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## Coefficient estimates for general subclasses of $m$ -fold symmetric analytic bi-univalent functions

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**Abstract:** In this work, we introduce and investigate two new subclasses of the bi-univalent functions in which both  $f$  and  $f^{-1}$  are  $m$ -fold symmetric analytic functions. For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for  $|a_{m+1}|$  and  $|a_{2m+1}|$ .

**Key words:** Analytic functions, univalent functions, bi-univalent functions,  $m$ -fold symmetric bi-univalent functions

### 1. Introduction

Let  $\mathcal{A}$  denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We also denote by  $\mathcal{S}$  the class of all functions in the normalized analytic function class  $\mathcal{A}$  that are univalent in  $\mathbb{U}$ .

It is well known that every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

The inverse function  $g = f^{-1}$  is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (2)$$

A function  $f \in \mathcal{A}$  is said to be bi-univalent in  $\mathbb{U}$  if both  $f$  and  $f^{-1}$  are univalent in  $\mathbb{U}$ . Let  $\Sigma$  denote the class of bi-univalent functions in  $\mathbb{U}$  given by (1). The class of analytic bi-univalent functions was first

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introduced and studied by Lewin [16], where it was proved that  $|a_2| < 1.51$ . Brannan and Clunie [2] improved Lewin’s result to  $|a_2| \leq \sqrt{2}$  and later Netanyahu [18] proved that  $|a_2| \leq 4/3$ . Brannan and Taha [3] and Taha [24] also investigated certain subclasses of bi-univalent functions and found nonsharp estimates on the first two Taylor–Maclaurin coefficients  $|a_2|$  and  $|a_3|$ . For a brief history and interesting examples of functions in the class  $\Sigma$ , see [21] (see also [3]). The aforementioned work of Srivastava et al. [21] essentially revived the investigation of various subclasses of the bi-univalent function class  $\Sigma$  in recent years; it was followed by such works as those by Frasin and Aouf [12], Xu et al. [25, 26], Hayami and Owa [15], and others (see, for example, [1, 4–10, 13, 17, 19, 20]).

Let  $m \in \mathbb{N} = \{1, 2, 3, \dots\}$ . A domain  $E$  is said to be  $m$ -fold symmetric if a rotation of  $E$  about the origin through an angle  $2\pi/m$  carries  $E$  on itself. It follows that a function  $f(z)$  analytic in  $\mathbb{U}$  is said to be  $m$ -fold symmetric ( $m \in \mathbb{N}$ ) if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

In particular, every  $f(z)$  is 1-fold symmetric and every odd  $f(z)$  is 2-fold symmetric. We denote by  $\mathcal{S}_m$  the class of  $m$ -fold symmetric univalent functions in  $\mathbb{U}$ .

A simple argument shows that  $f \in \mathcal{S}_m$  is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}). \tag{3}$$

Srivastava et al. [22] defined  $m$ -fold symmetric bi-univalent functions, analogues to the concept of  $m$ -fold symmetric univalent functions. For the normalized form of  $f$  given by (3), they obtained the series expansion for  $f^{-1}$  as follows:

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_{m+1} w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} \\ &\quad - \left[ \frac{1}{2} (m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots \end{aligned} \tag{4}$$

We denote by  $\Sigma_m$  the class of  $m$ -fold symmetric bi-univalent functions in  $\mathbb{U}$ . For  $m = 1$ , the formula (4) coincides with the formula (2) of the class  $\Sigma$ . For some examples of  $m$ -fold symmetric bi-univalent functions, see [22].

The object of the present paper is to introduce two new general subclasses of bi-univalent functions in which both  $f$  and  $f^{-1}$  are  $m$ -fold symmetric analytic functions and to obtain coefficient bounds for  $|a_{m+1}|$  and  $|a_{2m+1}|$  for functions in each of these new subclasses.

In order to establish our main results, we shall require the following lemma.

