nabla\alpha = 0 \tag{5}$$

which implies that if the pressure p does not depend explicitly on time, that is, if $p = p(\mathbf{x})$, then the Bernoulli function α is a time-dependent conserved function

$$\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla\alpha = 0 \, , \quad \alpha(t, \mathbf{x}) = p(\mathbf{x}) + \frac{1}{2}v^2(t, \mathbf{x}) \tag{6}$$

along the trajectories of the velocity field.

2.2. Navier–Stokes equations

The Navier–Stokes equation for a viscous incompressible fluid in a bounded domain $M \subset \mathbb{R}^3$ is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v} \tag{7}$$

where ν is the kinematic viscosity [6]. Eq. (7) results in the equation

$$\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{w}) = \nu\nabla^2\mathbf{w} \tag{8}$$

for the divergence-free vorticity field. Corresponding to the Bernoulli equation of Euler flow, we have, for the viscous case

$$\frac{\partial}{\partial t}(\frac{1}{2}v^2) + \mathbf{v} \cdot \nabla\alpha = \nu\mathbf{v}\cdot\nabla^2\mathbf{v} \tag{9}$$

from which it follows that

$$\frac{\partial \alpha}{\partial t} + \mathbf{v} \cdot \nabla\alpha = \frac{\partial p}{\partial t} + \nu\mathbf{v}\cdot\nabla^2\mathbf{v} \, . \tag{10}$$

Thus, if the time dependence of pressure is given by

$$\frac{\partial p}{\partial t}+\nu\mathbf{v}\cdot\nabla^2\mathbf{v}=0 \tag{11}$$

then the Bernoulli function becomes a conserved quantity of the viscous Lagrangian flow as well. Eq. (11) is an expression for balancing the viscous stress in direction of motion by an adjustment of pressure in time.

2.3. Generalities on Eulerian equations

More generally, we can consider the evolution equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{F} \tag{12}$$

for an arbitrary force field $\mathbf{F}$. To account for magnetohydrodynamic systems, we can replace the vorticity by a divergence-free frozen-field $\mathbf{B}$ satisfying the condition in Eq. (4). In this case, if the force field $\mathbf{F}$ satisfies the condition $\mathbf{B}\cdot(\mathbf{F}+\nabla(v^2/2))=0$, then the function $\mathbf{v}\cdot\mathbf{B}$ turns out to be an invariant of the Lagrangian flow [7].

With this remark, the following discussions can be extended to the Eulerian equations such as the equations describing the Boussinesq approximation to inhomogeneous Euler equations, equations of barotropic fluids, equations of nonrelativistic superconductivity, equations of ideal magnetohydrodynamics, and dynamo theory (see [8] and the references therein).

3. Geometry

3.1. Symplectic two-form

We will obtain from the Eulerian equations a nondegenerate closed two-form Ω_ν on $\mathbb{R}\times M$, that is, a symplectic structure [1, 3, 10, 13, 16].

Proposition 1 *Let ϕ be an a priori unspecified function on $\mathbb{R}\times M$. Then the two-form*

$$\Omega_\nu=\mathbf{w}\cdot(d\mathbf{x}\wedge d\mathbf{x})-(\mathbf{v}\times\mathbf{w}+\nabla\phi+\nu\nabla^2\mathbf{v})\cdot d\mathbf{x}\wedge dt \tag{13}$$

on $\mathbb{R}\times M$ is symplectic on the space of solutions of the Navier–Stokes equations provided $\mathbf{w}\cdot(\nu\nabla^2\mathbf{v}+\nabla\phi)\neq 0$.

Proof The three-form

$$d\Omega_\nu=(\nabla\cdot\mathbf{w})\ (d\mathbf{x}\wedge d\mathbf{x}\wedge d\mathbf{x})+[\mathbf{w}_{,t}-\nabla\times(\mathbf{v}\times\mathbf{w}-\nu\nabla\times\mathbf{w})]\cdot d\mathbf{x}\wedge d\mathbf{x}\wedge dt \tag{14}$$

vanishes for divergence-free vector field $\mathbf{w}$ satisfying the Navies–Stokes equations in the vorticity form (3). Therefore, Ω_ν is closed. For $\nu=0$ this reduces to the two-form

$$\Omega_e=\mathbf{w}\cdot(d\mathbf{x}\wedge d\mathbf{x})-(\mathbf{v}\times\mathbf{w}+\nabla\phi)\cdot d\mathbf{x}\wedge dt\ , \tag{15}$$

which is closed by the Euler equations in rotational form. For nondegeneracy, we compute

$$\frac{1}{2}\Omega_\nu\wedge\Omega_\nu=(\nu\mathcal{H}_w-\mathbf{w}\cdot\nabla\phi)\ d\mathbf{x}\wedge d\mathbf{x}\wedge d\mathbf{x}\wedge dt\neq 0 \tag{16}$$

where the scalar function

$$\mathcal{H}_w \equiv \mathbf{w} \cdot \nabla \times \mathbf{w} = -\mathbf{w} \cdot \nabla^2 \mathbf{v} \tag{17}$$

is known as the vortical helicity density [6]. For viscous flows $\nabla^2 \mathbf{v} \neq 0$ and, for a realistic fluid we have $\mathbf{w} \neq 0$ [15]. Hence $\mathcal{H}_w \neq 0$. This makes Ω_ν nondegenerate. For the Euler equations, Eq. (16) reduces to

$$\frac{1}{2}\Omega_e \wedge \Omega_e = -(\mathbf{w} \cdot \nabla\phi)\, d\mathbf{x} \wedge d\mathbf{x} \wedge d\mathbf{x} \wedge dt \neq 0 \tag{18}$$

which, for nondegeneracy, requires $\nabla\phi$ to be nonzero. □

The nonzero four-form in Eq. (18) is the symplectic or the Liouville volume element on $\mathbb{R}\times M$.

