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LUISA CONSIGLIERI

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Explicit estimates on a mixed Neumann–Robin–Cauchy problem

Luisa CONSIGLIERI*
R. Tomàs da Fonseca 26, Pateo Central 1600-256, Lisbon, Portugal

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Abstract: We deal with the existence of weak solutions for a mixed Neumann–Robin–Cauchy problem. The existence results are based on global-in-time estimates of approximating solutions, and the passage to the limit exploits compactness techniques. We investigate explicit estimates for solutions of the parabolic equations with nonhomogeneous boundary conditions and distributional right-hand sides. The parabolic equation is of divergence form with discontinuous coefficients. We consider a nonlinear condition on a part of the boundary such that the power laws (and the Robin boundary condition) appear as particular cases.

Key words: Mixed Neumann–Robin–Cauchy problem, discontinuous coefficient, bounded solution

1. Introduction

The existence of solutions to partial differential equations (PDEs) is not sufficient whenever the main objective is their application to other branches of science. In industrial applications, physical fields (such as temperature or potentials) verify PDEs in divergence form with a nondifferentiable leading coefficient. Thus, they do not correspond to the classical solutions. There is a growing demand for the existence of quantitative estimates with explicit constants due to the application of fixed-point arguments [14, 17, 26]. The knowledge of the values of the involved constants in the estimates is crucial.

The study of the Dirichlet–Cauchy problem is vast in the literature; see [2, 6, 15, 19–21, 23, 28] to mention a few. It is known that Dirichlet boundary conditions roughly approximate the reality. As a consequence, the study of the Cauchy problem under Neumann or Robin boundary conditions has its actuality in the works [1, 4, 9, 16, 18, 24, 25, 29].

This work is devoted to the determination of the involved constants for the boundary value problems concerned in the presence of radiative-type conditions on the boundary, which are typical of thermodynamic models evolved from engineering practice [3, 13].

The derivation of the estimates is not unique. It depends on the mathematical choice of what are the most relevant data. Of course, the most relevant data do not come from a mathematical choice, but from a bio-chemico-geo-physical choice. With this state of mind, we detail the proofs in order to be easily changed for other requisites.

The steady-state study can be found in [11, 12].

Let $[0,T]\subset\mathbb{R}$ be the time interval with $T>0$, and let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a bounded domain of class

*Correspondence: lconsiglieri@gmail.com

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$C^{0,1}$. The boundary $\partial\Omega$ is decomposed into two disjoint open subsets, namely Γ and $\partial\Omega\setminus\bar{\Gamma}$. Moreover, we set $Q_T = \Omega\times]0,T[$, and $\Sigma_T = \Gamma\times]0,T[$. Here we consider the nonlinear boundary condition version of the Cauchy problem studied in [7]:

$$\partial_t u - \nabla\cdot(\mathsf{A}\nabla u) = -\nabla\cdot(u\mathbf{E}) \text{ in } Q_T; \tag{1}$$

$$(\mathsf{A}\nabla u - u\mathbf{E})\cdot\mathbf{n} = 0 \text{ on } (\partial\Omega\setminus\bar{\Gamma})\times]0,T[; \tag{2}$$

$$(\mathsf{A}\nabla u - u\mathbf{E})\cdot\mathbf{n} + b(u)u = h \text{ on } \Gamma\times]0,T[, \tag{3}$$

for matrix and vector value functions A and $\mathbf{E}$, respectively. Here, $\mathbf{n}$ is the outward unit normal to the boundary $\partial\Omega$. For the sake of simplicity, we handle no right-hand side data, namely $\nabla\cdot\mathbf{f}+f$. These data can be clearly included in our main results, as well as some lower order terms in the differential operator.

The function b satisfies a $(\ell-2)$-growth condition that includes:

- Neumann: $b(u)\equiv 0$;
- Robin: $b(u)=b_*$, a constant that stands for either the heat convective transfer coefficient or the Rayleigh–Jeans radiation approximation;
- Blackbody radiation: $b(u)=\sigma|u|^3$, with σ representing the Stefan–Boltzmann constant;
- Wien displacement law: $b(u)=b_*u^4$.

2. Main results

Let us introduce the following Banach spaces, for $p,q>1$, in the framework of Bochner, Sobolev, and Lebesgue functional spaces:

$$\begin{aligned} V_{p,q} &= \{v\in W^{1,p}(\Omega):\ v|_\Gamma\in L^q(\Gamma)\};\\ L^{p,q}(Q_T) &= L^q(0,T;L^p(\Omega));\\ L^{p,q}(\Sigma_T) &= L^q(0,T;L^p(\Gamma));\\ W^{1,q}(0,T;X,Y) &= \{v\in L^q(0,T;X):\ v'\in L^q(0,T;Y)\}, \end{aligned}$$

where X and Y denote Banach spaces such that $X\hookrightarrow Y$.

Remark 2.1 *If there exists a Hilbert space H such that $X\hookrightarrow H=\bar{X}\hookrightarrow Y=X'$ then (see, for instance, [27, p. 106])*

$$W^{1,2}(0,T;X,Y)\hookrightarrow C([0,T];H).$$

Moreover, if $u,v\in W^{1,2}(0,T;X,X')$ then $(u(t),v(t))_H$ is absolutely continuous on $[0,T]$, and

$$\frac{d}{dt}(u(t),v(t))_H = \langle u'(t),v(t)\rangle + \langle v'(t),u(t)\rangle,\quad \textit{for a.e. } t\in[0,T]. \tag{4}$$

Throughout this paper, the hypotheses on the coefficients A and b are

(A) the $(n \times n)$ matrix-valued function $\mathsf{A} = [A_{ij}]_{i,j=1,\cdots,n}$ is measurable, uniformly elliptic, and uniformly bounded:

$$\exists a_\# > 0, \quad A_{ij}(x)\xi_i\xi_j \geq a_\# |\xi|^2, \quad \text{for a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n; \tag{5}$$

$$\exists a^\# > 0, \quad \|\mathsf{A}\|_{\infty,\Omega} \leq a^\#, \tag{6}$$

under the summation convention over repeated indices: $\mathsf{A}\mathbf{a}\cdot\mathbf{b} = A_{ij}a_jb_i = \mathbf{b}^\top\mathsf{A}\mathbf{a}$.

(B) $b : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. measurable with respect to $x \in \Gamma$ and continuous with respect to $\xi \in \mathbb{R}$, such that there exists $\ell \geq 2$ such that b has $(\ell - 2)$-growthness property, and it is monotone with respect to the last variable:

$$\exists b_\#, b^\# \geq 0, \qquad b_\#|\xi|^{\ell-2} \leq b(x,\xi) \leq b^\#|\xi|^{\ell-2}; \tag{7}$$

$$(b(x,\xi)\xi - b(x,\eta)\eta)(\xi - \eta) \geq 0, \tag{8}$$

for a.e. $x \in \Gamma$, and for all $\xi, \eta \in \mathbb{R}$.

Let us state our first existence result (the Dirichlet problem is established in [7, Lemma 3.2]).

Theorem 2.1 *Let* $\mathbf{E} \in L^{2/(1-n/q-\theta)}(0,T;\mathbf{L}^q(\Omega))$ *with* $\theta = 0$ *for* $q > n > 2$ *and any* $0 < \theta < 1 - 2/q$ *if* $n = 2$, $h \in L^{\ell'}(\Sigma_T)$, *with* ℓ' *standing for the conjugate exponent* $\ell' = \ell/(\ell-1)$, *and* $u_0 \in L^2(\Omega)$. *Under the assumptions (A)-(B) with* $b_\# > 0$, *there exists a function* u *in* $L^{2,\infty}(Q_T) \cap L^2(0,T;V_{2,\ell}) \cap L^\ell(\Sigma_T)$, *which is a solution of (1)–(3) in the sense that*

$$\langle \partial_t u, v\rangle + \int_{Q_T} \mathsf{A}\nabla u \cdot \nabla v \mathrm{dxdt} + \int_{\Sigma_T} b(u)uv \mathrm{dsdt} = \\ = \int_{Q_T} u\mathbf{E}\cdot\nabla v \mathrm{dxdt} + \int_{\Sigma_T} hv \mathrm{dsdt}, \tag{9}$$

for every $v \in L^2(0,T;V_{2,\ell}) \cap L^\ell(\Sigma_T)$. *Here,* $\partial_t u$ *belongs to* $L^2(0,T;(V_{2,\ell})') + L^{\ell/(\ell-1)}(\Sigma_T)$. *In particular, we have*

$$\|u\|^2_{2,\infty,Q_T} \leq \left(\|u_0\|^2_{2,\Omega} + \frac{2}{\ell' b_\#^{1/(\ell-1)}} \|h\|^{\ell'}_{\ell',\Sigma_T} \right) \exp\left[\mathcal{Q}\right]; \tag{10}$$

$$a_\#\|\nabla u\|^2_{2,Q_T} + b_\#\|u\|^\ell_{\ell,\Sigma_T} \leq \left(\|u_0\|^2_{2,\Omega} + \frac{2}{\ell' b_\#^{1/(\ell-1)}} \|h\|^{\ell'}_{\ell',\Sigma_T} \right) \left(\mathcal{Q}\exp\left[\mathcal{Q}\right] + 1\right), \tag{11}$$

with

$$\mathcal{Q} = \frac{2}{a_\#} S^{2(n+\theta q)/q}_{(2+\theta q)/(1+\theta q)} |\Omega|^\theta \|\mathbf{E}\|^2_{q,2,Q_T} + \\ + \left[(4/a_\#)^{q+n+\theta q} S^{2(n+\theta q)}_{(2+\theta q)/(1+\theta q)} |\Omega|^{q\theta}\right]^{(q-n-\theta q)^{-1}} \int_0^T \|\mathbf{E}\|^{2(1-n/q-\theta)^{-1}}_{q,\Omega} \mathrm{dt}.$$

