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Some normality criteria

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Abstract: In this article, we prove some normality criteria for a family of meromorphic functions, which involves sharing of a nonzero value by certain differential monomials generated by the members of the family. These results generalize some of the results of Schwick.

Key words: Meromorphic functions, holomorphic functions, shared values, normal families

1. Introduction and main results

The notion of normal families was introduced by Montel in 1907. Let us begin by recalling the definition. A family of meromorphic functions defined on a domain $D \subset \mathbb{C}$ is said to be normal in the domain if every sequence in the family has a subsequence that converges spherically uniformly on compact subsets of D to a meromorphic function or to ∞ (see [1, 6, 9, 14]).

One important aspect of the theory of complex analytic functions is to find normality criteria for families of meromorphic functions. Montel obtained a normality criterion, now known as the fundamental normality test, which says that *a family of meromorphic functions in a domain is normal if it omits three distinct complex numbers.* This result has undergone various extensions. In 1975, Zalcman [15] proved a remarkable result, now known as Zalcman's lemma, for families of meromorphic functions that are not normal in a domain. Roughly speaking, it says that *a nonnormal family can be rescaled at small scale to obtain a nonconstant meromorphic function in the limit.* This result of Zalcman gave birth to many new normality criteria. These normality criteria have been used extensively in complex dynamics for studying the Julia–Fatou dichotomy.

Schwick [11] gave a connection between normality and sharing values and proved a result that says that *a family of meromorphic functions on a domain $D \subset \mathbb{C}$ is normal if every function of the family and its first-order derivative share three distinct complex numbers.* Since then, many results of normality criteria concerning sharing values have been obtained [3, 5, 8, 12, 17–19].

Let f and g be meromorphic functions in a domain D and $p \in \mathbb{C}$. If the zeros of $f-p$ are the zeros of $g-p$ ignoring multiplicity, we write $f=p \Rightarrow g=p$. Hence, $f=p \iff g=p$ means that $f-p$ and $g-p$ have the same zeros ignoring multiplicity. If $f-p=0 \iff g-p=0$, then we say that f and g share the value p IM (see [13]).

Schwick [10] also proved a normality criterion that states that: *Let n, k be positive integers such that $n \geq k+3$, and let $\mathcal{F}$ be a family of functions meromorphic in a domain D. If each $f \in \mathcal{F}$ satisfies $(f^n)^{(k)}(z) \neq 1$*

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for $z \in D$, then $\mathcal{F}$ is a normal family. This result holds good for holomorphic functions in the case of $n \geq k+1$. Recently, Dethloff et al. [4] came up with new normality criteria, which improve the result given by Schwick [10].

Theorem 1.1 *Let $p \neq 0$ be a complex number, n be a nonnegative integer, and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every $f \in \mathcal{F}, f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - p$ is nowhere vanishing on D. Assume that*

(a) $n_j \geq t_j$ for all $1 \leq j \leq k$,

(b) $n + \sum_{j=1}^{k} n_j \geq 3 + \sum_{j=1}^{k} t_j$.

Then $\mathcal{F}$ is normal on D.

For the case of holomorphic functions they proved the following strengthened version:

Theorem 1.2 *Let $p \neq 0$ be a complex number, n be a nonnegative integer, and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of holomorphic functions in a domain D such that for every $f \in \mathcal{F}, f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - p$ is nowhere vanishing on D. Assume that*

(a) $n_j \geq t_j$ for all $1 \leq j \leq k$,

(b) $n + \sum_{j=1}^{k} n_j \geq 2 + \sum_{j=1}^{k} t_j$.

Then $\mathcal{F}$ is normal on D.

The main aim of this paper is to find normality criteria in terms of sharing values, which is motivated by [4].

Theorem 1.3 *Let $p \neq 0$ be a complex number, n be a nonnegative integer, and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers such that*

(a) $n_j \geq t_j$ for all $1 \leq j \leq k$,

(b) $n + \sum_{j=1}^{k} n_j \geq 3 + \sum_{j=1}^{k} t_j$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every pair of functions $f, g \in \mathcal{F}, f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ and $g^n(g^{n_1})^{(t_1)} \ldots (g^{n_k})^{(t_k)}$ share p IM on D. Then $\mathcal{F}$ is normal in D.

For families of holomorphic functions we have the following strengthened version:

Theorem 1.4 *Let $p \neq 0$ be a complex number, n be a nonnegative integer, and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers such that*

(a) $n_j \geq t_j$ for all $1 \leq j \leq k$,

(b) $n + \sum_{j=1}^{k} n_j \geq 2 + \sum_{j=1}^{k} t_j$.

Let $\mathcal{F}$ be a family of holomorphic functions in a domain D such that for every pair of functions $f, g \in \mathcal{F}, f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ and $g^n(g^{n_1})^{(t_1)} \ldots (g^{n_k})^{(t_k)}$ share p IM on D. Then $\mathcal{F}$ is normal in D.

The following examples show that the condition on p is necessary.

Example 1.5 *Let $\mathcal{F} = \{e^{mz} : m = 1, 2, \ldots\}$ be a family on $\Delta := \{z : |z| < 1\}$. Let $n, n_i's$, and $t_i's$ be as in Theorem 1.3. Then for every pair $f, g \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ and $g^n(g^{n_1})^{(t_1)} \ldots (g^{n_k})^{(t_k)}$ share 0 and ∞, but $\mathcal{F}$ is not normal.*

Example 1.6 *Let $\mathcal{F} = \{mz : m = 1, 2, \ldots\}$ be a family on $\Delta := \{z : |z| < 1\}$. Let $n, n_i's$ $t_i's$ be as in Theorem 1.3. Then for every pair $f, g \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ and $g^n(g^{n_1})^{(t_1)} \ldots (g^{n_k})^{(t_k)}$ share 0 and ∞, but $\mathcal{F}$ is not normal.*

The following example supports our result.