3.2. Exactness

For a one-form $\theta = \psi dt + \mathbf{v} \cdot d\mathbf{x}$ we compute

$$d\theta = \mathbf{w} \cdot (d\mathbf{x} \wedge d\mathbf{x}) - (\mathbf{v} \times \mathbf{w} + \nabla(\psi + \alpha) - \nu\nabla \times \mathbf{w}) \cdot d\mathbf{x} \wedge dt$$

where we solved the time derivative of the velocity field from the Navier–Stokes equations. The right-hand side is the same as the symplectic form provided we have $\psi + \alpha = -\phi$. Thus, the parametric family of symplectic two-forms is exact

$$\theta = -(\phi + \alpha)dt + \mathbf{v} \cdot d\mathbf{x}, \quad \Omega_\nu = d\theta \quad mod\ Eq.(7)$$

and this includes $\Omega_e = d\theta \quad mod\ Eq.$ (1) for the Euler equations.

3.3. Hamiltonian vector fields

The nondegeneracy of Ω_ν means that given a one-form $\beta \equiv \beta_a dx^a$ on $\mathbb{R}\times M$ with the local coordinates $(x^a) = (x^0 = t, \mathbf{x})$ the equations

$$i(X)(\Omega_\nu) = \beta\ , \quad (\Omega_\nu)_{ab} X^a = \beta_b \tag{19}$$

has a unique solution for the vector field $X = X^a \partial_a = X^0 \partial_t + \mathbf{X} \cdot \nabla$ and vice versa. Here, $i(X)(\cdot)$ denotes the interior product or the contraction with the vector field X [2], $(\Omega_\nu)_{ab}$ are the components of the skew-symmetric matrix of the symplectic two-form Ω_ν in the given coordinates and we employ the summation over repeated indices.

For an arbitrary smooth function f on $\mathbb{R}\times M$ the Hamiltonian vector field X_f defined by the symplectic two-form Ω_ν is

$$X_f = \frac{1}{\nu\mathcal{H}_w - \mathbf{w} \cdot \nabla\phi}[\frac{df}{dt}\mathbf{w} \cdot \nabla - (\mathbf{w} \cdot \nabla f)(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla) + (\nabla\phi - \nu\nabla \times \mathbf{w}) \times \nabla f \cdot \nabla] \tag{20}$$

and this satisfies the Hamilton's equations $i(X_f)(\Omega_\nu) = df$. Here $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the convective (or material) derivative. From the skew-symmetry of the matrix $(\Omega_\nu)_{ab}$ we have the conservation law

$$0 \equiv i(X_f)i(X_f)(\Omega_\nu) = (\Omega_\nu)_{ab} X^a X^b = X^a f_{,a} = X^0 f_{,t} + \mathbf{X} \cdot \nabla f = 0 \tag{21}$$

for the Hamiltonian function. In particular, for the function $f = t$ we obtain the Hamiltonian vector field

$$i(X_t)(\Omega_\nu) = dt \qquad X_t = (\nu\mathcal{H}_w - \mathbf{w} \cdot \nabla\phi)^{-1}\mathbf{w} \cdot \nabla$$

which is the vorticity field normalized with the Liouville volume.

3.4. Poisson bracket

The symplectic structure Ω_ν on $\mathbb{R}\times M$ induces a Lie algebraic structure on the space of smooth functions on $\mathbb{R}\times M$ with the Poisson bracket

$$\begin{aligned} \{f,g\}_\nu &\equiv \frac{\partial f}{\partial x^a}(\Omega_\nu^{-1})^{ab}\frac{\partial g}{\partial x^b} = -X_f(g) = \Omega_\nu(X_f,X_g) &(22)\\ &= \frac{1}{\nu\mathcal{H}_w-\mathbf{w}\cdot\nabla\phi}[\frac{dg}{dt}\mathbf{w}\cdot\nabla f-\frac{df}{dt}\mathbf{w}\cdot\nabla g-(\nabla f\times\nabla g)\cdot\nabla\phi &(23)\\ &\qquad +\nu(\nabla f\times\nabla g)\cdot(\nabla\times\mathbf{w})] \end{aligned}$$

where $(\Omega_\nu^{-1})^{ab}$ are components of the inverse of the matrix of symplectic two-form. Skew-symmetry of $(\Omega_\nu)_{ab}$ implies that $\{,\}_\nu$ is skew-symmetric and the fact that Ω_ν is closed corresponds to the Jacobi identity

$$\{\{f,g\}_\nu,h\}_\nu+\{\{h,f\}_\nu,g\}_\nu+\{\{g,h\}_\nu,f\}_\nu=0$$

for arbitrary functions f,g,h on $\mathbb{R}\times M$. The Lie algebra isomorphism

$$[X_f,X_g]=-X_{\{f,g\}} \tag{24}$$

between the algebra of Hamiltonian vector fields and the Poisson bracket algebra of functions is induced by the symplectic structure Ω_ν [10, 13].

3.5. Helicity conservation

A conservation law for Eulerian equations is a divergence expression of the form $\partial T/\partial t+\nabla\cdot\mathbf{P}=0$ with T being conserved density and $\mathbf{P}$ the corresponding flux. We call the four-vector $(T,\mathbf{P})$ on $\mathbb{R}\times M$ the current associated with the conserved quantity T. We will consider some evolution equations that may be related to Eulerian conservation laws under certain conditions. These are derived from the formal symplectic structure as differential identities and will include evolution equations for helicity and potential vorticity.