Hereafter, S_p denotes the Sobolev constant of continuity under the standard $W^{1,p}(\Omega)$-norm ($1 \leq p < n$). Meanwhile, $S_{p,q}$ denotes the Sobolev constant of continuity under the $V_{p,q}$-norm. Other constants occur in $\mathcal{Q}$ if we use the inequality [22]:

$$\|v\|_{pn/(n-p),\Omega} \leq S_p \|\nabla v\|_{p,\Omega} + S_1^{1/p_*} \|v\|_{p_*,\partial\Omega}.$$

Notice that $V_{p,\ell} = W^{1,p}(\Omega)$ whenever the radiation exponent $\ell \leq p_* = p(n-1)/(n-p)$ if $n > p$.

Remark 2.2 *Observe that Theorem 2.1 remains true if* $\mathbf{E} \in L^r(0,T;\mathbf{L}^q(\Omega))$ *with* $2/r + n/q \leq 1$ *if* $n > 2$, *and* $2/r + 2/q < 1$ *if* $n = 2$.

Next we establish a maximum principle due to the Moser technique (see, for instance, [5]), with the upper bound being different from the one established in [7, Theorem 2.1], which depends on the data in an exponential form, which is a shortcoming for physical applications.

Theorem 2.2 *Under* $2/r + n/q < 1$, $h \in L^\infty(\Sigma_T)$ *being such that* $h \geq 0$ *on* Σ_T, *and* $u_0 \in L^\infty(\Omega)$ *being such that* $u_0 > 0$ *in* Ω, *any solution in accordance with Theorem 2.1 satisfies* $0 \leq u \leq \mathcal{M}$ *in* Q_T, *and* $0 \leq u \leq (\mathcal{M} + P_1)/b_\#$ *on* Σ_T, *if provided by the smallness condition* $P_2 \leq P$, *with* $\mathcal{M}$, P_1, P_2, *and* P *being explicitly given in Proposition 4.2.*

We state the following existence result (see, for instance, [10, Section 4.4], where the divergence-free $\mathbf{E}$ is taken into account). We emphasize that the estimate (16) is not so pleasant as we might expect.

Theorem 2.3 *Let* $u_0 \in L^1(\Omega)$, $f \in L^1(Q_T)$, $h \in L^1(\Sigma_T)$, *and* $\mathbf{E} \in L^r(0,T;\mathbf{L}^q(\Omega))$ *for*

$$1 < r\left(1 - \frac{n}{q}\right) < 2. \tag{12}$$

Under the assumptions (A)–(B) with $b_\# > 0$, *there exists a function* u *in* $L^{1,\infty}(Q_T) \cap L^p(0,T;V_{p,\ell-1}) \cap L^{\ell-1}(\Sigma_T)$ *such that* $\partial_t u \in L^1(0,T;[W^{1,p'}(\Omega)]')$, *with*

$$\frac{n}{q} + \frac{p(n+1)-n}{r} = 1, \tag{13}$$

satisfying the variational problem

$$\langle \partial_t u, v\rangle + \int_{Q_T} \mathsf{A}\nabla u \cdot \nabla v \mathrm{dxdt} + \int_{\Sigma_T} b(u) u v \mathrm{dsdt} =$$

$$= \int_{Q_T} u\mathbf{E} \cdot \nabla v \mathrm{dxdt} + \int_{Q_T} f v \mathrm{dxdt} + \int_{\Sigma_T} h v \mathrm{dsdt} \tag{14}$$

for every $v \in L^\infty(0,T;W^{1,p'}(\Omega))$. *For* $r(2-p) < 2np$, *we have*

$$\|u\|_{1,\infty,Q_T} + b_\# \|u\|_{\ell-1,\Sigma_T}^{\ell-1} \leq \|u_0\|_{1,\Omega} + \|f\|_{1,Q_T} + \|h\|_{1,\Sigma_T} := \mathcal{Z}; \tag{15}$$

$$\|\nabla u\|_{p,Q_T}^p \leq \mathcal{B} + \frac{rn^2\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n} + 1\right)}{a_\#(n+2-p(n+1))(n-1)} \left(\frac{b^\#}{b_\#}\mathcal{Z}\right), \tag{16}$$

with

$$\mathcal{B} = r\left(T|\Omega| + (Z_2)^p \mathcal{Z}^{p(n+1)/n}\right) +$$

$$+\frac{rn^2}{a_\#(n+2-p(n+1))(n-1)}\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n}+1\right)(T|\Omega|+2\mathcal{Z}) +$$

$$+\frac{r}{2(a_\#)^2}\|\mathbf{E}\|^2_{q,r,Q_T}(Z_2)^{p(r-2)/r}\left((Z_1)^{2-p}\mathcal{Z}^p + \mathcal{Z}^{[p(n+1)-2]/n}\right) +$$

$$+\frac{2^{(r-2)/2}}{(a_\#)^r}\|\mathbf{E}\|^r_{q,r,Q_T}\left((Z_1)^{r-p}\mathcal{Z}^p + (Z_1)^{p(r-2)/2}\mathcal{Z}^{p-r(2-p)/(2n)}\right),$$

where $Z_1 = S_p(1+|\Omega|^{1/n}S_1)$ *and* $Z_2 = S_pS_1T^{1/p}|\Omega|^{1/p+1/n-1}$.

Observe that (12)–(13) mean $1 < p < (n+2)/(n+1)$.

Under similar proofs, we state the corresponding results of Theorems 2.1 and 2.3 under the assumption (7) with $b_\# = 0$. In the following, K_p denotes the constant of continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\Gamma)$ ($p < n$).

Theorem 2.4 *If the conditions of Theorem 2.1 are fulfilled under* $h \in L^{(2_*)',2}(\Sigma_T)$, *where* $2_* = 2(n-1)/(n-2)$ *if* $n > 2$ *and* 2_* *represents any real number greater than* 2, *and the assumption (7) with* $b_\# = 0$, *and* $2 \leq \ell \leq 3$, *then the variational problem (9) admits at least one solution* $u \in L^{2,\infty}(Q_T) \cap W^{1,2}(0,T;H^1(\Omega);[H^1(\Omega)]')$ *satisfying the following estimates:*

$$\|u\|_{2,\infty,Q_T} \leq \mathcal{A}\sqrt{\exp\left[\mathcal{Q}+T\right]}; \tag{17}$$

$$a_\#\|\nabla u\|^2_{2,Q_T} \leq 2\mathcal{A}^2\left((\mathcal{Q}+T)\exp\left[\mathcal{Q}+T\right]+1\right), \tag{18}$$

with $\mathcal{Q}$ *according to Theorem 2.1, and*

$$\mathcal{A}^2 = \|u_0\|^2_{2,\Omega} + (2/a_\# + 1)(K_s)^2|\Omega|^{2/s-1}\|h\|^2_{(2_*)',2,\Sigma_T},$$

where $s = 2$ *if* $n > 2$ *and* $s = 22_*/(2_*+1)$ *if* $n = 2$.