Example 1.7 *Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$, where $f_n(z) = n$. Then $\mathcal{F}$ satisfies conditions of Theorem 1.3 and $\mathcal{F}$ is normal.*

It is natural to ask what happens if we have a zero of $f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - p$. For this question we can extend Theorem 1.1 in the following manner.

Theorem 1.8 *Let $p \neq 0$ be a complex number, n be a nonnegative integer, and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers such that*

(*a*) $n_j \geq t_j$ *for all* $1 \leq j \leq k$,

(*b*) $n + \sum_{j=1}^{k} n_j \geq 3 + \sum_{j=1}^{k} t_j$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every $f \in \mathcal{F}, f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - p$ has at most one zero IM. Then $\mathcal{F}$ is normal in D.

Remark 1.9 *Theorem 1.1 is an immediate corollary of Theorem 1.3 and Theorem 1.8.*

2. Some notations

Let $\Delta = \{z : |z| < 1\}$ be the unit disk and $\Delta(z_0, r) := \{z : |z - z_0| < r\}$. We use the following standard functions of value distribution theory, namely

$$T(r, f), m(r, f), N(r, f) \text{ and } \overline{N}(r, f).$$

We let $S(r, f)$ be any function satisfying

$$S(r, f) = o\big(T(r, f)\big), \text{ as } r \to +\infty,$$

possibly outside of a set with finite measure.

3. Some lemmas

In order to prove our results we need the following lemmas. The following is a new version of Zalcman's lemma (see [15, 16]).

Lemma 3.1 *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disk Δ, with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least l and the poles of f are of multiplicity at least k. If $\mathcal{F}$ is not normal at z_0 in Δ, then for $-l < \alpha < k$, there exist*

1. *a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,*
2. *a sequence of functions $f_n \in \mathcal{F}$,*
3. *a sequence of positive numbers $\rho_n \to 0$,*

such that $g_n(\zeta) = \rho_n^\alpha f_n(z_n + \rho_n\zeta)$ converges to a nonconstant meromorphic function g on $\mathbb{C}$ with $g^\#(\zeta) \leq g^\#(0) = 1$. Moreover, g is of order at most two. Here $g^\#$ denotes the spherical derivative of g.

Lemma 3.2 *[2] Let f be an entire function. If the spherical derivative $f^\#(z)$ is bounded for all $z \in \mathbb{C}$, then f has order at most 1.*

Let f be a nonconstant meromorphic function in $\mathbb{C}$. A differential polynomial P of f is defined by $P(z) := \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} \left(f^{(j)}(z)\right)^{S_{ij}}$, where S_{ij}s are nonnegative integers and $\alpha_i(z) \not\equiv 0$ are small functions of f, which means $T(r, \alpha_i) = o\big(T(r, f)\big)$. The lower degree of the differential polynomial P is defined by

$$d(P) := \min_{1 \leq i \leq n} \sum_{j=0}^{p} S_{ij}.$$

The following result was proved by Dethloff et al. in [4].

Lemma 3.3 *Let $a_1, \ldots, a_q$ be distinct nonzero complex numbers. Let f be a nonconstant meromorphic function and let P be a nonconstant differential polynomial of f with $d(P) \geq 2$. Then*

$$T(r, f) \leq \left(\frac{q\theta(P) + 1}{qd(P) - 1}\right) \overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{qd(P) - 1} \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{P - a_j}\right) + S\left(r, f\right)$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgue measure, where $\theta(P) := \max_{1 \leq i \leq n} \sum_{j=0}^{p} jS_{ij}$.

Moreover, in the case of an entire function, we have

$$T(r, f) \leq \left(\frac{q\theta(P) + 1}{qd(P)}\right) \overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{qd(P)} \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{P - a_j}\right) + S(r, f)$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgue measure.

This result was proved by Hinchliffe in [7] for $q = 1$.

Lemma 3.4 *Let f be a transcendental meromorphic function. Let n be a nonnegative integer and $n_1, n_2, \ldots, n_k, t_1, t_2, \ldots, t_k$ be positive integers such that*

(*a*) $n_j \geq t_j$ *for all* $1 \leq j \leq k$,

(*b*) $n + \sum_{j=1}^{k} n_j \geq 3 + \sum_{j=1}^{k} t_j$.

Then $f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ assumes every nonzero complex value $p \in \mathbb{C}$ infinitely often.

Proof On the contrary, assume that $f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ takes the value p only finitely many times. Then

$$N\left(r, \frac{1}{(f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - p}\right) = O(\log r) = S(r, f). \tag{3.1}$$

Without loss of generality, we may assume $p = 1$. Let $P = f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$. Consider $(f^{n_i})^{(t_i)} = \sum c_{m_0, m_1, \ldots, m_{t_i}} f^{m_0}(f')^{m_1} \ldots (f^{(t_i)})^{m_{t_i}}$, where $c_{m_0, m_1, \ldots, m_{t_i}}$ are constants and $m_0, m_1, \ldots, m_{t_i}$ are nonnegative integers such that $\sum_{j=0}^{t_i} m_j = n_i, \sum_{j=1}^{t_i} j m_j = t_i$. It is easy to calculate

$$d(P) = n + \sum_{j=1}^{k} n_j \text{ and } \theta(P) = \sum_{j=1}^{k} t_j.$$

Clearly, $d(P) > 2$, so by Lemma 3.3, we get

$$T(r, f) \leq \left(\frac{\sum_{j=1}^{k} t_j + 1}{n + \sum_{j=1}^{k} n_j - 1}\right) \overline{N}\left(r, \frac{1}{f}\right) + \left(\frac{1}{n + \sum_{j=1}^{k} n_j - 1}\right) \overline{N}\left(r, \frac{1}{P-1}\right) + S(r, f),$$