Since Ω_ν is closed and $d\theta=\Omega_\nu$, these differential forms satisfy the relation

$$d(\theta\wedge\Omega_\nu)-\Omega_\nu\wedge\Omega_\nu\equiv 0 \tag{25}$$

identically. Here, we compute the three-form

$$\begin{aligned}\theta\wedge\Omega_\nu &= \mathcal{H}\,d\mathbf{x}\wedge d\mathbf{x}\wedge d\mathbf{x}+ &(26)\\ &((v^2-\phi-\alpha)\mathbf{w}-\mathcal{H}\mathbf{v}-\mathbf{v}\times(\nabla\phi-\nu\nabla\times\mathbf{w}))\cdot d\mathbf{x}\wedge d\mathbf{x}\wedge dt\end{aligned}$$

where the scalar component, namely, the coefficient of the term $d\mathbf{x}\wedge d\mathbf{x}\wedge d\mathbf{x}$ is the helicity density

$$\mathcal{H}\equiv\mathbf{v}\cdot\nabla\times\mathbf{v}=\mathbf{v}\cdot\mathbf{w}\,. \tag{27}$$

Proposition 2 *$\mathcal{H}$ is an Eulerian conserved quantity for the Euler equation ($\nu=0$) and for the Navier–Stokes equation if $\mathcal{H}_w=0$.*

Proof We recall Eq. (25), which results in the divergence expression

$$\frac{\partial\mathcal{H}}{\partial t}+\nabla\cdot(\mathcal{H}\mathbf{v}+(p-\frac{1}{2}v^2)\mathbf{w})=\nu(\mathbf{v}\cdot\nabla^2\mathbf{w}-\mathcal{H}_w) \tag{28}$$

for the evolution of helicity. Note that the above equation is independent of the function ϕ introduced artificially in the definition of Ω_ν. Eq. (28) can also be written in the form

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot (\mathcal{H}\mathbf{v} + (p - \frac{1}{2}v^2)\mathbf{w} - \nu \mathbf{v} \times (\nabla \times \mathbf{w})) = -2\nu \mathcal{H}_w \tag{29}$$

from which one can conclude the conservation of helicity even for viscous fluids provided the vortical helicity vanishes. In the case of the Euler equations, we have $\nu = 0$ and Eq. (29) implies

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot (\mathcal{H}\mathbf{v} + (p - \frac{1}{2}v^2)\mathbf{w}) = 0 \tag{30}$$

for the conservation law of helicity density $\mathcal{H}$ without further assumption. □

3.6. Further conservation laws

Although the conservation of Bernoulli's function requires some additional conditions to hold, we can obtain, for incompressible fluids in general, an Eulerian conservation law involving Bernoulli's function and the vorticity by considering the three-form $d\alpha \wedge \Omega_\nu$.

Proposition 3 *The potential vorticities* $\mathbf{w} \cdot \nabla \alpha$ *and* $\mathbf{w} \cdot \nabla \phi$ *are Eulerian conservation laws for incompressible viscous fluids.*

Proof The identity $d(d\alpha \wedge \Omega_\nu) \equiv 0$ gives the evolution equation

$$\frac{\partial (\mathbf{w} \cdot \nabla \alpha)}{\partial t} + \nabla \cdot [(\mathbf{w} \cdot \nabla \alpha)\mathbf{v} + \nu \mathbf{v} \cdot \nabla \times \mathbf{w} + \nu \nabla \alpha \times (\nabla \times \mathbf{w})] = 0$$

for the conservation of the potential vorticity $\mathbf{w} \cdot \nabla \alpha$. Similarly, for $\mathbf{w} \cdot \nabla \phi$ the conservation law reads

$$\frac{\partial (\mathbf{w} \cdot \nabla \phi)}{\partial t} + \nabla \cdot \left[\nabla \phi \times (\mathbf{v} \times \mathbf{w} + \nu \nabla \times \mathbf{w}) - \frac{\partial \phi}{\partial t}\mathbf{w}\right] = 0$$

which can be put into the form

$$\frac{d}{dt}(\mathbf{w} \cdot \nabla \alpha) - \mathbf{w} \cdot \nabla \frac{d\phi}{dt} - \nu \nabla \phi \cdot \nabla^2 \mathbf{w} = 0$$

with the convective time derivative d/dt. □

3.7. Symplectic dilation

A particularly interesting solution of Eq. (19) is obtained when we let the one-form β be the canonical one-form θ. The vector field J_ν satisfying the equation

$$i(J_\nu)(\Omega_\nu) = \theta \tag{31}$$

can be uniquely determined to be

$$J_\nu = \frac{1}{\mathbf{w} \cdot \nabla \phi - \nu \mathcal{H}_w}[\mathcal{H}(\partial_t + \mathbf{v} \cdot \nabla) - (\psi + v^2)\mathbf{w} \cdot \nabla + \mathbf{v} \times (\nabla \phi - \nu \nabla \times \mathbf{w}) \cdot \nabla] . \tag{32}$$

It follows from Eq. (31) and $d\Omega_\nu = 0$ that J_ν fulfills the condition

$$\mathcal{L}_{J_\nu}(\Omega_\nu) = di(J_\nu)(\Omega_\nu) = d\theta = \Omega_\nu \tag{33}$$

of being an infinitesimal symplectic dilation for Ω_ν [18]. As a consequence of Eq. (33) and the derivation property of the Lie derivative we see that J_ν expands the Liouville volume in Eq. (18). That means J_ν is not divergence free with respect to the symplectic volume. J_ν is also said to be the Liouville vector field of Ω_ν [10]. The symplectic divergence of the dilation J_ν may be computed from

$$(div_{\Omega_\nu} J_\nu)(\frac{1}{2}\Omega_\nu \wedge \Omega_\nu) = \mathcal{L}_{J_\nu}(\frac{1}{2}\Omega_\nu \wedge \Omega_\nu) = \Omega_\nu \wedge \Omega_\nu \tag{34}$$

where we used Eq. (33). The second equality is the same as the identity in Eq. (25) resulting in the helicity evolution. Thus, we have

Proposition 4 *The evolution equation in Eq. (30) for helicity density can be expressed by the equation*

$$div_{\Omega_\nu} J_\nu - 2 \equiv 0 \tag{35}$$

involving the symplectic-divergence of the symplectic dilation in Eq. (32).