Theorem 2.5 *If the conditions of Theorem 2.3 are fulfilled under the assumption (7) with* $b_\# = 0$, *then the variational problem (14) admits at least one solution* $u \in L^{1,\infty}(Q_T) \cap L^p(0,T;W^{1,p}(\Omega))$, *under (13) and* $2 \leq \ell \leq p+1$, *such that* $\partial_t u \in L^1(0,T;[W^{1,p'}(\Omega)]')$, *satisfying the following estimates:*

$$\|u\|_{1,\infty,Q_T} \leq \|u_0\|_{1,\Omega} + \|f\|_{1,Q_T} + \|h\|_{1,\Sigma_T} := \mathcal{Z}; \tag{19}$$

$$\|\nabla u\|^p_{p,Q_T} \leq \alpha_\ell +$$

$$+\beta_\ell\left(\mathcal{B} + 2^{\ell-2}\beta\left((S_1^{\frac{n(p-1)}{n-p(n-1)}}|\Omega|^{\frac{n(p-1)^2}{(n-p(n-1))p}} + S_1^{n(1-1/p)})T\mathcal{Z}\right)^{\ell-1}\right), \tag{20}$$

with

$$\begin{cases} \alpha_\ell = 0,\ \beta_\ell = (1-2^{2p-1}\beta)^{-1} & \text{if } \ell = p+1 \text{ and } \beta < 2^{1-2p} \\ \alpha_\ell = (2^{2\ell-3}\beta)^{p/(p-\ell+1)},\ \beta_\ell = p/(p-\ell+1) & \text{if } \ell < p+1 \end{cases}$$

$$\beta = b^\# T^{1-(\ell-1)/p}|\Gamma|^{1-(\ell-1)/p_*}K_p^{\ell-1}\frac{rn^2\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n}+1\right)}{a_\#(n+2-p(n+1))(n-1)}.$$

Remark 2.3 *The Neumann problem, i.e.* $b_\# = b^\# = 0$*, clearly verifies the smallness condition* $\beta = 0$*. Then (20) is satisfied under* $\alpha_\ell = 0$*,* $\beta_\ell = 1$*, and* $\ell = p+1$*.*

Finally, we restrict to the minimum principle (cf. Proposition 4.1). The explicit upper bound correspondent to $\mathcal{M}$ in Theorem 2.2 is not straightforward, remaining an open problem.

Theorem 2.6 *If* $h \in L^2(\Sigma_T)$ *is such that* $h \geq 0$ *on* Σ_T*, and* $u_0 \in L^2(\Omega)$ *is such that* $u_0 > 0$ *in* Ω*, then any solution in accordance with Theorem 2.4 is nonnegative in* Q_T*, and its trace is nonnegative on* Σ_T*.*

3. Proof of Theorem 2.1

The proof of existence is divided into three canonical steps: existence of approximate solutions (regularization), derivation of uniform estimates, and passage to the limit.

For each $m \in \mathbb{N}$, if we consider the truncating function

$$T_m(s) = \min\{m, \max\{-m, s\}\} \quad \text{for } s \in \mathbb{R}, \tag{21}$$

then there exists at least a weak solution u_m of

$$\int_0^T \langle \partial_t u_m, v\rangle \mathrm{dt} + \int_{Q_T} (\mathsf{A}\nabla u_m + T_m(u_m)\mathbf{E})\cdot\nabla v \mathrm{dxdt} + \\ + \int_{\Sigma_T} b(u_m)u_m v\mathrm{dsdt} = \int_{\Sigma_T} hv\mathrm{dsdt}, \qquad \forall v \in L^2(0,T;V_{2,\ell}), \tag{22}$$

which belongs to $L^{2,\infty}(Q_T)\cap L^2(0,T;V_{2,\ell})$ such that $\partial_t u_m \in L^2(0,T;(V_{2,\ell})')$. The existence is true due to the Faedo–Galerkin method [27, Theorem 4.1, p. 120].

In order to pass to the limit as m tends to infinity, we seek for estimates independent on m.

3.1. Proof of the estimates (10)–(11) for u_m

Let us take $v = \chi_{]0,\tau[}u_m$ as a test function in (22), where $\chi_{]0,\tau[}$ is the characteristic function of the open interval $]0,\tau[$, with τ being a fixed number lesser than T. Applying the assumptions (5) and (7) with $b_\# > 0$, it follows that

$$\frac{1}{2}\int_\Omega |u_m|^2(\tau)\mathrm{dx} + a_\#\int_{Q_\tau}|\nabla u_m|^2\mathrm{dxdt} + b_\#\int_{\Sigma_\tau}|u_m|^\ell \mathrm{dsdt} \leq \\ \leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + \int_0^\tau \|u_m\|_{2q/(q-2),\Omega}\|\mathbf{E}\|_{q,\Omega}\|\nabla u_m\|_{2,\Omega}\mathrm{dt} + \|h\|_{\ell',\Sigma_\tau}\|u_m\|_{\ell,\Sigma_\tau} \leq \\ \leq \frac{1}{2}\|u_0\|^2_{2,\Omega} + I + \frac{1}{\ell' b_\#^{1/(\ell-1)}}\|h\|^{\ell'}_{\ell',\Sigma_\tau} + \frac{b_\#}{\ell}\|u_m\|^\ell_{\ell,\Sigma_\tau}, \tag{23}$$

by considering the property $|T_m(u)| \leq |u|$, and the Hölder and Young inequalities.

For any $0 < \lambda < 1$ such that

$$\lambda = \frac{2s}{(s-2)q} \quad \left(\Leftrightarrow \frac{q-2}{2q} = \frac{\lambda}{s} + \frac{1-\lambda}{2}\right),$$

the Hölder inequality yields

$$\|u_m\|_{2q/(q-2),\Omega} \le \|u_m\|_{s,\Omega}^{\lambda} \|u_m\|_{2,\Omega}^{1-\lambda}. \tag{24}$$

Taking $\lambda = n/q + \theta$ with $\theta = 0$ if $n > 2$ and any $0 < \theta < 1 - 2/q$ if $n = 2$ to uniform Sobolev constants for dimensions $n > 2$ and $n = 2$, we use the Sobolev embedding $W^{1,ns/(s+n)}(\Omega) \hookrightarrow L^s(\Omega)$ followed by the Hölder inequality

$$\|u_m\|_{s,\Omega} \le S_{ns/(s+n)} |\Omega|^{1/s+1/n-1/2} \left(\|\nabla u_m\|_{2,\Omega} + \|u_m\|_{2,\Omega}\right), \tag{25}$$

where $s = 2(n + \theta q)(n - 2 + \theta q)^{-1}$. Gathering (24) and (25) we deduce

$$I \le \frac{a_\#}{2} \|\nabla u_m\|_{2,Q_\tau}^2 + \int_0^\tau \|u_m\|_{2,\Omega}^2 \left(\frac{1}{a_\#} [S_{ns/(s+n)} |\Omega|^{1/s+1/n-1/2}]^{2\lambda} \|\mathbf{E}\|_{q,\Omega}^2 + \right.$$

$$\left. + \frac{1-\lambda}{2} \left(\frac{2(\lambda+1)}{a_\#} \right)^{\frac{\lambda+1}{1-\lambda}} [S_{ns/(s+n)} |\Omega|^{1/s+1/n-1/2}]^{2\lambda/(1-\lambda)} \|\mathbf{E}\|_{q,\Omega}^{2/(1-\lambda)} \right) \mathrm{dt}.$$

Introducing the above inequality in (23) we find

$$\|u_m\|_{2,\Omega}^2(\tau) + a_\# \|\nabla u_m\|_{2,Q_\tau}^2 + b_\# \|u_m\|_{\ell,\Sigma_\tau}^{\ell} \le \|u_0\|_{2,\Omega}^2 + \frac{2}{\ell' b_\#^{1/(\ell-1)}} \|h\|_{\ell',\Sigma_\tau}^{\ell'} +$$

$$+ \int_0^\tau \|u_m\|_{2,\Omega}^2 \left(\frac{2}{a_\#} S_{2n(n+\theta q)/(n^2+(n+2)\theta q)}^{2(n/q+\theta)} |\Omega|^{2\theta/n} \|\mathbf{E}\|_{q,\Omega}^2 + \right.$$

$$\left. + [(4/a_\#)^{1+n/q+\theta} S_{\frac{2n(n+\theta q)}{n^2+(n+2)\theta q}}^{2(n/q+\theta)} |\Omega|^{2\theta/n}]^{q(q-n-\theta q)^{-1}} \|\mathbf{E}\|_{q,\Omega}^{2(1-n/q-\theta)^{-1}} \right) \mathrm{dt}.$$

By applying the Gronwall inequality, we conclude (10) for u_m, and consequently (11).