and this gives

$$\left(\frac{n + \sum_{j=1}^{k} n_j - \sum_{j=1}^{k} t_j - 2}{n + \sum_{j=1}^{k} n_j - 1}\right) T(r, f) \leq \left(\frac{1}{n + \sum_{j=1}^{k} n_j - 1}\right) \overline{N}\left(r, \frac{1}{P-1}\right) + S(r, f),$$

and this gives

$$\left(\frac{1}{n + \sum_{j=1}^{k} n_j - 1}\right) T(r, f) \leq \left(\frac{1}{n + \sum_{j=1}^{k} n_j - 1}\right) N\left(r, \frac{1}{P-1}\right) + S(r, f).$$

By using (3.1), we get $T(r, f) = S(r, f)$, which is a contradiction. □

Lemma 3.5 *Let f be a transcendental entire function. Let n be a nonnegative integer and $n_1, n_2, \ldots, n_k, t_1, t_2, \ldots, t_k$ be positive integers such that*

(*a*) $n_j \geq t_j$ *for all* $1 \leq j \leq k$, (*b*) $n+\sum_{j=1}^{k} n_j \geq 2+\sum_{j=1}^{k} t_j$.

Then $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ *assumes every nonzero complex value* $p\in\mathbb{C}$ *infinitely often.*

We can prove this lemma by arguments similar to the proof of Lemma 3.4.

Lemma 3.6 *[17, 18], Let* $R=\frac{P}{Q}$ *be a rational function and* Q *be nonconstant. Then* $\left(R^{(k)}\right)_\infty \leq (R)_\infty - k$, *where* k *is a positive integer,* $(R)_\infty =$ *deg*$(P)-$ *deg*(Q), *and deg*(P) *denotes the degree of* P.

Lemma 3.7 *[17] Let* $R=a_m z^m+\dots+a_1 z+a_0+\frac{P}{B}$, *where* $a_0,a_1,\dots,a_{m-1},a_m(\neq 0)$ *are constants,* m *is a positive integer, and* P, B *are polynomials with deg*$(P)<$ *deg*(B). *If* $k\leq m$, *then* $\left(R^{(k)}\right)_\infty = (R)_\infty - k$.

Lemma 3.8 *Let* f *be a nonconstant rational function,* $p\in\mathbb{C}\setminus\{0\}$, n *be a nonnegative integer, and* $n_1,n_2,\dots,n_k,t_1,t_2,\dots,t_k$ *be positive integers such that*

(*a*) $n_j \geq t_j$ *for all* $1\leq j\leq k$,

(*b*) $n+\sum_{j=1}^{k} n_j \geq 3+\sum_{j=1}^{k} t_j$.

Then $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ *has at least two distinct* p*-points.*

Proof On the contrary, assume that $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ has at most one p-point. Now there are two cases to consider.

Case 1: Suppose $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ has exactly one p-point. First we assume that f is a nonconstant polynomial. Since $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ has exactly one p-point, we can set

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-p=A(z-z_0)^l,$$

where A is a nonzero constant and l is a positive integer satisfying $l\geq n+\sum n_j-\sum t_j\geq 3$. Then

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'=Al(z-z_0)^{l-1}.$$

Since a zero of f is a zero of $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ with multiplicity greater than 1, it is also a zero of $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'$. Since $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'$ has exactly one zero, namely z_0, and f is a nonconstant polynomial, it follows that z_0 is a zero of f and so is a zero of $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$, which is a contradiction. Therefore, f is a rational function that is not a polynomial. Let

$$f(z)=A\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\dots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n'_1}(z-\beta_2)^{n'_2}\dots(z-\beta_t)^{n'_t}}, \tag{3.2}$$

where A is a nonzero constant and m_is and n_js are integers. We put

$$M=n\sum_{i=1}^{s} m_i, \quad N=n\sum_{j=1}^{t} n'_j, \tag{3.3}$$

and

$$M_i = n_i \sum_{j=1}^{s} m_j, \quad N_i = n_i \sum_{j=1}^{t} n'_j, i = 1, 2, \ldots, k. \tag{3.4}$$

From (3.2), we get

$$f^{n_i}(z) = A^{n_i} \frac{(z-\alpha_1)^{n_i m_1}(z-\alpha_2)^{n_i m_2} \ldots (z-\alpha_s)^{n_i m_s}}{(z-\beta_1)^{n_i n'_1}(z-\beta_2)^{n_i n'_2} \ldots (z-\beta_t)^{n_i n'_t}}, \tag{3.5}$$

and so

$$(f^{n_i})^{(t_i)}(z) = \frac{(z-\alpha_1)^{n_i m_1 - t_i}(z-\alpha_2)^{n_i m_2 - t_i} \ldots (z-\alpha_s)^{n_i m_s - t_i} g_i(z)}{(z-\beta_1)^{n_i n'_1 + t_i}(z-\beta_2)^{n_i n'_2 + t_i} \ldots (z-\beta_t)^{n_i n'_t + t_i}}, \tag{3.6}$$

where $g_i(z)$ is a polynomial. From (3.5) and (3.6), we get

$$(f^{n_i})_\infty = M_i - N_i \text{ and } \left((f^{n_i})^{(t_i)}\right)_\infty = M_i - N_i - t_i(s+t) + \deg g_i(z).$$