With this interpretation we intend to call J_ν the current associated with the helicity. The dynamical content of the helicity current can be revealed from a comparison of the symplectic structures obtained from the Navier–Stokes and the Euler equations. The canonical one-forms are the same. Thus, the dynamics is encoded into the symplectic two-forms. They define the current vectors by Eq. (31) for the same canonical one-form. With this definition, the dynamical properties of the fluid, such as viscosity, become implicit in the helicity current. Thus, we can conclude that the pairs

$$(\theta, J_e), \qquad (\theta, J_\nu)$$

are geometric representatives of the dynamics of ideal and viscous fluid motions on the space of trajectories.

3.8. Hamiltonian automorphisms

The helicity current is not a Hamiltonian vector field. However, it takes a Hamiltonian vector field into a Hamiltonian vector field by its action via a Lie derivative. To see this, we compute

$$\begin{aligned} i([J_\nu, X_f])(\Omega_\nu) &\equiv \mathcal{L}_{J_\nu}(i(X_f)(\Omega_\nu)) - i(X_f)(\mathcal{L}_{J_\nu}(\Omega_\nu)) & (36)\\ &= d(J_\nu(f) - f) & (37)\end{aligned}$$

where Eq. (36) is an identity [2, 13] and we used Eq. (33). Thus, $[J_\nu, X_f]$ is Hamiltonian with the function $J_\nu(f) - f$. Replacing X_f with $[J_\nu, X_f]$ in Eq. (36) and using Eq. (37) we get

$$\begin{aligned} i([J_\nu, [J_\nu, X_f]])(\Omega_\nu) &\equiv \mathcal{L}_{J_\nu}(i([J_\nu, X_f])(\Omega_\nu)) - i([J_\nu, X_f])(\mathcal{L}_{J_\nu}(\Omega_\nu)) & (38)\\ &= d(J_\nu(J_\nu(f)) - 2J_\nu(f) + f) & (39)\\ &= d(J_\nu - 1)^2(f) & (40)\end{aligned}$$

which is also Hamiltonian. Thus, by repeated applications of the Lie derivative with respect to the symplectic dilation J_ν one can generate an infinite hierarchy of Hamiltonian vector fields.

Proposition 5 *Let X_f be a Hamiltonian vector field $i(X_f)(\Omega_\nu) \equiv df$. Then for each $k = 0, 1, 2, ...$ the vector fields $(\mathcal{L}_{J_\nu})^k(X_f)$ are Hamiltonian with respect to Ω_ν and for the Hamiltonian functions $(J_\nu - 1)^k(f)$.*

This infinite hierarchy of Hamiltonian vector fields is anchored to X_f. Since

$$\mathcal{L}_{[J_\nu, X_f]}(\Omega_\nu) \equiv \mathcal{L}_{J_\nu}\mathcal{L}_{X_f}(\Omega_\nu) - \mathcal{L}_{X_f}\mathcal{L}_{J_\nu}(\Omega_\nu) = 0$$

they are Hamiltonian automorphisms of Ω_ν on $\mathbb{R}\times M$.

4. Lagrangian descriptions

4.1. Hamiltonian velocity fields

The formal symplectic structure obtained from the Eulerian equations immediately implies that the normalized vorticity field is Hamiltonian on $\mathbb{R}\times M$ with the Hamiltonian function t. Our real interest is the geometry of the velocity field $\mathbf{v}$ or its suspension $\partial_t + \mathbf{v}\cdot\nabla$. With respect to the two-form Ω_ν the suspended velocity field is, in general, not even locally Hamiltonian. (Locally Hamiltonian vector fields are obtained from Eq. (19) for closed nonexact one-forms β). To see this, first observe that the symplectic two-form is invariant under the flows of locally Hamiltonian vector fields because $\mathcal{L}_X(\Omega_\nu) = di(X)(\Omega_\nu) = d\beta \equiv 0$ where we used the identity $\mathcal{L}_X = i(X)\circ d + d\circ i(X)$ for the Lie derivative, $d\Omega_\nu = 0$ and β is closed.

Proposition 6 *The suspended velocity field $\partial_t + v$ is Hamiltonian*

$$i(\partial_t + v)(\Omega_\nu) = -d\alpha \tag{41}$$

with the Hamiltonian function being the negative of the Bernoulli's function whenever $\mathbf{w}$ is a frozen in field.

Proof To obtain conditions under which $\partial_t + v$ is Hamiltonian we compute, from Eq. (13),

$$\begin{aligned} i(\partial_t + v)(\Omega_\nu) &= (\mathbf{v}\times\mathbf{w} + \nabla\phi + \nu\nabla^2\mathbf{v})\cdot d\mathbf{x} \\ &\quad + \mathbf{w}\times\mathbf{v}\cdot d\mathbf{x} - \mathbf{v}\cdot(\nabla\phi + \nu\nabla^2\mathbf{v})dt \\ &= (\nabla\phi + \nu\nabla^2\mathbf{v})\cdot(d\mathbf{x} - \mathbf{v}dt) \\ &= -(\mathbf{v}\cdot\nabla(\phi+\alpha) + \frac{\partial v^2/2}{\partial t})dt + (\nabla\phi + \nu\nabla^2\mathbf{v})\cdot d\mathbf{x} \\ &= -d\alpha + (\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla\psi)dt + (\nu\nabla^2\mathbf{v} - \nabla\psi)\cdot d\mathbf{x} \end{aligned} \tag{42}$$

where in the third line we used the Navier–Stokes equations to replace $\nu\nabla^2\mathbf{v}$ in the coefficient of dt and $-\psi = \phi + \alpha$. If we require the right-hand side to be an exact one-form, the second and the third terms must be the time derivative and gradient of some function, respectively. However, we have already introduced the arbitrary function ϕ, which implies the arbitrariness of ψ. Therefore, we set