**Lemma 1** [11] *If  $p \in \mathcal{P}$ , then  $|c_k| \leq 2$  ( $k \in \mathbb{N}$ ), where the Carathéodary class  $\mathcal{P}$  is the family of all functions  $p$  analytic in  $\mathbb{U}$  for which*

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (z \in \mathbb{U}).$$

**2. Coefficient estimates for the function class  $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$**

**Definition 2** For  $\lambda \geq 1$  and  $\mu \geq 0$ , a function  $f \in \Sigma_m$  given by (3) is said to be in the class  $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$  if the following conditions are satisfied:

$$\left| \arg \left( (1 - \lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \tag{5}$$

and

$$\left| \arg \left( (1 - \lambda) \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2} \tag{6}$$

where  $0 < \alpha \leq 1$ ;  $m \in \mathbb{N}$ ;  $z, w \in \mathbb{U}$ ; and  $g = f^{-1}$  is defined by (4).

**Remark 3** In the following special cases of Definition 2, we show how the class of analytic bi-univalent functions  $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$  for suitable choices of  $\lambda$ ,  $\mu$ , and  $m$  lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.

(i) For  $\mu = 1$ , we obtain the  $m$ -fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^1(\alpha, \lambda) = \mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$$

introduced by Sümer Eker [23]. In addition, for  $m = 1$  we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^1(\alpha, \lambda) = \mathcal{B}_{\Sigma}(\alpha, \lambda)$$

introduced by Frasin and Aouf [12].

(ii) For  $\mu = 1$  and  $\lambda = 1$ , we have the  $m$ -fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^1(\alpha, 1) = \mathcal{H}_{\Sigma,m}^{\alpha}$$

introduced by Srivastava et al. [22]. In addition, for  $m = 1$  we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^1(\alpha, 1) = \mathcal{H}_{\Sigma}^{\alpha}$$

introduced by Srivastava et al. [21].

(iii) For  $\mu = 0$  and  $\lambda = 1$ , we get a new class

$$\mathcal{N}_{\Sigma,m}^0(\alpha, 1) = \mathcal{S}_{\Sigma,m}^{\alpha}$$

of  $m$ -fold symmetric strongly bi-starlike functions of order  $\alpha$ . In addition, for  $m = 1$  we have the strongly bi-starlike function class

$$\mathcal{N}_{\Sigma,1}^0(\alpha, 1) = \mathcal{S}_{\Sigma}^*[\alpha]$$

introduced by Brannan and Taha [3].

(iv) For  $\lambda = 1$ , we have a new class

$$\mathcal{N}_{\Sigma,m}^{\mu}(\alpha, 1) = \mathcal{P}_{\Sigma,m}(\alpha, \mu),$$

which consists of  $m$ -fold symmetric bi-Bazilevič functions.

(v) For  $m = 1$ , we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{\mu}(\alpha, \lambda) = \mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$$

introduced by Çağlar et al. [8].

**Theorem 4** Let the function  $f(z)$  given by (3) be in the class  $\mathcal{N}_{\Sigma,m}^{\mu}(\alpha, \lambda)$ . Then

$$|a_{m+1}| \leq \begin{cases} \frac{2\alpha}{\sqrt{(\mu+m\lambda)^2+m\alpha(\mu+2m\lambda-m\lambda^2)}} & , \quad 1 \leq \lambda < 1 + \sqrt{\frac{\mu+m}{m}} \\ \frac{2\alpha}{\mu+m\lambda} & , \quad \lambda \geq 1 + \sqrt{\frac{\mu+m}{m}} \end{cases} \quad (7)$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{(\mu+m\lambda)^2} + \frac{2\alpha}{\mu+2m\lambda}. \quad (8)$$

**Proof** It follows from (5) and (6) that

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} = [p(z)]^{\alpha} \quad (9)$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} = [q(w)]^{\alpha}, \quad (10)$$

where

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \dots \in \mathcal{P} \quad (11)$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \dots \in \mathcal{P}. \quad (12)$$