Since by Lemma 3.6, $\left((f^{n_i})^{(t_i)}\right)_\infty \leq (f^{n_i})_\infty - t_i$, we get

$$\deg(g_i) \leq t_i(s+t-1). \tag{3.7}$$

From (3.2) and (3.6), we get

$$\begin{aligned} f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} & \qquad\qquad (3.8) \\ &= A^n \frac{(z-\alpha_1)^{m_1 n' - t'}(z-\alpha_2)^{m_2 n' - t'} \ldots (z-\alpha_s)^{m_s n' - t'} g(z)}{(z-\beta_1)^{n'_1 n' + t'}(z-\beta_2)^{n'_2 n' + t'} \ldots (z-\beta_t)^{n'_t n' + t'}} \\ &= \frac{p_1}{q_1}, \end{aligned}$$

where $n' = n + \sum_{j=1}^{k} n_j$, $t' = \sum_{j=1}^{k} t_j$ and p_1, q_1, $g(z)$ are polynomials with

$$\deg(g(z)) \leq (s+t-1) \sum_{j=1}^{k} t_j = t'(s+t-1). \tag{3.9}$$

Since $f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}$ has exactly one p-point and it is at z_0, we get from (3.8) that

$$\begin{aligned} f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} & \\ &= p + \frac{B(z-z_0)^l}{(z-\beta_1)^{n'_1 n' + t'}(z-\beta_2)^{n'_2 n' + t'} \ldots (z-\beta_t)^{n'_t n' + t'}} \\ &= \frac{p_1}{q_1}, \qquad\qquad (3.10) \end{aligned}$$

where B is a nonzero constant and l is a positive integer. From (3.8), we also obtain that

$$\begin{aligned} \left(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)}\right)' & \\ &= \frac{(z-\alpha_1)^{m_1 n' - t' - 1}(z-\alpha_2)^{m_2 n' - t' - 1} \ldots (z-\alpha_s)^{m_s n' - t' - 1} g_1(z)}{(z-\beta_1)^{n'_1 n' + t' + 1}(z-\beta_2)^{n'_2 n' + t' + 1} \ldots (z-\beta_t)^{n'_t n' + t' + 1}}, \qquad (3.11) \end{aligned}$$

where $g_1(z)$ is a polynomial. From (3.10), we obtain that

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'$$
$$=\frac{(z-z_0)^{l-1}g_2(z)}{(z-\beta_1)^{n_1'n'+t'+1}(z-\beta_2)^{n_2'n'+t'+1}\dots(z-\beta_t)^{n_t'n'+t'+1}}, \tag{3.12}$$

where $g_2(z)$ is a polynomial. From (3.8) and (3.11), we obtain

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty=M+\sum_{i=1}^{k}M_i-st'+\deg(g(z))-N-\sum_{i=1}^{k}N_i-tt',$$
$$\left(\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'\right)_\infty=M+\sum_{i=1}^{k}M_i-st'+\deg\left(g_1(z)\right)-N-\sum_{i=1}^{k}N_i-tt'-s-t.$$

By Lemma 3.6, we get

$$\left(\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'\right)_\infty\le\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty-1. \tag{3.13}$$

Hence, we obtain

$$\begin{aligned}\deg\left(g_1\left(z\right)\right)&\le s+t+\deg(g(z))-1\\&\le s+t+(s+t-1)t'-1\\&=(s+t-1)(t'+1).\end{aligned} \tag{3.14}$$

Now we consider the following subcases.

Subcase 1. When $l<N+\sum_{i=1}^{k}N_i+tt'$.

From (3.10), we have $\deg(p_1)=\deg(q_1)$, and from (3.8) and (3.9), we get that

$$\begin{aligned}\deg(q_1)&=N+\sum_{i=1}^{k}N_i+tt'=\deg(p_1)\\&\le M+\sum_{i=1}^{k}M_i+(t-1)t'.\end{aligned}$$

Hence, $\left(M+\sum_{i=1}^{k}M_i\right)-(N+\sum_{i=1}^{k}N_i)\ge t'$. This implies $\sum_{j=1}^{s}m_j-\sum_{j=1}^{t}n_j'\ge 1$. Therefore $(f)_\infty\ge 1$ and $(f^{n_i})_\infty\ge n_i$. Therefore, we can write f^{n_i} as follows:

$$f^{n_i}=a_mz^m+\ldots+a_1z+a_0+\tfrac{p}{B},$$

where $m(\ge n_i)$ is an integer, $a_m,\ldots,a_1,a_0$ are constants such that $a_m\ne 0$, and p, B are polynomials with $\deg(p)<\deg(B)$. Now by using Lemma 3.7, we get

$$\left((f^{n_i})^{(t_i)}\right)_\infty=(f^{n_i})_\infty-t_i\ge n_i-t_i. \tag{3.15}$$

Since $(f)_\infty \geq 1$, from (3.15), we see that $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty \geq n' - t' \geq 3$, which contradicts the fact that $\deg(p_1) = \deg(q_1)$.

Subcase 2. When $l = N + \sum_{i=1}^{k} N_i + tt'$.

Then from (3.10), we get $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty \leq 0$. Now we show that

$$\sum_{i=1}^{s} m_i \leq \sum_{i=1}^{t} n_i'. \tag{3.16}$$

Otherwise, $(f^n)_\infty = n\sum_{i=1}^{s} m_i - n\sum_{i=1}^{t} n_i' \geq n$ and $\left((f^{n_i})^{(t_i)}\right)_\infty = (f^{n_i})_\infty - t_i \geq n_i - t_i$ and so $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty \geq n + \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} t_i \geq 3$, which is a contradiction.