3.2. Passage to the limit in (22) as $m \to \infty$

According to Section 3.1 we may extract a subsequence of $\{u_m\}$ still denoted by $\{u_m\}$ such that $u_m \rightharpoonup u$ in $L^2(0,T;V_{2,\ell})$, and $u_m \rightharpoonup u$ *-weakly in $L^\infty(0,T;L^2(\Omega))$. In particular, $u_m(T) \rightharpoonup z$ in $L^2(\Omega)$, and using (7) there exists a positive constant C such that

$$\|b(u_m)u_m\|_{\ell/(\ell-1),\Sigma_T} \le \|u_m\|_{\ell,\Sigma_T}^{\ell-1} \le C.$$

Thus, at least a subsequence $b(u_m)u_m$ weakly converges to w in $L^{\ell/(\ell-1)}(\Sigma_T)$.

Let us pass to the limit in (22) by (4) (see [27, Theorem 4.1, p. 120]). Indeed, by (4) we have the integration per parts formula for all $\psi \in C^\infty([0,T])$ and $v \in V_{2,\ell}$,

$$(u_m(T), \psi(T)v) + (u_0, \psi(0)v) = \int_0^T \langle \partial_t u_m(t), \psi(t)v \rangle + \langle \psi'(t)v, u_m(t) \rangle \mathrm{dt} =$$

$$= -\int_{Q_T} \psi(t)(\mathsf{A}\nabla u_m(t) + T_m(u_m(t))\mathbf{E}) \cdot \nabla v \mathrm{dxdt} +$$

$$+ \int_{\Sigma_T} \psi(t)(h(t) - b(u_m(t))u_m(t)) v \mathrm{dsdt} + \int_0^T \psi'(t)(v, u_m(t))\mathrm{dt},$$

where $(\cdot,\cdot)$ stands for the inner product of $L^2(\Omega)$.

Passing to the limit as $m\to\infty$ in the above equality, we see that the triple (z,u,w) satisfies

$$(z,\psi(T)v)+(u_0,\psi(0)v)=-\int_{Q_T}\psi(t)(\mathsf{A}\nabla u(t)+u(t)\mathbf{E})\cdot\nabla v\mathrm{dxdt}+$$
$$+\int_{\Sigma_T}\psi(t)(h(t)-w(t))v\mathrm{dsdt}+\int_0^T\psi'(t)(v,u(t))\mathrm{dt}.$$

If $\psi(T)=\psi(0)=0$, we find

$$\langle\partial_t u,v\rangle=-\int_\Omega(\mathsf{A}\nabla u+u\mathbf{E})\cdot\nabla v\mathrm{dx}+\int_\Gamma(h-w)v\mathrm{ds},\quad \text{a.e. in }]0,T[. \tag{26}$$

If $\psi(T)=1$ and $\psi(0)=0$, $u(T)=z$ (for details see [30, Lemma 30.5, pp. 776-777]).

It remains to prove that $w=b(u)u$. Observe that the weak convergences are not sufficient to that, since the argument of the Minty trick fails, in spite of the coercivity (5) of A and the monotonicity property (7) of b, because of the existence of the term $T_m(u_m)\mathbf{E}\cdot\nabla v$. In order to apply the Aubin–Lions lemma, let us estimate $\partial_t u_m$ in $L^1(0,T;(V_{2,\ell})')$.

For every $v\in V_{2,\ell}$, and for almost all $t\in]0,T[$, we have

$$|\langle\partial_t u_m(t),v\rangle|\le a^\#\|\nabla u_m(t)\|_{2,\Omega}\|\nabla v\|_{2,\Omega}+$$
$$+(b^\#\|u_m(t)\|^{\ell-1}_{\ell,\Gamma}+\|h(t)\|_{\ell',\Gamma})\|v\|_{\ell,\Gamma}+\|u_m(t)\|_{2q/(q-2),\Omega}\|\mathbf{E}(t)\|_{q,\Omega}\|\nabla v\|_{2,\Omega}.$$

Using (24)–(25), it follows that

$$\|\partial_t u_m(t)\|_{(V_{2,\ell})'}\le a^\#\|\nabla u_m(t)\|_{2,\Omega}+(b^\#\|u_m(t)\|^{\ell-1}_{\ell,\Gamma}+\|h(t)\|_{\ell',\Gamma})+$$
$$+S^{n/q+\theta}_{(2+\theta q)/(1+\theta q)}|\Omega|^{\theta/n}\left(\|\nabla u_m(t)\|^\lambda_{2,\Omega}\|u_m(t)\|^{1-\lambda}_{2,\Omega}+\|u_m(t)\|_{2,\Omega}\right)\|\mathbf{E}(t)\|_{q,\Omega},$$

where $\lambda=n/q+\theta$ with $\theta=0$ if $n>2$ and any $0<\theta<1-2/q$ if $n=2$. Since the inclusion of the spaces $L^r(0,T)\subseteq L^2(0,T)\subseteq L^{\ell/(\ell-1)}(0,T)\subseteq L^1(0,T)$ holds, applying the Minkowski and Young inequalities we deduce

$$\|\partial_t u_m\|_{L^1(0,T;(V_{2,\ell})')}\le T^{1/2}(a^\#+S_{(2+\theta q)/(1+\theta q)})\|\nabla u_m\|_{2,Q_T}+$$
$$+T^{1/\ell}b^\#(\|u_m\|^{\ell-1}_{\ell,\Sigma_T}+\|h\|_{\ell',\Sigma_T})+$$
$$+T^{1-1/r}\left(|\Omega|^{\frac{\theta}{n(1-\lambda)}}T^{-\lambda}+S^{n/q+\theta}_{(2+\theta q)/(1+\theta q)}|\Omega|^{\theta/n}\right)\|\mathbf{E}\|_{q,r,Q_T}\|u_m\|_{2,\infty,Q_T}.$$

The sequence on the right-hand side of this last relation is uniformly bounded due to the estimates (10)–(11). By the Aubin–Lions lemma, $\{u_m\}$ is relatively compact in $L^{q,\ell/(\ell-1)}(Q_T)$ for any $q<2n/(n-2)$, and $L^{q,\ell/(\ell-1)}(\Sigma_T)$ for any $q<2(n-1)/(n-2)$.

Passing to the limit as $m\to\infty$ in (22), we see that the function u satisfies (9).

4. Minimum and maximum principles

The objective of this section is the proof of Theorem 2.2 by making recourse of the minimum and maximum principles. It will be consequence of Propositions 4.1 and 4.2.

Proposition 4.1 (Minimum principle) *Let u solve (9). Under $b_\# \geq 0$ in (7), $h \geq 0$ on Σ_T, and $u_0 \geq 0$ in Ω, we have that $u \geq 0$ in Q_T as well as its trace on Σ_T.*

Proof The classical choice of $v = u^- = \min\{u, 0\}$ as a test function in (22) implies that for almost all values τ in $]0, T[$

$$\int_\Omega (u^-)^2(\tau)\mathrm{dx} + \frac{a_\#}{2}\int_{Q_\tau} |\nabla u^-|^2 \mathrm{dxdt} \leq \frac{1}{2a_\#}\int_{Q_\tau} (u^-)^2 |\mathbf{E}|^2 \mathrm{dxdt}, \tag{27}$$

by considering the assumptions (5) and (7) with $b_\# \geq 0$, and the Young inequality. Therefore, by applying the Gronwall inequality, we conclude that $u^- = 0$ in Q_T.

Introducing this fact in (27), and letting $\tau \to T$, it follows that the trace function $u \geq 0$ on Σ_T. □

In order to state our maximum principle, we begin by establishing some preliminary results. The first one deals with the well-known interpolation result, which is a direct consequence of the Hölder inequality.

Lemma 4.1 *If $w \in L^{q,q_1}(Q_T) \cap L^{r,r_1}(Q_T)$, then $w \in L^{p,p_1}(Q_T)$, where*

$$\frac{1}{p} = \frac{\lambda}{q} + \frac{1-\lambda}{r}, \qquad \frac{1}{p_1} = \frac{\lambda}{q_1} + \frac{1-\lambda}{r_1}, \qquad (\lambda \geq 1).$$

Moreover,

$$\|w\|_{p,p_1,Q_T} \leq \|w\|^{\lambda}_{q,q_1,Q_T} \|w\|^{1-\lambda}_{r,r_1,Q_T}.$$

We improve in Lemma 4.2 (see also Remark 4.1) a result established in [5].