Now, equating the coefficients in (9) and (10), we have

$$(\mu+m\lambda)a_{m+1} = \alpha p_m \quad (13)$$

$$(\mu+2m\lambda)\left[\frac{\mu-1}{2}a_{m+1}^2 + a_{2m+1}\right] = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2 \quad (14)$$

$$-(\mu+m\lambda)a_{m+1} = \alpha q_m \quad (15)$$

$$(\mu+2m\lambda)\left[\left(m+\frac{\mu+1}{2}\right)a_{m+1}^2 - a_{2m+1}\right] = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \quad (16)$$

From (13) and (15), we obtain

$$p_m = -q_m \quad (17)$$

and

$$2(\mu+m\lambda)^2 a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2). \quad (18)$$

Also, from (14), (16), and (18), we find that

$$a_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{(\mu + m\lambda)^2 + m\alpha (\mu + 2m\lambda - m\lambda^2)}. \tag{19}$$

Applying Lemma 1 for (13) and (19), we get the desired estimate on the coefficient  $|a_{m+1}|$  as asserted in (7).

Next, in order to find the bound on the coefficient  $|a_{2m+1}|$ , we subtract (16) from (14). Observing (17) we get

$$a_{2m+1} = \frac{m + 1}{2} a_{m+1}^2 + \frac{\alpha (p_{2m} - q_{2m})}{2(\mu + 2m\lambda)}. \tag{20}$$

It follows from (13) and (20) that

$$a_{2m+1} = \frac{(m + 1) \alpha^2 p_m^2}{2(\mu + m\lambda)^2} + \frac{\alpha (p_{2m} - q_{2m})}{2(\mu + 2m\lambda)}. \tag{21}$$

Applying Lemma 1 for (21), we get the desired estimate on the coefficient  $|a_{2m+1}|$  as asserted in (8). This completes the proof of Theorem 4.  $\square$

By setting  $\mu = 1$  in Theorem 4, we obtain the following consequence.

**Corollary 5** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$ . Then*

$$|a_{m+1}| \leq \begin{cases} \frac{2\alpha}{\sqrt{(1+m\lambda)^2 + m\alpha(1+2m\lambda - m\lambda^2)}} & , \quad 1 \leq \lambda < 1 + \sqrt{\frac{m+1}{m}} \\ \frac{2\alpha}{1+m\lambda} & , \quad \lambda \geq 1 + \sqrt{\frac{m+1}{m}} \end{cases} \tag{22}$$

and

$$|a_{2m+1}| \leq \frac{2(m + 1) \alpha^2}{(1 + m\lambda)^2} + \frac{2\alpha}{1 + 2m\lambda}.$$

**Remark 6** *The above estimates for  $|a_{m+1}|$  show that the inequality (22) is an improvement of the estimates obtained by Sümer Eker [23].*

**Corollary 7** (see [23]) *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{A}_{\Sigma,m}^{\alpha,\lambda}$ . Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1 + m\lambda)^2 + m\alpha (1 + 2m\lambda - m\lambda^2)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m + 1) \alpha^2}{(1 + m\lambda)^2} + \frac{2\alpha}{1 + 2m\lambda}.$$

By setting  $\mu = 1$  and  $\lambda = 1$  in Theorem 4, we obtain the following consequence.

**Corollary 8** (see [22]) *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{H}_{\Sigma,m}^{\alpha}$ . Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(1+m)(1+m+m\alpha)}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2}{1+m} + \frac{2\alpha}{1+2m}.$$

By setting  $\mu = 0$  and  $\lambda = 1$  in Theorem 4, we obtain the following consequence.

**Corollary 9** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{S}_{\Sigma,m}^{\alpha}$ . Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{1+\alpha}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2} + \frac{\alpha}{m}.$$

By setting  $\lambda = 1$  in Theorem 4, we obtain the following consequence.

**Corollary 10** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{P}_{\Sigma,m}(\alpha, \mu)$ . Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(\mu+m)^2 + m\alpha(\mu+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{(\mu+m)^2} + \frac{2\alpha}{\mu+2m}.$$

By setting  $m = 1$  in Theorem 4, we obtain the following consequence.