Since $\alpha_i \neq z_0$ for $i = 1, 2, \dots, s$ from (3.11) and (3.12), we see that $(z - z_0)^{l-1}$ is a factor of g_1. Therefore, by (3.14), we get $l - 1 \leq \deg(g_1) \leq (s + t - 1)(t' + 1)$. Now we have

$$\begin{aligned}
N + \sum_{i=1}^{k} N_i &= l - t\sum_{i=1}^{k} t_i \\
&\leq (s + t - 1)\left(\sum_{i=1}^{k} t_i + 1\right) + 1 - t\sum_{i=1}^{k} t_i \\
&= s\left(\sum_{i=1}^{k} t_i + 1\right) + t - \sum_{i=1}^{k} t_i \\
&\leq \sum_{i=1}^{s} m_i\left(\sum_{i=1}^{k} n_i + 1\right) + \sum_{i=1}^{t} n_i' - \sum_{i=1}^{k} t_i \\
&\leq \sum_{i=1}^{k} M_i + 2\sum_{i=1}^{t} n_i' - \sum_{i=1}^{k} t_i \\
&\leq \sum_{i=1}^{k} N_i + 2\sum_{i=1}^{t} n_i' - \sum_{i=1}^{k} t_i,
\end{aligned}$$

which is a contradiction when $n > 2$. For the case $n \in \{1, 2\}$, we use the condition $n + \sum_{i=1}^{k} n_i \geq 3 + \sum_{i=1}^{k} t_i$, to get

$$\begin{aligned}
N + \sum_{i=1}^{k} N_i &\leq \sum_{i=1}^{s} m_i\left(\sum_{i=1}^{k} t_i + 1\right) + t - \sum_{i=1}^{k} t_i \\
&\leq \sum_{i=1}^{k} n_i \sum_{i=1}^{s} m_i + \sum_{i=1}^{t} n_i' - \sum_{i=1}^{k} t_i \\
&\leq \sum_{i=1}^{k} M_i + \sum_{i=1}^{t} n_i' - \sum_{i=1}^{k} t_i
\end{aligned}$$

$$\leq \sum_{i=1}^{k} N_i + \frac{N}{n} - \sum_{i=1}^{k} t_i$$

$$\leq N + \sum_{i=1}^{k} N_i - \sum_{i=1}^{k} t_i,$$

which is again a contradiction. For $n = 0$ we have $N = 0$ by (3.3). Now use $\sum_{i=1}^{k} n_i \geq 3 + \sum_{i=1}^{k} t_i$, $s \leq \frac{\sum_{i=1}^{k} M_i}{\sum_{i=1}^{k} n_i}$, and $t \leq \frac{\sum_{i=1}^{k} N_i}{\sum_{i=1}^{k} n_i}$ to get

$$\begin{aligned}\sum_{i=1}^{k} N_i &\leq (t'+1)s + t - t' \\ &\leq (t'+1)\frac{\sum_{i=1}^{k} M_i}{\sum_{i=1}^{k} n_i} + \frac{\sum_{i=1}^{k} N_i}{\sum_{i=1}^{k} n_i} - t' \\ &\leq \left(\frac{\sum_{i=1}^{k} t_i + 2}{\sum_{i=1}^{k} n_i}\right)\sum_{i=1}^{k} N_i - t' \\ &< \sum_{i=1}^{k} N_i - \sum_{i=1}^{k} t_i,\end{aligned}$$

which is again absurd.

Subcase 3. When $l > N + \sum_{i=1}^{k} N_i + tt'$.

Then from (3.10), we have $(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)})_\infty > 0$. Now we claim that

$$\sum_{i=1}^{s} m_i > \sum_{i=1}^{t} n_i'. \tag{3.17}$$

If $\sum_{i=1}^{s} m_i \leq \sum_{i=1}^{t} n_i'$, then $(f)_\infty \leq 0, (f^{n_i})_\infty \leq 0,$ and $(f^n)_\infty \leq 0$. Hence, by Lemma 3.6, we obtain that

$$\begin{aligned}\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty &= (f^n)_\infty + \left((f^{n_1})^{(t_1)}\right)_\infty + \dots + \left((f^{n_k})^{(t_k)}\right)_\infty \\ &\leq 0 + \sum_{i=1}^{\infty}(f^{n_i})_\infty - t_i < 0,\end{aligned}$$

which is a contradiction.

Again from (3.10) and (3.12), we get

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)_\infty = l - \left(N + \sum_{i=1}^{k} N_i + tt'\right) \text{ and}$$

$$\left(\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)'\right)_\infty = l - 1 + \deg(g_2) - \left(\sum_{i=1}^{t} n_i'\right)(n' + tt' + t),$$

and from this with Lemma 3.6, we obtain $\deg(g_2) \leq t$.

Since for each $i = 1, 2, \ldots, s, \alpha_i \neq z_0$. From (3.11) and (3.12), we observe that $(z-\alpha_1)^{m_1 n' - t' - 1}(z-\alpha_2)^{m_2 n' - t' - 1} \ldots (z-\alpha_s)^{m_s n' - t' - 1}$ is a factor of g_2. Therefore,

$$M + \sum_{i=1}^{k} M_i - st' - s \leq \deg(g_2) \leq t, \tag{3.18}$$

and from (3.18), we get that

$$\begin{aligned} M + \sum_{i=1}^{k} M_i &\leq s + t + st' = t + (t'+1)s \\ &\leq \sum_{i=1}^{t} n_i' + \left(\sum_{i=1}^{k} n_i + 1\right) \sum_{i=1}^{s} m_i \\ &< \sum_{i=1}^{s} m_i + \left(\sum_{i=1}^{k} n_i + 1\right) \sum_{i=1}^{s} m_i \\ &= \frac{2}{n} M + \sum_{i=1}^{k} M_i, \end{aligned}$$