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla\psi = 0\ , \qquad \nu\nabla^2\mathbf{v} - \nabla\psi = 0 \tag{43}$$

as the condition for the right-hand side of Eq. (42) to be exact. The integrability condition $\nabla\times\nabla\psi = 0$ implies $\nu\nabla^2\mathbf{w} = 0$. This holds in cases of either ideal flow $\nu = 0$ or $\nabla^2\mathbf{w} = \nabla\times\nabla\times\mathbf{w} = 0$ for the viscous case. In either case, $\mathbf{w}$ becomes a frozen in field by the vorticity form of Euler and Navier–Stokes equations, respectively. Elimination of the function ψ in Eq. (43) leads to Eq. (11), which guarantees the conservation of Bernoulli's function α under the time-dependent flow of velocity field. □

4.2. Differential invariants

The symplectic structure provides the velocity field with differential invariants that in turn enable us to obtain Eulerian conservation laws. Recall that a differential $p-$form ξ is said to be a relative invariant for a vector field X if there exists a $(p-1)-$form ζ such that

$$\mathcal{L}_X(\xi) \equiv i(X)d\xi + di(X)\xi = d\zeta \tag{44}$$

where $\mathcal{L}_X(\cdot)$ is the Lie derivative. If $\zeta = 0$, ξ is said to be an absolute invariant [2, 10].

Proposition 7 *Let $\phi = -\alpha$, or equivalently $\psi = 0$. Then θ and $\theta \wedge \Omega_\nu$ become absolute invariants of $\partial_t + v$ on level surfaces of the function $v^2/2 - p$. In this case, the helicity density is a conserved function of the Lagrangian motion.*

Proof From Eq. (42) and assuming that Eq. (43) holds we have

$$\begin{aligned} \mathcal{L}_{\partial_t+v}(\theta) &= i(\partial_t + v)(\Omega_\nu) + d(v^2 - \phi - \alpha) \\ &= d(\frac{v^2}{2} - p - \phi - \alpha), \end{aligned}$$

that is, θ is a relative invariant for incompressible fluid satisfying the conditions for the Bernoulli's function to be conserved. In this case, $\mathcal{L}_{\partial_t+v}(\Omega_e) = d\mathcal{L}_{\partial_t+v}(\theta) = 0$ and the derivation property of the Lie derivative implies the relative invariance

$$\begin{aligned} \mathcal{L}_{\partial_t+v}(\theta \wedge \Omega_\nu) &= \mathcal{L}_{\partial_t+v}(\theta) \wedge \Omega_\nu + \theta \wedge \mathcal{L}_{\partial_t+v}(\Omega_\nu) \\ &= d((\frac{v^2}{2} - p - \phi - \alpha)\Omega_\nu) \end{aligned} \tag{45}$$

of the three-form $\theta \wedge \Omega_\nu$. Recall that the function ϕ was arbitrarily introduced into Ω_ν and its dynamical significance is yet to be specified. To this end, it will be convenient to let $\phi = -\alpha$ in accordance with the fact that the helicity evolution is independent of this function. Then both θ and $\theta \wedge \Omega_\nu$ become absolute invariants on surfaces $v^2/2 - p =$constant. The condition for relative invariance makes the helicity density an Eulerian conserved quantity (c.f. Eq. (30)), while on surfaces $v^2/2 - p =$constant the helicity density becomes a conserved function of the Lagrangian motion. □

For ideal fluids, the potential vorticity $\mathbf{w} \cdot \nabla\alpha$ is also conserved under the flow of the velocity field. From the right-hand side of Eq. (18) we conclude that the absolute invariance of $\Omega_e \wedge \Omega_e$ is a statement for the conservation of the Liouville density

$$\frac{\partial}{\partial t}(\mathbf{w} \cdot \nabla\alpha) + \mathbf{v} \cdot \nabla(\mathbf{w} \cdot \nabla\alpha) = 0 \tag{46}$$

along the trajectories of the velocity field.

5. Symplectic dilation and contact structures

5.1. Contact structure

A contact structure on a three-dimensional manifold is a field of nonintegrable, two-dimensional hyperplanes in its tangent spaces. Locally, this may be described as the kernel of a one-form σ satisfying $\sigma \wedge d\sigma \neq 0$ everywhere. The contact form σ determines a unique vector field E by the conditions

$$i(E)(\sigma) = 1 \,, \quad i(E)(d\sigma) = 0 \tag{47}$$

which is called the Reeb vector field [3, 11]. According to the definition of Ref. [19] a hypersurface in a symplectic manifold admits a contact one-form if and only if there exists a symplectic dilation that is defined on its neighborhood and is transversal to the hypersurface. It is also remarked that such hypersurfaces arise as level sets of Hamiltonian functions of Hamiltonian vector fields and that the periodic solutions of Hamiltonian vector fields exist on contact hypersurfaces [11, 14, 18].

We shall describe two examples of contact hypersurfaces in $\mathbb{R}\times M$. Then we shall consider their relation with the symplectic structure Ω_ν and discuss some physical significance. In this section, we will assume $\mathcal{H}_w = 0$ and $\phi = -\alpha$.