**Corollary 11** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$ . Then*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{\sqrt{(\mu+\lambda)^2 + \alpha(\mu+2\lambda-\lambda^2)}} & , \quad 1 \leq \lambda < 1 + \sqrt{\mu+1} \\ \frac{2\alpha}{\mu+\lambda} & , \quad \lambda \geq 1 + \sqrt{\mu+1} \end{cases} \quad (23)$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\mu+\lambda)^2} + \frac{2\alpha}{\mu+2\lambda}.$$

**Remark 12** *The above estimates for  $|a_2|$  show that the inequality (23) is an improvement of the estimates obtained by Çağlar et al. [8].*

**3. Coefficient estimates for the function class  $\mathcal{N}_{\Sigma,m}^{\mu}(\beta, \lambda)$**

**Definition 13** For  $\lambda \geq 1$  and  $\mu \geq 0$ , a function  $f \in \Sigma_m$  given by (3) is said to be in the class  $\mathcal{N}_{\Sigma,m}^{\mu}(\beta, \lambda)$  if the following conditions are satisfied:

$$\Re \left( (1 - \lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \right) > \beta \tag{24}$$

and

$$\Re \left( (1 - \lambda) \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} \right) > \beta \tag{25}$$

where  $0 \leq \beta < 1$ ;  $m \in \mathbb{N}$ ;  $z, w \in \mathbb{U}$ ; and  $g = f^{-1}$  is defined by (4).

**Remark 14** In the following special cases of Definition 13, we show how the class of analytic bi-univalent functions  $\mathcal{N}_{\Sigma,m}^{\mu}(\beta, \lambda)$  for suitable choices of  $\lambda$ ,  $\mu$ , and  $m$  lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.

(i) For  $\mu = 1$ , we obtain the  $m$ -fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^1(\beta, \lambda) = \mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$$

introduced by Sümer Eker [23]. In addition, for  $m = 1$  we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^1(\beta, \lambda) = \mathcal{B}_{\Sigma}(\beta, \lambda)$$

introduced by Frasin and Aouf [12].

(ii) For  $\mu = 1$  and  $\lambda = 1$ , we have the  $m$ -fold symmetric bi-univalent function class

$$\mathcal{N}_{\Sigma,m}^1(\beta, 1) = \mathcal{H}_{\Sigma,m}(\beta)$$

introduced by Srivastava et al. [22]. In addition, for  $m = 1$  we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^1(\beta, 1) = \mathcal{H}_{\Sigma}(\beta)$$

introduced by Srivastava et al. [21].

(iii) For  $\mu = 0$  and  $\lambda = 1$ , we get the class

$$\mathcal{N}_{\Sigma,m}^0(\beta, 1)$$

of  $m$ -fold symmetric bi-starlike functions of order  $\beta$  (see [14]). In addition, for  $m = 1$  we have the bi-starlike function class

$$\mathcal{N}_{\Sigma,1}^0(\beta, 1) = \mathcal{S}_{\Sigma}^*(\beta)$$

introduced by Brannan and Taha [3].

(iv) For  $\lambda = 1$ , we have a new class

$$\mathcal{N}_{\Sigma,m}^{\mu}(\beta, 1) = \mathcal{P}_{\Sigma,m}(\beta, \mu),$$



which consists of  $m$ -fold symmetric bi-Bazilevič functions.

(v) For  $m = 1$ , we have the bi-univalent function class

$$\mathcal{N}_{\Sigma,1}^{\mu}(\beta, \lambda) = \mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$$

introduced by Çağlar et al. [8].

**Theorem 15** Let the function  $f(z)$  given by (3) be in the class  $\mathcal{N}_{\Sigma,m}^{\mu}(\beta, \lambda)$ . Then

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(\mu+2m\lambda)(\mu+m)}} & , \quad 0 \leq \beta < \frac{m(\mu+2m\lambda-m\lambda^2)}{(\mu+2m\lambda)(\mu+m)} \\ \frac{2(1-\beta)}{\mu+m\lambda} & , \quad \frac{m(\mu+2m\lambda-m\lambda^2)}{(\mu+2m\lambda)(\mu+m)} \leq \beta < 1 \end{cases} \quad (26)$$

and

$$|a_{2m+1}| \leq \begin{cases} \min \left\{ \frac{2(m+1)(1-\beta)}{(\mu+2m\lambda)(\mu+m)}, \frac{2(m+1)(1-\beta)^2}{(\mu+m\lambda)^2} + \frac{2(1-\beta)}{\mu+2m\lambda} \right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2m\lambda} & , \quad \mu \geq 1 \end{cases} \quad (27)$$