which is a contradiction when $n > 2$. For the case $n \in \{1, 2\}$, we use the condition $n + \sum_{i=1}^{k} n_i \geq 3 + \sum_{i=1}^{k} t_i$ to get

$$\begin{aligned} M + \sum_{i=1}^{k} M_i &\leq \sum_{i=1}^{t} n_i' + \left(\sum_{i=1}^{k} t_i + 1\right) \sum_{i=1}^{s} m_i \\ &\leq \frac{N}{n} + \sum_{i=1}^{k} n_i \sum_{i=1}^{s} m_i \\ &< \frac{M}{n} + \sum_{i=1}^{k} M_i, \end{aligned}$$

which is a contradiction. For $n = 0$ we have $M = 0$ by (3.3). Now use $\sum_{i=1}^{k} n_i \geq 3 + \sum_{i=1}^{k} t_i$, $s \leq \frac{\sum_{i=1}^{k} M_i}{\sum_{i=1}^{k} n_i}$, and $t \leq \frac{\sum_{i=1}^{k} N_i}{\sum_{i=1}^{k} n_i}$ to get

$$\begin{aligned} \sum_{i=1}^{k} M_i &\leq (t'+1)s + t \\ &\leq (t'+1)\frac{\sum_{i=1}^{k} M_i}{\sum_{i=1}^{k} n_i} + \frac{\sum_{i=1}^{k} N_i}{\sum_{i=1}^{k} n_i} \\ &< \left(\frac{\sum_{i=1}^{k} t_i + 2}{\sum_{i=1}^{k} n_i}\right) \sum_{i=1}^{k} M_i \\ &< \sum_{i=1}^{k} M_i, \end{aligned}$$

which is again a contradiction.

Case 2. Suppose $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} - p$ has no zero. Then f cannot be a polynomial, so f is a rational function that is not a polynomial. Now we put $l = 0$ in (3.10) and proceed as in Subcase 1. □

4. Proof of main results

Proof [**Proof of Theorem 1.3**] Since normality is a local property, we assume that $D = \Delta$. Suppose that $\mathcal{F}$ is not normal in Δ. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0 in Δ. Without loss of generality, we assume that $z_0 = 0$. Then by Lemma 3.1, for

$$\alpha = -\frac{\sum_{i=1}^{k} t_i}{n + \sum_{i=1}^{k} n_i}$$

there exist

1. a sequence of complex numbers $z_j \to 0$, $|z_j| < r < 1$,
2. a sequence of functions $f_j \in \mathcal{F}$,
3. a sequence of positive numbers $\rho_j \to 0$,

such that $g_j(\zeta) = \rho_j^{\alpha} f_j(z_j + \rho_j\zeta)$ converges to a nonconstant meromorphic function $g(\zeta)$ on $\mathbb{C}$ with $g^{\#}(\zeta) \leq g^{\#}(0) = 1$. Moreover, g is of order at most two.

We see that

$$\begin{aligned} & f_j^n(z_j+\rho_j\zeta)(f_j^{n_1})^{(t_1)}(z_j+\rho_j\zeta)\dots(f_j^{n_k})^{(t_k)}(z_j+\rho_j\zeta) \\ &\qquad = g_j^n(\zeta)(g_j^{n_1})^{(t_1)}(\zeta)\dots(g_j^{n_k})^{(t_k)}(\zeta) \to g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta), \end{aligned} \tag{4.1}$$

as $j \to \infty$, locally spherically uniformly.

Let

$$g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta) \equiv p. \tag{4.2}$$

Then g is a nonvanishing entire function. Using Lemma 3.2, we write $g(\zeta) = \exp(c\zeta + d)$, where $c(\neq 0), d$ are constants. Then from (4.2), we get

$$(n_1c)^{t_1}\dots(n_kc)^{t_k}\exp\left(\left(n+\sum_{i=1}^{k}n_i\right)c\zeta + \left(n+\sum_{i=1}^{k}n_i\right)d\right) \equiv p,$$

which is not possible. Hence, $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta) \not\equiv p$.

Therefore, by Lemma 3.4 and Lemma 3.8, $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta) - p$ has at least two distinct zeros, say ζ_0 and ζ_0^*. Now we choose $\delta > 0$ small enough so that $\Delta(\zeta_0, \delta) \cap \Delta(\zeta_0^*, \delta) = \emptyset$ and $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta) - p$ has no other zeros in $\Delta(\zeta_0, \delta) \cup \Delta(\zeta_0^*, \delta)$. By Hurwitz's theorem, there exist two sequences $\{\zeta_j\} \subset \Delta(\zeta_0, \delta)$ and $\{\zeta_j^*\} \subset \Delta(\zeta_0^*, \delta)$ converging to ζ_0 and ζ_0^* respectively and from (4.1), for sufficiently large j, we have

$$g_j^n(\zeta_j)(g_j^{n_1})^{(t_1)}(\zeta_j)\dots(g_j^{n_k})^{(t_k)}(\zeta_j) = p \text{ and } g_j^n(\zeta_j^*)(g_j^{n_1})^{(t_1)}(\zeta_j^*)\dots(g_j^{n_k})^{(t_k)}(\zeta_j^*) = p. \tag{4.3}$$

Since, by assumption that $f_j^n(f_j^{n_1})^{(t_1)}\ldots(f_j^{n_k})^{(t_k)}$ and $f_m^n(f_m^{n_1})^{(t_1)}\ldots(f_m^{n_k})^{(t_k)}$ share p in $D=\Delta$, for each pair f_j and f_m in $\mathcal{F}$, then by (4.3), for any m and for all j we get

$$g_m^n(\zeta_j)(g_m^{n_1})^{(t_1)}(\zeta_j)\ldots(g_m^{n_k})^{(t_k)}(\zeta_j)=p \text{ and } g_m^n(\zeta_j^*)(g_m^{n_1})^{(t_1)}(\zeta_j^*)\ldots(g_m^{n_k})^{(t_k)}(\zeta_j^*)=p.$$