Lemma 4.2 *If $w \in L^{2,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega))$, then $w \in L^{\sigma 2q/(q-2),\sigma 2r/(r-2)}(Q_T)$ for all $q, r > 2$ and $1 - 2/q < \sigma \leq 1 + 2(1 - n/q - 2/r)/n$ that satisfy*

$$\sigma \leq n(q-2)/[q(n-2)] \quad (q > 2), \qquad (\sigma > 2/q - 2/r \textit{ if } n = 2).$$

Moreover,

$$\|w\|^2_{\sigma 2q/(q-2),\sigma 2r/(r-2),Q_T} \leq T^{\nu(\sigma)} \left(\|w\|^2_{2,\infty,Q_T} + C_n(\sigma)\left(\|\nabla w\|^2_{2,Q_T} + \|w\|^2_{2,\Sigma_T}\right)\right),$$

where

$$\textit{if } n > 2 \quad \begin{cases} \nu(\sigma) = (1 - n/q - 2/r)/\sigma + n(1/\sigma - 1)/2 \\ C_n(\sigma) = 2(S_{2,2})^2 \end{cases}$$

$$\textit{if } n = 2 \quad \begin{cases} \nu(\sigma) = (1 - 1/q - 1/r)/\sigma - 1/2 \\ C_2(\sigma) = 2\left(S_{2s/(s+2),2s/(s+2)}\right)^2 \left(|\Omega|^{1/s} + |\Gamma|^{1/s}\right) \end{cases}$$

with $s = \nu^{-1}(2\sigma^{-1}(1/q - 1/r) + 1)$.

Proof For any $\sigma > 1 - 2/q$ and $s > \sigma 2q/(q-2)$ such that $\sigma 2r/(r-2) \leq 2/\lambda$, applying successively the Hölder inequality, Lemma 4.1, and the Young inequality, we find

$$\|w\|^2_{\sigma 2q/(q-2),\sigma 2r/(r-2),Q_T} \leq T^{\nu}\|w\|^2_{\sigma 2q/(q-2),2/\lambda,Q_T} \leq T^{\nu}\|w\|^{2\lambda}_{s,2,Q_T}\|w\|^{2(1-\lambda)}_{2,\infty,Q_T} \leq$$
$$\leq T^{\nu}\left(\lambda\|w\|^2_{s,2,Q_T} + (1-\lambda)\|w\|^2_{2,\infty,Q_T}\right),$$

with

$$\lambda = \frac{2s}{s-2}\left[\frac{1}{2} - \frac{1}{\sigma}\left(\frac{1}{2} - \frac{1}{q}\right)\right] \quad \text{and} \quad \nu = \frac{1}{\sigma}\left(1 - \frac{2}{r}\right) - \lambda.$$

If $n > 2$, we choose $s = 2^* = 2n/(n-2)$ and then the Sobolev embedding can be applied, concluding the desired result.

If $n = 2$, we choose $s = 2 + 4\nu^{-1}\left[1/2 - (1/2 - 1/q)/\sigma\right] > 2$, which implies that $\lambda = \nu + 1 - (1 - 2/q)^\sigma$. Therefore, we use the Sobolev embedding $W^{1,2s/(s+2)}(\Omega) \hookrightarrow L^s(\Omega)$ followed by the Hölder inequality in order to determine the constant $C_2(\sigma)$, namely,

$$\|w\|_{s,\Omega}^2 \leq (S_{2s/(s+2),2s/(s+2)})^2 \left(|\Omega|^{1/s}\|\nabla w\|_{2,\Omega} + |\Gamma|^{1/s}\|w\|_{2,\Gamma}\right)^2,$$

finishing then the proof of Lemma 4.2. □

Remark 4.1 *The existence of* σ *satisfying* $1 - 2/q < \sigma \leq 1 + 2(1 - n/q - 2/r)/n$ *is given by* $r > 2$*, while* $\min\{n(q-2)/[q(n-2)], 1 + 2(1 - n/q - 2/r)/n\} = 1 + 2(1 - n/q - 2/r)/n$ *if and only if*

$$\frac{n}{q} + \frac{2-n}{r} \leq 1.$$

For $n = 2$*, the existence of* σ *satisfying* $\max\{2/q - 2/r, 1 - 2/q\} \leq 1 < \sigma \leq 2(1 - 1/q - 1/r)$ *is guaranteed by* $q, r > 2$*.*

Set

$$\|w\|_{\infty,Q_T} = \lim_{N\to\infty} \|w\|_{p\chi^N, q\chi^N, Q_T}, \quad \forall w \in \cap_{1\leq p,q<\infty} L^{p,q}(Q_T), \tag{28}$$

where $p, q \geq 1$, and $\chi > 1$.

Next we improve the technical result, which involves an additional term.

Lemma 4.3 *Let* $p, q > 1$*, and* $\chi < 1$*. If* $w \in \cap_{1\leq p,q<\infty} L^{p,q}(Q_T)$ *verifies*

$$\|w\|_{p(1/\chi)^{m+1}, q(1/\chi)^{m+1}, Q_T} \leq (P\chi^{-2m})^{\chi^m} \|w\|_{p(1/\chi)^m, q(1/\chi)^m, Q_T} + P_1 P_2^{\chi^m}, \tag{29}$$

for some constants $P \geq 1$*,* $P_1 \geq 0$*, and* $0 < P_2 \leq P$*, and for any* $m \in \mathbb{N}_0$*, then*

$$\operatorname*{ess\,sup}_{Q_T} |w| \leq P^{1/(1-\chi)} \chi^{-\chi(1-\chi)^{-2}} \|w\|_{p,q,Q_T} + P_1 \sum_{i\geq 0} P^{\frac{\chi^{i+1}}{1-\chi}} \chi^{\frac{i\chi^{i+2} - (i+1)\chi^{i+1}}{(1-\chi)^2}} P_2^{\chi^i}. \tag{30}$$

Proof By induction, we have for all $N \in \mathbb{N}$

$$\|w\|_{p\chi^{-N}, q\chi^{-N}, Q_T} \leq P^{a_{0,N}} \chi^{-b_{0,N}} \|w\|_{p,q,Q_T} + P_1 \sum_{i=0}^{N-1} P^{a_{i+1,N}} \chi^{-b_{i+1,N}} P_2^{\chi^i},$$

where

$$a_{j_0,N} = \sum_{j=j_0}^{N-1} \chi^j = \frac{\chi^{j_0} - \chi^N}{1-\chi};$$

$$b_{j_0,N} = \sum_{j=j_0}^{N-1} j\chi^j = \frac{j_0\chi^{j_0} + (1-j_0)\chi^{j_0+1} - N\chi^N + (N-1)\chi^{N+1}}{(1-\chi)^2}.$$

Using d'Alembert's ratio criterium, the second series in (30) is convergent if there holds $P_2^{\chi}\chi^{(i+1)\chi^{i+1}} < P^{\chi^{i+1}}$. Indeed, this inequality is true for all $i\in\mathbb{N}_0$, for $\chi\leq 1$, $0<P_2\leq P$, and $P\geq 1$.

Letting $N\to\infty$, we find (30) by the definition (28). □

Finally, we are in position to establish the upper bound of any solution of (22), if $2/r+n/q<1$.

Proposition 4.2 (Maximum principle) *Let u solve (9). Under $b_\#>0$ in (7), and $2/r+n/q<1$, we have*

$$u \leq P^{\sigma/(\sigma-1)}\sigma^{\sigma(\sigma-1)^{-2}}T^{\nu(1)/2}\left(\|u\|^2_{2,\infty,Q_T}+C_n(1)\left(\|\nabla u\|^2_{2,Q_T}+\|u\|^2_{2,\Sigma_T}\right)\right)^{1/2}$$

$$+P_1\sum_{i\geq 0}P^{\sigma^{-i}/(\sigma-1)}\sigma^{\left((i+1)\sigma^{-i+1}-i\sigma^{-i}\right)(\sigma-1)^{-2}}P_2^{\sigma^{-i}} := \mathcal{M} \quad \text{in } Q_T; \tag{31}$$

$$u\leq \frac{1}{b_\#}\left(\mathcal{M}+P_1\right) \qquad \text{on } \Sigma_T, \tag{32}$$

if provided by the smallness condition $P_2\leq P$, with

$$P=\left(\frac{2T^{\nu(\sigma)}\max\{1,C_n(\sigma)\}}{a_\#\min\{1,a_\#,b_\#\}}\right)^{1/2}\|\mathbf{E}\|_{q,r,Q_T}\geq 1;$$

$$P_1=\left(\max\{1,\|u_0\|_{\infty,\Omega},\|h\|_{\infty,\Sigma_T}\}\right)^{1/2};$$

$$P_2=\left(\frac{T^{\nu(\sigma)}\max\{1,C_n(\sigma)\}}{\min\{1,a_\#,b_\#\}}\left(|\Omega|+(b_\#(\ell-2)+\max\{1,1/b_\#\})|\Sigma_T|\right)\right)^{1/2};$$

$$\sigma=1+\frac{2}{n}\left(1-\frac{2}{r}-\frac{n}{q}\right),$$

where ν and C_n are introduced in Lemma 4.2.