**Proof** It follows from (24) and (25) that

$$(1-\lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} = \beta + (1-\beta)p(z) \quad (28)$$

and

$$(1-\lambda) \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} = \beta + (1-\beta)q(w), \quad (29)$$

where  $p(z)$  and  $q(w)$  have the forms (11) and (12), respectively. Now, equating the coefficients in (28) and (29), we have

$$(\mu+m\lambda)a_{m+1} = (1-\beta)p_m \quad (30)$$

$$(\mu+2m\lambda) \left[ \frac{\mu-1}{2} a_{m+1}^2 + a_{2m+1} \right] = (1-\beta)p_{2m} \quad (31)$$

$$-(\mu+m\lambda)a_{m+1} = (1-\beta)q_m \quad (32)$$

$$(\mu+2m\lambda) \left[ \left( m + \frac{\mu+1}{2} \right) a_{m+1}^2 - a_{2m+1} \right] = (1-\beta)q_{2m}. \quad (33)$$

From (30) and (32), we obtain

$$p_m = -q_m \quad (34)$$

and

$$2(\mu+m\lambda)^2 a_{m+1}^2 = (1-\beta)^2 (p_m^2 + q_m^2). \quad (35)$$

Also, from (31) and (33), we have

$$(\mu+2m\lambda)(\mu+m)a_{m+1}^2 = (1-\beta)(p_{2m} + q_{2m}). \quad (36)$$

Therefore, from equalities (35) and (36) we find that

$$2|a_{m+1}|^2 \leq \frac{(1-\beta)^2 (|p_m|^2 + |q_m|^2)}{2(\mu+m\lambda)^2}$$

and

$$|a_{m+1}|^2 \leq \frac{(1-\beta)(|p_{2m}| + |q_{2m}|)}{(\mu+2m\lambda)(\mu+m)},$$

respectively, and applying Lemma 1, we get the desired estimate on the coefficient  $|a_{m+1}|$  as asserted in (26).

Next, in order to find the bound on the coefficient  $|a_{2m+1}|$ , by subtracting (33) from (31), we get

$$a_{2m+1} = \frac{m+1}{2}a_{m+1}^2 + \frac{(1-\beta)(p_{2m} - q_{2m})}{2(\mu+2m\lambda)}. \tag{37}$$

Upon substituting the value of  $a_{m+1}^2$  from (35) into (37), it follows that

$$a_{2m+1} = \frac{(m+1)(1-\beta)^2(p_m^2 + q_m^2)}{4(\mu+m\lambda)^2} + \frac{(1-\beta)(p_{2m} - q_{2m})}{2(\mu+2m\lambda)}. \tag{38}$$

Applying Lemma 1 for (38), we find that

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{(\mu+m\lambda)^2} + \frac{2(1-\beta)}{(\mu+2m\lambda)}. \tag{39}$$

On the other hand, upon substituting the value of  $a_{m+1}^2$  from (36) into (37), it follows that

$$a_{2m+1} = \frac{1-\beta}{2(\mu+2m\lambda)(\mu+m)} [(\mu+2m+1)p_{2m} + (1-\mu)q_{2m}]. \tag{40}$$

Applying Lemma 1 for (40), we have

$$|a_{2m+1}| \leq \frac{1-\beta}{(\mu+2m\lambda)(\mu+m)} [\mu+2m+1 + |1-\mu|]. \tag{41}$$

By investigating the bound on  $|a_{2m+1}|$  according to  $\mu$  in (41) and comparing with (39), we get the desired estimate on the coefficient  $|a_{2m+1}|$  as asserted in (27). This completes the proof of the Theorem 15.  $\square$

By setting  $\mu = 1$  in Theorem 15, we obtain the following consequence.