We fix m and letting $j\to\infty$, and noting $z_j+\rho_j\zeta_j\to 0$, $z_j+\rho_j\zeta_j^*\to 0$, we obtain

$$f_m^n(0)(f_m^{n_1})^{(t_1)}(0)\ldots(f_m^{n_k})^{(t_k)}(0)-p=0.$$

Since the zeros are isolated, for sufficiently large j we have $z_j+\rho_j\zeta_j=0$, $z_j+\rho_j\zeta_j^*=0$. Hence, $\zeta_j=-z_j/\rho_j$ and $\zeta_j^*=-z_j/\rho_j$, which is not possible as $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_0^*,\delta)=\emptyset$. This completes the proof. □

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3.

Proof [**Proof of Theorem 1.8**] We may again assume that $D=\Delta$. Suppose that $\mathcal{F}$ is not normal in Δ. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0 in Δ. Without loss of generality, we assume that $z_0=0$. Then by Lemma 3.1, for

$$\alpha=-\frac{\sum_{i=1}^k t_i}{n+\sum_{i=1}^k n_i}$$

there exist

1. a sequence of complex numbers $z_j\to 0$, $|z_j|<r<1$,
2. a sequence of functions $f_j\in\mathcal{F}$,
3. a sequence of positive numbers $\rho_j\to 0$,

such that $g_j(\zeta)=\rho_j^{\alpha}f_j(z_j+\rho_j\zeta)$ converges to a nonconstant meromorphic function $g(\zeta)$ on $\mathbb{C}$ with $g^{\#}(\zeta)\le g^{\#}(0)=1$. Moreover, g is of order at most two.

We see that

$$\begin{aligned}&f_j^n(z_j+\rho_j\zeta)(f_j^{n_1})^{(t_1)}(z_j+\rho_j\zeta)\ldots(f_j^{n_k})^{(t_k)}(z_j+\rho_j\zeta)\\&\qquad=g_j^n(\zeta)(g_j^{n_1})^{(t_1)}(\zeta)\ldots(g_j^{n_k})^{(t_k)}(\zeta)\to g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\ldots(g^{n_k})^{(t_k)}(\zeta),\end{aligned}\tag{4.4}$$

as $j\to\infty$, locally spherically uniformly.

From the proof of the above result, we see that $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\ldots(g^{n_k})^{(t_k)}(\zeta)\not\equiv p$. Now we claim that $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\ldots(g^{n_k})^{(t_k)}(\zeta)-p$ has at most one zero IM. Suppose that $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\ldots(g^{n_k})^{(t_k)}(\zeta)-p$ has two distinct zeros, say ζ_0 and ζ_0^*, and choose $\delta>0$ small enough so that $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_0^*,\delta)=\emptyset$ and $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\ldots(g^{n_k})^{(t_k)}(\zeta)-p$ has no other zeros in $\Delta(\zeta_0,\delta)\cup\Delta(\zeta_0^*,\delta)$. By Hurwitz's theorem, there exist two sequences $\{\zeta_j\}\subset\Delta(\zeta_0,\delta),\{\zeta_j^*\}\subset\Delta(\zeta_0^*,\delta)$ converging to ζ_0 and ζ_0^* respectively and from (4.4), for sufficiently large j, we have

$$g_j^n(\zeta_j)(g_j^{n_1})^{(t_1)}(\zeta_j)\ldots(g_j^{n_k})^{(t_k)}(\zeta_j)=p \text{ and } g_j^n(\zeta_j^*)(g_j^{n_1})^{(t_1)}(\zeta_j^*)\ldots(g_j^{n_k})^{(t_k)}(\zeta_j^*)=p.\tag{4.5}$$

Since $z_j\to 0$ and $\rho_j\to 0$, we get for sufficiently large j, $z_j+\rho_j\zeta_j\in\Delta(\zeta_0,\delta)$ and $z_j+\rho_j\zeta_j^*\in\Delta(\zeta_0^*,\delta)$. Therefore, $f_j^n(f_j^{n_1})^{(t_1)}\ldots(f_j^{n_k})^{(t_k)}-p$ has two distinct zeros, which contradicts the fact that

$f_j^n(f_j^{n_1})^{(t_1)}\dots(f_j^{n_k})^{(t_k)}-p$ has at most one zero. However, Lemma 3.4 and Lemma 3.8 confirm that there does not exist such a nonconstant meromorphic function. This contradiction shows that $\mathcal{F}$ is normal in Δ and this proves the theorem. □

5. Extensions of Theorem 1.3 and Theorem 1.4

It is natural to ask whether one can replace the value p by a holomorphic function $\alpha(z)$ in Theorem 1.3. In this direction we extend Theorem 1.3 in the following manner:

Theorem 5.1 *Let $\alpha(z)$ be a holomorphic function defined in a domain $D\subset\mathbb{C}$ such that $\alpha(z)\neq 0$. Let n be a nonnegative integer and $n_1,n_2,\dots,n_k,t_1,t_2,\dots,t_k$ be positive integers such that*

(a) $n_j\geq t_j$ *for all* $1\leq j\leq k$,

(b) $n+\sum_{j=1}^k n_j\geq 3+\sum_{j=1}^k t_j$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every pair of functions $f,g\in\mathcal{F}$, $f^n(z)(f^{n_1})^{(t_1)}(z)\dots(f^{n_k})^{(t_k)}(z)$ and $g^n(z)(g^{n_1})^{(t_1)}(z)\dots(g^{n_k})^{(t_k)}(z)$ share $\alpha(z)$ IM on D. Then $\mathcal{F}$ is normal in D.