Proof Set $\theta=1-2/r-n/q>0$. Arguing as in [5], the first step involves showing that for almost all values τ in $]0,T[$

$$\frac{1}{\beta+1}\int_\Omega (u^+)^{\beta+1}(\tau)\mathrm{dx}+\frac{a_\#\beta}{2}\int_{Q_\tau}(u^+)^{\beta-1}|\nabla u^+|^2\mathrm{dxdt}+$$

$$+\frac{b_\#}{\beta+1}\int_{\Sigma_\tau}(u^+)^{\beta+1}\mathrm{dsdt}\leq\frac{\ell-2}{\beta+\ell-1}b_\#|\Sigma_\tau|+\frac{1}{\beta+1}\int_\Omega(u^+)^{\beta+1}(0)\mathrm{dx}+$$

$$+\frac{\beta}{2a_\#}\int_{Q_\tau}(u^+)^{\beta+1}|\mathbf{E}|^2\mathrm{dxdt}+\frac{|\Sigma_\tau|}{(\beta+1)(b_\#)^\beta}\|h\|^{\beta+1}_{\infty,\Sigma_\tau}, \tag{33}$$

where $\beta\geq 1$, and $u^+=\max\{u,0\}$. Let us take $v=\chi_{]0,\tau[}\mathcal{G}(u)$ as a test function in (22), where $\chi_{]0,\tau[}$ is the characteristic function of the open interval $]0,\tau[$, with τ being a fixed number lesser than T, and

$$\mathcal{G}(u)=\begin{cases}(u^+)^\beta & \text{for } -\infty<u\leq M\\ M^{\beta-1}u & \text{for } M\leq u<+\infty\end{cases}.$$

Applying (5), it follows that

$$\int_{Q_\tau} \partial_t[\mathcal{H}(u)]\mathrm{dxdt} + \frac{a_\#}{2}\int_{Q_\tau} \mathcal{G}'(u)|\nabla u^+|^2\mathrm{dxdt} + \int_{\Sigma_\tau} b(u)u\mathcal{G}(u)\mathrm{dsdt} \leq$$
$$\leq \frac{1}{2a_\#}\int_{Q_\tau} |u^+\mathbf{E}|^2\mathcal{G}'(u)\mathrm{dxdt} + \int_{\Sigma_\tau} h(u^+)^\beta\mathrm{dsdt},$$

with $\mathcal{H}'(u) = \mathcal{G}(u)$, and considering Remark 2.1. As the last boundary integral in the above inequality is new, we analyze it separately. Applying the Hölder and Young inequalities, we deduce

$$\int_{\Sigma_\tau} h(u^+)^\beta\mathrm{dsdt} \leq \frac{|\Sigma_\tau|}{(\beta+1)(b_\#)^\beta}\|h\|^{\beta+1}_{\infty,\Sigma_\tau} + \frac{b_\#\beta}{\beta+1}\int_{\Sigma_\tau}(u^+)^{\beta+1}\mathrm{dsdt}.$$

Letting the parameter M tend to infinity (since $\mathcal{G}'(u) \leq \beta(u^+)^{\beta-1}$), and applying (7) with $b_\# > 0$, we compute the boundary integral on the left-hand side as follows

$$\int_{\Sigma_\tau}(u^+)^{\beta+1}\mathrm{dsdt} \leq \frac{\ell-2}{\beta+\ell-1}|\Sigma_\tau| + \int_{\Sigma_\tau}(u^+)^{\beta+\ell-1}\mathrm{dsdt},$$

finding (33).

The second step involves showing that (33) implies

$$\|w^\sigma\|^{2/\sigma}_{2q/(q-2),2r/(r-2),Q_T} \leq \frac{T^\nu\max\{1,C_n\}}{\min\{1,a_\#,b_\#\}}\Big(|\Omega|\|u_0\|^{\beta+1}_{\infty,\Omega}+$$
$$+\frac{(\beta+1)^2}{2a_\#}\|\mathbf{E}\|^2_{q,r,Q_T}\|w\|^2_{\frac{2q}{q-2},\frac{2r}{r-2},Q_T} + (\ell-2)b_\#|\Sigma_T| + \frac{|\Sigma_T|}{(b_\#)^\beta}\|h\|^{\beta+1}_{\infty,\Sigma_T}\Big), \tag{34}$$

with $w = (u^+)^{(\beta+1)/2}$.

Multiplying (33) by $\beta+1$ we have

$$\|w\|^2_{2,\infty,Q_T} + a_\#\|\nabla w\|^2_{2,Q_T} + b_\#\|w\|^2_{2,\Sigma_T} \leq |\Omega|\|u_0\|^{\beta+1}_{\infty,\Omega}+$$
$$+\frac{(\beta+1)^2}{2a_\#}\|\mathbf{E}\|^2_{q,r,Q_T}\|w\|^2_{\frac{2q}{q-2},\frac{2r}{r-2},Q_T} + (\ell-2)b_\#|\Sigma_T| + \frac{|\Sigma_T|}{(b_\#)^\beta}\|h\|^{\beta+1}_{\infty,\Sigma_T}. \tag{35}$$

On the other hand, by taking $\sigma = 1+2\theta/n$ $(n \geq 2)$, that is $\nu = 0$, Lemma 4.2 guarantees that (34) holds.

Next, returning to u^+, (34) becomes

$$\varphi_{N+1} = \|u^+\|_{\sigma^{N+1}2q/(q-2),\sigma^{N+1}2r/(r-2),Q_T} \leq (P\sigma^N)^{\sigma^{-N}}\varphi_N + P_1P_2^{\sigma^{-N}},$$

with $N\in\mathbb{N}$ and $(\beta+1)/2 = \sigma^N$ standing for the iterative argument (cf. Lemma 4.3). Therefore, we conclude (31) making recourse of Lemma 4.2 with $\sigma = 1$.

Finally, introducing the upper bound $\mathcal{M}$ in (35) we find

$$b_\#\|u^+\|_{\sigma^N,\Sigma_T} \leq \sigma^{N\sigma^{-N}}\left(\frac{|\Sigma_T|}{2a_\#}\|\mathbf{E}\|^2_{q,r,Q_T}\right)^{\sigma^{-N}}\mathcal{M} + P_1P_2^{\sigma^{-N}}.$$

Applying directly the definition (28) we conclude (32). □

5. Proof of Theorem 2.3

Let us reformulate Lemma 4.2 under exponents p being lesser than or equal to $n/(n-1)$.

Lemma 5.1 *If $w \in L^{1,\infty}(Q_T) \cap L^{p_1}(0,T;W^{1,p}(\Omega))$, then $w \in L^{\bar{p},\bar{q}}(Q_T)$ for all $1 \le p \le n/(n-1)$, $1 \le p_1 < \bar{q}$ and*

$$\frac{1}{\bar{p}} + \frac{p_1}{\bar{q}}\left(1 + \frac{1}{n} - \frac{1}{p}\right) = 1. \tag{36}$$

For $\lambda = p_1/\bar{q} < 1$ we have

$$\|w\|_{\bar{p},\bar{q},Q_T} \le \left(S_p(1 + |\Omega|^{1/n}S_1)\right)^{\lambda} \|\nabla w\|^{\lambda}_{p,p_1,Q_T}\|w\|^{1-\lambda}_{1,\infty,Q_T} +$$
$$+ (S_pS_1)^{\lambda} T^{1/\bar{q}} |\Omega|^{\lambda(1/p+1/n-1)} \|w\|_{1,\infty,Q_T}. \tag{37}$$

Proof Let us begin by establishing the following correlation between the Sobolev constants, for all $1 \le p \le n/(n-1)$,

$$\|w\|_{pn/(n-p),\Omega} \le S_p\left(\|\nabla w\|_{p,\Omega} + |\Omega|^{1/p-(n-1)/n}\|w\|_{n/(n-1),\Omega}\right) \le$$
$$\le S_p\left((1 + |\Omega|^{1/n}S_1)\|\nabla w\|_{p,\Omega} + |\Omega|^{1/p-(n-1)/n}S_1\|w\|_{1,\Omega}\right).$$

Thanks to Lemma 4.1 with the above inequality we conclude (37). □

As in the proof of Theorem 2.1, let us first take the existence of approximate solutions u_m in $L^{2,\infty}(Q_T) \cap L^2(0,T;V_{2,\ell})$ such that $\partial_t u_m \in L^2(0,T;(V_{2,\ell})')$, for each $m \in \mathbb{N}$, of the variational problem

$$\int_0^T \langle \partial_t u_m, v\rangle \mathrm{dt} + \int_{Q_T} (\mathsf{A}\nabla u_m + T_m(u_m)\mathbf{E}) \cdot \nabla v \mathrm{dxdt} +$$
$$+ \int_{\Sigma_T} b(u_m)u_m v \mathrm{dsdt} = \int_{Q_T} \frac{mf}{m + |f|} v \mathrm{dxdt} + \int_{\Sigma_T} hv \mathrm{dsdt}, \tag{38}$$

for all $v \in L^2(0,T;V_{2,\ell})$.