5.2. Spatial hypersurfaces

First, we consider spatial hypersurfaces

$$M_c = \{(t,x) \in \mathbb{R}\times M \mid t = c = constant\}$$

as level sets of the function t. Recall that t is the Hamiltonian function for the normalized vorticity field $X_t = (\mathbf{w}\cdot\alpha)^{-1}\mathbf{w}\cdot\nabla$. Then the transversality condition

$$i(J_\nu)(i(X_t)(\Omega_\nu)) = i(J_\nu)(dt) = \frac{2\mathcal{H}}{\mathbf{w}\cdot\nabla\alpha} \neq 0 \tag{48}$$

for M_c holds for nonvanishing values of the helicity density. In other words, for $\mathcal{H} \neq 0$ the helicity current is not contained in the tangent spaces to M_c. This is obvious from the explicit expression in Eq. (32) for J in which $\mathcal{H}$ occurs as the component in the time direction. In this case, the contact form on M_c is obtained as follows. Let $i : M_c \to \mathbb{R}\times M : (t = c, x) \mapsto (t, x)$ be the inclusion of time slices M_c into space-time. A function on $\mathbb{R}\times M$ gives a function on M_c when composed by i. This operation can then be extended to differential forms. If $\gamma = f_a(x)dx^a$ is a one-form on $\mathbb{R}\times M$ its pull-back $i^*\gamma$ to M_c by the inclusion map i is defined to be

$$i^*\gamma = i^*(f_a(x)dx^a) = i^*(f_a(x))i^*(dx^a) = (f_a \circ i)(x)di(x^a) \tag{49}$$

where we used the commutativity of the operators d and i^* [2].

Proposition 8 *On the spatial hypersurfaces M_c, the one-form*

$$\sigma \equiv i^*\theta \;=\; \mathbf{v}(t = c, \mathbf{x}) \cdot d\mathbf{x} = i(J_\nu)(\Omega_\nu)|_{M_c} \tag{50}$$

defines a contact structure.

In particular, we compute

$$d\sigma = di^*\theta \; = \; i^*d\theta = i^*\Omega_\nu = \mathbf{w}(t=c,\mathbf{x})\cdot d\mathbf{x}\wedge d\mathbf{x}$$

and it follows that σ is a contact form on M_c

$$\sigma\wedge d\sigma = i^*\theta\wedge i^*\Omega_\nu = i^*(\theta\wedge\Omega_\nu) = 2\mathcal{H}\, d\mathbf{x}\cdot d\mathbf{x}\wedge d\mathbf{x}\neq 0 \tag{51}$$

provided the helicity density is nonzero. Note that as a result of the transversality condition we have $i(J_\nu)(\sigma) = 0$. In fact, $i(J_\nu)(\theta) = 0$ follows from the very definition of the symplectic dilation J_ν. Projections on M_c of equilibrium solutions for the Lagrangian motion correspond to Legendrian submanifolds of the contact structure.

We shall now show that the investigation of the normalized vorticity field X_t as a Hamiltonian system on $\mathbb{R}\times M$ is equivalent to the study of the Reeb vector field on the level surfaces M_c of the Hamiltonian function t. Since $t\circ i$ is the constant function on time slices M_c we have

$$i^*(i(X_t)(\Omega_\nu)) = i^*(dt) = di^*t = d(t\circ i) = 0 \tag{52}$$

as the pull-back of the Hamilton's equations for X_t to the spatial hypersurfaces. On the other hand, for any vector field X on $\mathbb{R}\times M$ we have the identity

$$i^*(i(X)(\Omega_\nu)) = i(i_*(X))(i^*\Omega_\nu) = i(i_*(X))(d\sigma) \tag{53}$$

where i_*X denotes the push-forward of X to M_c. That means i_*X is the pull-back of X by i^{-1} and hence is a vector field on M_c [2]. Since the one-dimensional kernel of $d\sigma$ (in the tangent spaces of M_c) is spanned by the Reeb vector field, Eq. (52) for X_t is possible only if the push-forward i_*X_t is proportional to the Reeb vector field of the contact structure on M_c. In fact, it is easy to check that the vector field

$$\mathbf{E}(c,\mathbf{x}) = \frac{1}{\mathcal{H}}\mathbf{w}(t=c,\mathbf{x}) = (\frac{\mathbf{w}\cdot\nabla\alpha}{\mathcal{H}}\mathbf{X}_t)(t=c,\mathbf{x}) \tag{54}$$

with $\mathcal{H}$ being evaluated at $t=c$ satisfies the criteria in Eq. (47) for the contact one form in Eq. (50).

5.3. Comparison with steady flow

To this end, we want to remark that the contact structure on spatial hypersurfaces M_c must be distinguished from similar geometric constructions on M obtained from the Euler equation

$$\mathbf{v}(\mathbf{x})\times\mathbf{w}(\mathbf{x}) = \nabla\alpha(\mathbf{x}) \tag{55}$$

for the steady flow of incompressible fluid. In this latter case there is also a contact structure on M provided the (time-independent) helicity density is nonzero. However, the difference between the two cases is not merely the time-dependence of fields. They imply qualitatively different descriptions of the flows. For example, from Eq. (55) of the steady flow we obtain

$$\mathbf{v}(\mathbf{x})\cdot\nabla\alpha(\mathbf{x}) = 0\;,\quad \mathbf{w}(\mathbf{x})\cdot\nabla\alpha(\mathbf{x}) = 0\;,\quad [\mathbf{v}(\mathbf{x}),\mathbf{w}(\mathbf{x})] = 0 \tag{56}$$

which means that the fields $\mathbf{v}$ and $\mathbf{w}$ span the tangent spaces of the (two-dimensional) Bernoulli surfaces $\alpha(\mathbf{x}) = constant$ and their flow lines commute on these surfaces [3]. On the other hand, the pull-back of the unsteady Euler equation to the hypersurfaces M_c gives

$$\mathbf{v}_{,t}(t=c,\mathbf{x}) - \mathbf{v}(t=c,\mathbf{x})\times\mathbf{w}(t=c,\mathbf{x}) = \nabla\alpha(t=c,\mathbf{x})\;. \tag{57}$$