**Corollary 16** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{A}_{\Sigma,m}^\lambda(\beta)$ . Then*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(1+m)}} & , \quad 0 \leq \beta < \frac{m(1+2m\lambda-m\lambda^2)}{(1+2m\lambda)(1+m)} \\ \frac{2(1-\beta)}{1+m\lambda} & , \quad \frac{m(1+2m\lambda-m\lambda^2)}{(1+2m\lambda)(1+m)} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m\lambda}.$$

**Remark 17** Corollary 16 is an improvement of the following estimates obtained by Sümer Eker [23].

**Corollary 18** (see [23]) Let the function  $f(z)$  given by (3) be in the class  $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$ . Then

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+2m\lambda)(1+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{(1+m\lambda)^2} + \frac{2(1-\beta)}{1+2m\lambda}.$$

By setting  $\mu = 1$  and  $\lambda = 1$  in Theorem 15, we obtain the following consequence.

**Corollary 19** Let the function  $f(z)$  given by (3) be in the class  $\mathcal{H}_{\Sigma,m}(\beta)$ . Then

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(1+2m)(1+m)}} & , \quad 0 \leq \beta < \frac{m}{1+2m} \\ \frac{2(1-\beta)}{1+m} & , \quad \frac{m}{1+2m} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)}{1+2m}.$$

**Remark 20** Corollary 19 is an improvement of the following estimates obtained by Srivastava et al. [22].

**Corollary 21** (see [22]) Let the function  $f(z)$  given by (3) be in the class  $\mathcal{H}_{\Sigma,m}(\beta)$ . Then

$$|a_{m+1}| \leq \sqrt{\frac{4(1-\beta)}{(1+2m)(1+m)}}$$

and

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2}{1+m} + \frac{2(1-\beta)}{1+2m}.$$

By setting  $\mu = 0$  and  $\lambda = 1$  in Theorem 15, we obtain the following consequence.

**Corollary 22** Let the function  $f(z)$  given by (3) be in the class  $\mathcal{N}_{\Sigma,m}^0(\beta, 1)$ . Then

$$|a_{m+1}| \leq \begin{cases} \frac{1}{m} \sqrt{2(1-\beta)} & , \quad 0 \leq \beta < \frac{1}{2} \\ \frac{2(1-\beta)}{m} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \frac{(m+1)(1-\beta)}{m^2} & , \quad 0 \leq \beta < \frac{2m+1}{2(m+1)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m} & , \quad \frac{2m+1}{2(m+1)} \leq \beta < 1 \end{cases}.$$

By setting  $\lambda = 1$  in Theorem 15, we obtain the following consequence.

**Corollary 23** *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{P}_{\Sigma,m}(\beta, \mu)$ . Then*

$$|a_{m+1}| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(\mu+2m)(\mu+m)}} & , \quad 0 \leq \beta < \frac{m}{\mu+2m} \\ \frac{2(1-\beta)}{\mu+m} & , \quad \frac{m}{\mu+2m} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq \begin{cases} \min \left\{ \frac{2(m+1)(1-\beta)}{(\mu+2m)(\mu+m)}, \frac{2(m+1)(1-\beta)^2}{(\mu+m)^2} + \frac{2(1-\beta)}{\mu+2m} \right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2m} & , \quad \mu \geq 1 \end{cases} .$$

By setting  $m = 1$  in Theorem 15, we obtain the following consequence.

**Corollary 24** (see [8]) *Let the function  $f(z)$  given by (3) be in the class  $\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$ . Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}} & , \quad 0 \leq \beta < \frac{\mu+2\lambda-\lambda^2}{(\mu+2\lambda)(\mu+1)} \\ \frac{2(1-\beta)}{\mu+\lambda} & , \quad \frac{\mu+2\lambda-\lambda^2}{(\mu+2\lambda)(\mu+1)} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \min \left\{ \frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}, \frac{4(1-\beta)^2}{(\mu+\lambda)^2} + \frac{2(1-\beta)}{\mu+2\lambda} \right\} & , \quad 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{\mu+2\lambda} & , \quad \mu \geq 1 \end{cases} .$$

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