Next we deal with the derivation of uniform estimates, and the passage to the limit in (38).

5.1. $L^{1,\infty}(Q_T)$- and $L^{\ell-1}(\Sigma_T)$- estimates (15) for u_m

Let $\varepsilon \in]0,m[$ be arbitrary. Choosing $v = \chi_{]0,\tau[}T_1(u_m/\varepsilon)$ as a test function in (22), where $\chi_{]0,\tau[}$ is the characteristic function of the open interval $]0,\tau[$, with τ being a fixed number lesser than T, and T_1 being the truncating function (21) with $m = 1$, we obtain

$$\int_0^\tau \frac{d}{dt}\int_\Omega \left[\int_0^{u_m} T_1(z/\varepsilon)\mathrm{dz}\right] \mathrm{dxdt} + b_\# \int_{\Sigma_\tau[|u_m|>\varepsilon]} |u_m|^{\ell-1} \mathrm{dsdt} \le$$
$$\le \int_{Q_\tau[|u_m|<\varepsilon]} |\mathbf{E}||\nabla u_m| \mathrm{dxdt} + \int_{Q_\tau} |f| \mathrm{dxdt} + \int_{\Sigma_\tau} |h| \mathrm{dsdt},$$

taking (5) and (7) into account. Passing to the limit as ε tends to zero, (15) holds.

5.2. L^p-estimate (16) to the gradient of u_m

Thanks to the estimate [10, Lemma 4.4.5]

$$\|\nabla u_m\|_{p,Q_T}^p \leq \left(\int_{Q_T} \frac{|\nabla u_m|^2}{(1+|u_m|)^{\delta+1}}\mathrm{dxdt}\right)^{p/2} \Big(|Q_T|^{n/[p(n+1)]}+ +\|u_m\|_{p(n+1)/n,Q_T}\Big)^{p(n+1)(2-p)/(2n)},$$

for any $1 < p < (n+2)/(n+1)$, and $\delta = (2-p)(n+1)/n - 1 \in]0,1[$, considering that by Lemma 5.1 with $\bar{p} = \bar{q} = p(n+1)/n$ and $p_1 = p < (n+2)/(n+1)$ implies

$$\|u_m\|_{p(n+1)/n,Q_T} \leq (Z_1)^{n/(n+1)}\|\nabla u_m\|_{p,Q_T}^{n/(n+1)}\mathcal{Z}^{1/(n+1)} + (Z_2)^{n/(n+1)}\mathcal{Z}, \tag{39}$$

we deduce

$$\frac{p}{2}\|\nabla u_m\|_{p,Q_T}^p \leq \frac{p}{2}(Z_1)^{2-p}\left(\int_{Q_T} \frac{|\nabla u_m|^2}{(1+|u_m|)^{\delta+1}}\mathrm{dxdt}\right)\mathcal{Z}^{(2-p)/n}+ + \left(\int_{Q_T} \frac{|\nabla u_m|^2}{(1+|u_m|)^{\delta+1}}\mathrm{dxdt}\right)^{p/2}\left(|Q_T|^{n/[p(n+1)]} + (Z_2)^{n/(n+1)}\mathcal{Z}\right)^{p(n+1)(2-p)/(2n)},$$

observing that the term on the LHS is rearranged by using $(a+b)^\varkappa \leq a^\varkappa + b^\varkappa$ with $\varkappa = p(n+1)(2-p)/(2n) < 1$, and the Young inequality $ab \leq pa^{2/p}/2 + (1-p/2)b^{2/(2-p)}$. Reusing the Young inequality, and using $(a+b)^\varkappa \leq 2(a^\varkappa + b^\varkappa)$ with $\varkappa = p(n+1)/n < 2$, we rewrite the above inequality as

$$\|\nabla u_m\|_{p,Q_T}^p \leq \left(\int_{Q_T} \frac{|\nabla u_m|^2}{(1+|u_m|)^{\delta+1}}\mathrm{dxdt}\right)\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n} + 1\right)+ + \frac{2(2-p)}{p}\left(|Q_T| + (Z_2)^p\mathcal{Z}^{p(n+1)/n}\right). \tag{40}$$

Thus, it remains to estimate the integral term. From L^1-data theory (see, for instance, [8, 10] and the references therein), let us choose

$$v = -\mathrm{sign}(u_m)(1+|u_m|)^{-\delta} \in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q_T), \quad \text{for } \delta > 0,$$

as a test function in (22). Using (5) and (7), it follows that

$$a_\# \int_{Q_T} \frac{\delta|\nabla u_m|^2}{(1+|u_m|)^{\delta+1}}\mathrm{dxdt} \leq \frac{1}{1-\delta}(|Q_T| + \|u_m\|_{1,\infty,Q_T})+ +\|f\|_{1,Q_T} + \|h\|_{1,\Sigma_T} + b^\# \int_{\Sigma_T} |u_m|^{\ell-1}\mathrm{dsdt}+ + \frac{\delta}{2a_\#}\left\|\frac{u_m}{(1+|u_m|)^{(\delta+1)/2}}\right\|_{\frac{2q}{q-2},\frac{2r}{r-2},Q_T}^2 \|\mathbf{E}\|_{q,r,Q_T}^2 + \frac{a_\#\delta}{2}\left\|\frac{\nabla u_m}{(1+|u_m|)^{(\delta+1)/2}}\right\|_{2,Q_T}^2. \tag{41}$$

Since (13) implies that $\bar{p} = (1-\delta)q/(q-2)$, $\bar{q} = (1-\delta)r/(r-2)$, and $p_1 = p$ satisfy (36), then we compute

$$\left\|\frac{u_m}{(1+|u_m|)^{(\delta+1)/2}}\right\|^2_{2q/(q-2),2r/(r-2),Q_T} \leq \|u_m\|^{1-\delta}_{\bar{p},\bar{q},Q_T} \leq$$
$$\leq (Z_1)^{p(r-2)/r}\|\nabla u_m\|^{p(r-2)/r}_{p,Q_T}\mathcal{Z}^{(1-\delta)(1-\lambda)} + (Z_2)^{p(r-2)/r}\mathcal{Z}^{1-\delta},$$

with $\lambda = p(r-2)/[(1-\delta)r] < 1$ because $r(2-p) < 2np$. Inserting these two above inequalities into (40), we conclude

$$\frac{2}{r}\|\nabla u_m\|^p_{p,Q_T} \leq \frac{2(2-p)}{p}\left(|Q_T| + (Z_2)^p\mathcal{Z}^{p(n+1)/n}\right) +$$
$$+\frac{2}{a_\#\delta}\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n}+1\right)\left(\frac{|Q_T|+\mathcal{Z}(2-\delta)}{1-\delta} + \mathcal{Z}b^\#/b_\#\right)+$$
$$+\frac{1}{(a_\#)^2}\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n}+1\right)\|\mathbf{E}\|^2_{q,r,Q_T}(Z_2)^{p(r-2)/r}\mathcal{Z}^{1-\delta}+$$
$$+\frac{2}{r(a_\#)^r}\left((Z_1)^{2-p}\mathcal{Z}^{(2-p)/n}+1\right)^{r/2}\|\mathbf{E}\|^r_{q,r,Q_T}(Z_1)^{p(r-2)/2}\mathcal{Z}^{\frac{r(1-\delta)-p(r-2)}{2}},$$

and therefore replacing δ by its value (16) holds.