The qualitative analysis of this equation implies quite different and complicated results for the surfaces $\alpha(t = c, \mathbf{x}) = constant$ as well as the expression for the push-forward of the suspended velocity field to the contact hypersurfaces M_c. The behavior of the flows lines on M_c is also different. For example, if we take the curl of Eq. (57)

$$\mathbf{w}_{,t}(t = c, \mathbf{x}) + [\mathbf{v}(t = c, \mathbf{x}), \mathbf{w}(t = c, \mathbf{x})] = 0 \tag{58}$$

we see that the flows of the velocity and the vorticity fields do not necessarily commute on M_c.

5.4. Bernoulli hypersurfaces

Let $B \subset \mathbb{R}\times M$ be the level sets of the Bernoulli function α, which is the Hamiltonian function for the suspended velocity field of the Euler flow. Let $i : B \to \mathbb{R}\times M$ be the inclusion. The transversality condition reads

$$i(X_\alpha)(\theta) = i(J)(d\alpha) = -v^2 \neq 0 \tag{59}$$

where we used the fact that α is conserved under the Lagrangian motion. The transversality of the helicity current to the Bernoulli surfaces is the same as the nonvanishing of the kinetic energy of the fluid. On B, the Euler equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{w} = 0\ , \quad \alpha = constant\ . \tag{60}$$

Obviously, the pull-back of the symplectic two-form

$$i^*\Omega_e = \mathbf{w} \cdot (d\mathbf{x} \wedge d\mathbf{x}) - \mathbf{v} \times \mathbf{w} \cdot d\mathbf{x} \wedge dt \tag{61}$$

to the Bernoulli surfaces is degenerate. Its one-dimensional kernel in the tangent spaces of B is the span of the vorticity field. Since $\partial_t + v$ is also tangent to B the two-dimensional tangent hypersurfaces (in the three-dimensional tangent spaces of B) on which $i^*\Omega_e$ is nondegenerate can be defined to be the complement of $span\{\partial_t + v, \mathbf{w} \cdot \nabla\}$ in the tangent spaces of $\mathbb{R}\times M$. $i^*\Omega_e$ is also exact on B

$$i^*\Omega_e = d(\mathbf{v} \cdot d\mathbf{x}) \ \ mod\ Eq.(60) \tag{62}$$

by the pulled-back of Euler equation. Therefore, the contact structure on the Bernoulli surfaces is defined by the time-dependent one-form

$$\sigma_t = \mathbf{v}(t, \mathbf{x}) \cdot d\mathbf{x} \tag{63}$$

whose derivative is the two-form in Eq. (61). The nonintegrability condition

$$\sigma_t \wedge d\sigma_t = 2\mathcal{H}\ d\mathbf{x} \cdot d\mathbf{x} \wedge d\mathbf{x} + (v^2\mathbf{w} - 2\mathcal{H}\mathbf{v}) \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt \tag{64}$$

of the tangent hyperplanes defined as above requires either the helicity density or the kinetic energy to be nonzero.

Proposition 9 *Let $B = \{(t, \mathbf{x}) \in \mathbb{R}\times M : \alpha = p(\mathbf{x}) + v^2(t, \mathbf{x})/2 = constant\}$ be level sets of Bernoulli surfaces. The following statements are equivalent:*

i) the kinetic energy is nonzero.

ii) J_ν is transversal to B.

iii) σ_t defines a contact structure on B.

Recall that the nonvanishing of the kinetic energy is also required by the transversality condition. For the contact one-form σ_t on the Bernoulli surfaces we can find the Reeb vector field up to an arbitrary function

$$E = m(t,x)(\partial_t + v) + n(t,x)\mathbf{w}\cdot\nabla \ , \quad mv^2 + 2n\mathcal{H} = 1 \ . \tag{65}$$

This arbitrariness is a manifestation of the fact that contrary to the case of spatial hypersurfaces the inclusion of the Bernoulli hypersurfaces into $\mathbb{R}\times M$ is defined only implicitly.

6. Discussion and conclusions

6.1. Summary

We used the Euler equations to construct a four-dimensional symplectic structure and realized the suspended velocity field as well as the normalized vorticity field as Hamiltonian vector fields for Hamiltonian functions being the Bernoulli function and the time, respectively. We discussed extensions of this structure to viscous flows of the Navier–Stokes equations as well. The symplectic dilation associated to the symplectic structure satisfies the conditions of transversality to Bernoulli and spatial hypersurfaces indicating the existence of contact structures thereon. We constructed corresponding contact structures, showed that the dynamics of Hamiltonian vector fields reduces to that of the Reeb vector field for each, and gave a comparison with the contact structure coming from the stationary Euler equations.

In the rest of this section we shall discuss the fluid dynamical implications of this construction in the framework of symplectization of contact structures, or of Hamiltonization of Reeb vector fields. If σ is a contact form on M then the two-form

$$\Omega = e^t(d_M\sigma + dt\wedge\sigma) = d(e^t\sigma) \tag{66}$$

defines an exact symplectic structure on $\mathbb{R}\times M$ and is the foremost example of symplectization of a contact structure σ defined on spatial hypersurfaces. Note that Ω has ∂_t as its symplectic dilation and that the Reeb vector field on M defined by σ is a Hamiltonian vector field for the symplectization with the Hamiltonian function e^t [3, 11].