5.3. Estimate of $\partial_t u_m$ in $L^1(0,T;[W^{1,p'}(\Omega)]')$

For every $v \in W^{1,p'}(\Omega) \hookrightarrow C(\bar{\Omega})$, and for almost all $t \in]0,T[$, we have

$$|\langle\partial_t u_m(t), v\rangle| \leq a^\#\|\nabla u_m(t)\|_{p,\Omega}\|\nabla v\|_{p',\Omega}+$$
$$+(b^\#\|u_m(t)\|_{\ell-1,\Gamma} + \|h\|_{1,\Gamma})\|v\|_{\infty,\Gamma} + \|u_m(t)\|_{pq/(q-p),\Omega}\|\mathbf{E}(t)\|_{q,\Omega}\|\nabla v\|_{p',\Omega}.$$

Similarly to (24)–(25), we have

$$\|u_m\|_{pq/(q-p),\Omega} \leq \|u_m\|^\lambda_{np/(n-p),\Omega}\|u_m\|^{1-\lambda}_{1,\Omega}, \quad \lambda = \frac{np}{np+p-n}\left(1-\frac{1}{p}+\frac{1}{q}\right);$$
$$\|u_m\|_{np/(n-p),\Omega} \leq S_{p,\ell-1}\left(\|\nabla u_m\|_{p,\Omega} + \|u_m\|_{\ell-1,\Gamma}\right),$$

then it follows that

$$\|\partial_t u_m(t)\|_{[W^{1,p'}(\Omega)]'} \leq (a^\# + S_{p,\ell-1})\|\nabla u_m(t)\|_{p,\Omega} + \|h\|_{1,\Gamma}+$$
$$+(b^\# C_\infty + S_{p,\ell-1})\|u_m(t)\|_{\ell-1,\Gamma} + \|u_m(t)\|_{1,\Omega}\|\mathbf{E}(t)\|^{1/(1-\lambda)}_{q,\Omega},$$

where C_∞ denotes the constant of continuity of the Morrey embedding $W^{1,p'}(\Omega) \hookrightarrow C(\bar{\Omega})$. Since (13) means that $1/(1-\lambda) = q(p(n+1)-n)/[p(q-n)] = r/p$, we deduce

$$\|\partial_t u_m\|_{L^1(0,T;[W^{1,p'}(\Omega)]')} \leq (a^\# + S_{p,\ell-1})T^{1-1/p}\|\nabla u_m\|_{p,Q_T} + \|h\|_{1,\Sigma_T}+$$
$$+(b^\# C_\infty + S_{p,\ell-1})T^{1-1/(\ell-1)}\|u_m\|_{\ell-1,p,\Sigma_T} + \|u_m\|_{1,\infty,Q_T}T^{1-1/p}\|\mathbf{E}\|^{r/p}_{q,r,Q_T}.$$

The sequence on the right-hand side of this last relation is uniformly bounded due to the estimates (15)–(16).

5.4. Passage to the limit in (22) as $m \to \infty$

By Sections 5.1 and 5.2 we may extract a subsequence of $\{u_m\}$ still denoted by $\{u_m\}$ such that $u_m \rightharpoonup u$ in $L^\iota(0,T;V_{p,\ell-1})$ for $\iota = \min\{p, \ell-1\}$.

Since $V_{p,\ell-1} \hookrightarrow\hookrightarrow L^q(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{1,p'}(\Omega)$ for any $q < pn/(n-p)$, and $V_{p,\ell-1} \hookrightarrow\hookrightarrow L^q(\Gamma) \hookrightarrow L^p(\Gamma) \hookrightarrow W^{1,p'}(\Omega)$ for any $q < p(n-1)/(n-p)$, according to Section 5.3 the Aubin–Lions lemma yields that $\{u_m\}$ is relatively compact in $L^{q,\iota}(Q_T)$ for any $q < pn/(n-p)$, and $L^{q,\iota}(\Sigma_T)$ for any $q < p(n-1)/(n-p)$. In particular, $|u_m|^{\ell-2}$ converges to $|u|^{\ell-2}$ a.e. on Σ_T. As the Nemytskii operator b is continuous, $b(u_m)$ strongly converges to $b(u)$ in $L^{(\ell-1)/(\ell-2)}(\Sigma_T)$. Therefore, (38) passes to the limit as m tends to infinity, concluding that u solves (14).

6. The case of $b_\# = 0$

The following proofs pursue the ones that are established in Sections 3 and 5. Therefore, we only focus our attention on the quantitative estimates.

6.1. Proof of Theorem 2.4

We observe that (23) reads

$$\frac{1}{2}\int_\Omega |u_m|^2(\tau)\mathrm{dx} + a_\# \int_{Q_\tau} |\nabla u_m|^2 \mathrm{dxdt} \leq \frac{1}{2}\|u_0\|^2_{2,\Omega} +$$

$$+ \int_0^\tau \|u_m\|_{2q/(q-2),\Omega}\|\mathbf{E}\|_{q,\Omega}\|\nabla u_m\|_{2,\Omega}\mathrm{dt} + \int_0^\tau \|h\|_{(2_*)',\Gamma}\|u_m\|_{2_*,\Gamma}\mathrm{dt}.$$

The last term of the RHS is computed as follows

$$\int_0^\tau \|h\|_{(2_*)',\Gamma}\|u_m\|_{2_*,\Gamma}\mathrm{dt} \leq K_s|\Omega|^{1/s-1/2}\int_0^\tau \|h\|_{(2_*)',\Gamma}\left(\|\nabla u_m\|_{2,\Omega} + \|u_m\|_{2,\Omega}\right)\mathrm{dt}$$

$$\leq \left(\frac{1}{a_\#} + \frac{1}{2}\right)(K_s)^2|\Omega|^{2/s-1}\|h\|^2_{(2_*)',2,\Sigma_\tau} + \frac{a_\#}{4}\|\nabla u_m\|^2_{2,Q_\tau} + \frac{1}{2}\int_0^\tau \|u_m\|^2_{2,\Omega}\mathrm{dt},$$

where $s = 2_*n/(2_* + n - 1)$. Thus, we may proceed as in Section 3.1 to conclude (17) and subsequently (18). The remaining proof follows *mutatis mutandis.*

6.2. Proof of Theorem 2.5

The estimate (19) is a direct consequence of Section 5.1. To show that (20) holds, it suffices to pay attention in (41) to the boundary integral, which obeys the following lemma.

Lemma 6.1 *If $p(n-1) < n$ and $\ell \leq p+1$, then*

$$\int_{\Sigma_T} |v|^{\ell-1}\mathrm{dsdt} \leq T^{1-(\ell-1)/p}|\Gamma|^{1-(\ell-1)/p_*}K_p^{\ell-1}\Big(2\|\nabla v\|_{p,Q_T} +$$

$$+ (S_1^{\frac{n(p-1)}{n-p(n-1)}}|\Omega|^{\frac{n(p-1)^2}{(n-p(n-1))p}} + S_1^{n(1-1/p)})\|v\|_{1,p,Q_T}\Big)^{\ell-1}$$

for every $v \in L^p(0,T;W^{1,p}(\Omega))$.

Proof We apply firstly the Hölder inequality and secondly the trace embedding for $\ell-1\leq p_*$ to obtain

$$\|v\|_{\ell-1,\Gamma}\leq|\Gamma|^{1/(\ell-1)-1/p_*}K_p(\|\nabla v\|_{p,\Omega}+\|v\|_{p,\Omega}).$$

We separately apply the interpolative inequality and after the Sobolev embedding and the Hölder inequality, obtaining

$$\begin{aligned}\|v\|_{p,\Omega}&\leq\|v\|^{\lambda}_{n/(n-1),\Omega}\|v\|^{1-\lambda}_{1,\Omega}\qquad(\lambda=n(p-1)/p<1)\\&\leq S_1^{\lambda}|\Omega|^{\lambda(1-1/p)}\|\nabla v\|^{\lambda}_{p,\Omega}\|v\|^{1-\lambda}_{1,\Omega}+S_1^{\lambda}\|v\|_{1,\Omega}.\end{aligned}$$

Thus, inserting this last inequality into the previous one, we deduce

$$\|v\|_{\ell-1,\Gamma}\leq|\Gamma|^{1/(\ell-1)-1/p_*}K_p\left(2\|\nabla v\|_{p,\Omega}+(S_1^{\lambda/(1-\lambda)}|\Omega|^{\frac{\lambda}{1-\lambda}(1-1/p)}+S_1^{\lambda})\|v\|_{1,\Omega}\right).$$

Integrating in time, using $\ell-1\leq p$ and applying the Hölder inequality, the proof is complete. □

Then Lemma 6.1 implies that (16) for u_m is rewritten as

$$\|\nabla u_m\|^p_{p,Q_T}\leq\mathcal{B}+2^{2\ell-3}\beta\|\nabla u_m\|^{\ell-1}_{p,Q_T}+$$
$$+2^{\ell-2}\beta\left((S_1^{\frac{n(p-1)}{n-p(n-1)}}|\Omega|^{\frac{n(p-1)^2}{(n-p(n-1))p}}+S_1^{n(1-1/p)})T\mathcal{Z}\right)^{\ell-1}.$$

If $\ell-1=p$, supposing that $\beta<2^{1-2p}$, then we find (20). If $\ell-1<p$, we conclude (20) by considering the Young inequality. The proof of Theorem 2.5 follows the argument of Section 5.

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