6.2. Symplectization in fluid dynamics

The geometric fluid dynamics provides unusual but nevertheless natural examples of symplectization. Recall that we obtain the contact one-form on time slices by pulling the canonical one-form θ back to M_c by the inclusion. Conversely, the symplectization of the contact structure on time slices follows from the inclusion map

$$i : M_c \to \mathbb{R}\times M : (t=c,\mathbf{x}) \mapsto (t,\mathbf{x}) \ . \tag{67}$$

In this case, the time variable t is introduced naturally by the action of the invariant differential operators.

From a physical point of view, the symplectization of the time slices M_c corresponds to the construction of trajectories of the velocity field from streamlines. These are solutions of the nonautonomous and autonomous equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t,\mathbf{x}) \ , \quad \frac{d\mathbf{x}(\tau)}{d\tau} = \mathbf{v}(t=c,\mathbf{x}(\tau)) \ , \tag{68}$$

respectively. The solutions to the first equation on $\mathbb{R}\times M$ can be constructed by solving the autonomous system on M at each time t and then joining them by the inclusion

$$i : (streamlines) \mapsto (trajectories) \ . \tag{69}$$

The symplectization to $\mathbb{R}\times M$ of the contact structures on M_c means that the inclusion in Eq. (69) for solutions of the differential equations (68) extends to the inclusion of geometric structures

$$i\ :\ \begin{pmatrix} \textit{contact structure on} \\ \textit{the space of streamlines} \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{symplectic structure on} \\ \textit{the space of trajectories} \end{pmatrix}$$

on M_c into those on $\mathbb{R}\times M$.

The symplectization of the contact structure on the Bernoulli surfaces may be given a similar interpretation in the language of the solutions of differential equations. In this case, it will be appropriate to consider the solutions of the Euler equation. The inclusion

$$i\ :\ \begin{pmatrix} Bernoulli \\ surfaces \end{pmatrix} \longrightarrow \left(\ space - time \ \right)$$

implies the construction

$$i\ :\ \begin{pmatrix} \textit{solution of} \\ \mathbf{v}_{,t} - \mathbf{v}\times\mathbf{w} = 0 \\ \textit{on } \alpha = \textit{constant} \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{solution of} \\ \textit{the Euler equation} \end{pmatrix}$$

of the solutions of the Euler equation from the solutions of a homogeneous equation on each Bernoulli surface. This may be interpreted as a nonlinear analogue of the result in the theory of linear differential equations that the general solution of an inhomogeneous equation is the sum of the solution of its homogeneous part and the particular solution.

References

[1] Abraham R, Marsden JE. Foundations of Mechanics, Boston, MA, USA: Addison Wesley, 1987.

[2] Abraham R, Marsden JE, Ratiu T. Manifolds, tensor analysis, and applications, Boston, MA, USA: Addison Wesley, 1983.

[3] Arnold VI. Mathematical Methods of Classical Mechanics, Graduate Text in Mathematics, Vol: 60, Second Edition, Berlin, Germany: Springer, 1989.

[4] Arnold VI, Khesin BA. Topological Methods in Hydrodynamics, Applied Mathematical Sciences, Vol. 125, Berlin, Germany: Springer, 1994.

[5] Chorin AJ, Marsden JE. A Mathematical Introduction to Fluid Mechanics, Texts in Applied Mathematics vol. 4, Second Edition, Berlin, Germany: Springer, 1990.

[6] Frisch U, Turbulence: the legacy of A. N. Kolmogorov, Cambridge, UK: Cambridge University Press, 1995.

[7] Gümral H. Lagrangian description, symplectic structure and invariants of 3D fluid flow. Phys Lett 1997; 232: 417-424.

[8] Gümral H. Helicity invariants in 3D: kinematical aspects. Physica D 2000; 139: 335-359.

[9] Hofer H, Kriener M. Holomorphic curves in contact dynamics. In: Differential equations: La Pietra 1996: Conference on Differential equations marking the 70th birthdays of Peter Lax and Louis Nirenberg, July 3-7, 1996, Villa La Pietra, Florence, Italy, M. Giaquinta, J. Shatah, S.R.S. Varadhan, editors. Providence, RI, USA: AMS, 1999.

[10] Libermann P, Marle CM. Symplectic Geometry and Analytical Mechanics, Dordrecht, Netherlands: D. Reidel Publishing Company, 1987.

[11] McDuff D, Salamon D. Introduction to Symplectic Topology, Oxford, UK: Oxford University Press, 1995.

[12] Meyer RE. Introduction to Mathematical Fluid Dynamics. New York, NY, USA: Dover, 1982.

[13] Marsden JE, Ratiu T. Introduction to Symmetry and Mechanics, Texts in Applied Mathematics, Vol. 17, Berlin, Germany: Springer, 1994.

[14] Rabinowitz PH. Periodic solutions of a Hamiltonian system on a prescribed energy surface. J Diff Eq 1979; 33: 336-352.

[15] Saffman PG. Dynamics of vorticity. J Fluid Mech 1981; 106: 49-58.

[16] Souriau JM. Structure des systèmes dynamiques (Dunod, 1970), Structure of Dynamical Systems: A Symplectic View of Physics, translated by C.H. Cushman-de Vries, Translation Editors: R.H. Cushman and G.M. Tuynman, Basel, Switzerland: Birkhäuser, 1997.

[17] Weinstein A. Lectures on Symplectic Manifolds. In: Regional conference series in mathematics 29, Providence, RI, USA: AMS, 1977.

[18] Weinstein A. Periodic orbits for convex Hamiltonian systems. Ann Math 1978; 108: 507-518.

[19] Weinstein A. On the hypothesis of Rabinowitz' periodic orbit theorem. J Diff Eq 1979; 33: 353-358.