Proof

Once again we assume that $D=\Delta$. Suppose that $\mathcal{F}$ is not normal in Δ. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0 in Δ. Without loss of generality, we assume that $z_0=0$. Then by Lemma 3.1, for

$$\alpha=-\frac{\sum_{i=1}^k t_i}{n+\sum_{i=1}^k n_i}$$

there exist

1. a sequence of complex numbers $z_j\to 0$, $|z_j|<r<1$,
2. a sequence of functions $f_j\in\mathcal{F}$,
3. a sequence of positive numbers $\rho_j\to 0$,

such that $g_j(\zeta)=\rho_j^{\alpha}f_j(z_j+\rho_j\zeta)$ converges to a nonconstant meromorphic function $g(\zeta)$ on $\mathbb{C}$ with $g^{\#}(\zeta)\leq g^{\#}(0)=1$. Moreover, g is of order at most two.

We see that

$$\begin{aligned}
&f_j^n(z_j+\rho_j\zeta)(f_j^{n_1})^{(t_1)}(z_j+\rho_j\zeta)\dots(f_j^{n_k})^{(t_k)}(z_j+\rho_j\zeta)-\alpha(z_j+\rho_j\zeta)\\
&\quad=g_j^n(\zeta)(g_j^{n_1})^{(t_1)}(\zeta)\dots(g_j^{n_k})^{(t_k)}(\zeta)-\alpha(z_j+\rho_j\zeta)\\
&\quad\to g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta)-\alpha(0),
\end{aligned}\tag{5.1}$$

as $j\to\infty$, locally spherically uniformly.

Let

$$g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta)\equiv\alpha(0).\tag{5.2}$$

Then g is an entire function having no zero, so by Lemma 3.2, we write $g(\zeta) = \exp(c\zeta + d)$, where $c(\neq 0), d$ are constants. Then from (5.2), we get

$$(n_1c)^{t_1}\dots(n_kc)^{t_k}\exp\left(\left(n+\sum_{i=1}^{k}n_i\right)c\zeta+\left(n+\sum_{i=1}^{k}n_i\right)d\right)\equiv\alpha(0),$$

which is not possible. Hence, $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta)\not\equiv\alpha(0)$.

Therefore by Lemma 3.4 and Lemma 3.8, $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta)-\alpha(0)$ has at least two distinct zeros, say ζ_0 and ζ_0^*. Now we choose $\delta > 0$ small enough so that $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_0^*,\delta) = \emptyset$ and $g^n(\zeta)(g^{n_1})^{(t_1)}(\zeta)\dots(g^{n_k})^{(t_k)}(\zeta)-\alpha(0)$ has no other zeros in $\Delta(\zeta_0,\delta)\cup\Delta(\zeta_0^*,\delta)$.

By Hurwitz's theorem, there exist two sequences $\{\zeta_j\}\subset\Delta(\zeta_0,\delta), \{\zeta_j^*\}\subset\Delta(\zeta_0^*,\delta)$ converging to ζ_0 and ζ_0^* respectively and from (5.1), for sufficiently large j, we have

$$\begin{aligned}&g_j^n(\zeta_j)(g_j^{n_1})^{(t_1)}(\zeta_j)\dots(g_j^{n_k})^{(t_k)}(\zeta_j)=\alpha(z_j+\rho_j\zeta_j)\\&g_j^n(\zeta_j^*)(g_j^{n_1})^{(t_1)}(\zeta_j^*)\dots(g_j^{n_k})^{(t_k)}(\zeta_j^*)=\alpha(z_j+\rho_j\zeta_j).\end{aligned}\tag{5.3}$$

Since, by assumption that $f_j^n(f_j^{n_1})^{(t_1)}\dots(f_j^{n_k})^{(t_k)}$ and $f_m^n(f_m^{n_1})^{(t_1)}\dots(f_m^{n_k})^{(t_k)}$ share $\alpha(z)$ IM in $D=\Delta$, for each pair f_j and f_m in $\mathcal{F}$, then by (5.3), for any m and for all j we get

$$g_m^n(\zeta_j)(g_m^{n_1})^{(t_1)}(\zeta_j)\dots(g_m^{n_k})^{(t_k)}(\zeta_j)=\alpha(z_j+\rho_j\zeta_j)$$

and

$$g_m^n(\zeta_j^*)(g_m^{n_1})^{(t_1)}(\zeta_j^*)\dots(g_m^{n_k})^{(t_k)}(\zeta_j^*)=\alpha(z_j+\rho_j\zeta_j).$$

We fix m and letting $j\to\infty$, and noting $z_j+\rho_j\zeta_j\to 0$, $z_j+\rho_j\zeta_j^*\to 0$, we obtain

$$f_m^n(0)(f_m^{n_1})^{(t_1)}(0)\dots(f_m^{n_k})^{(t_k)}(0)-\alpha(0)=0.$$

Since the zeros are isolated, for sufficiently large j we have $z_j+\rho_j\zeta_j=0$, $z_j+\rho_j\zeta_j^*=0$. Hence, $\zeta_j=-z_j/\rho_j$ and $\zeta_j^*=-z_j/\rho_j$, which is not possible as $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_0^*,\delta)=\emptyset$. This completes the proof. □

For families of holomorphic functions we have the following result:

Theorem 5.2 *Let $\alpha(z)$ be a holomorphic function defined in a domain $D\subset\mathbb{C}$ such that $\alpha(z)\neq 0$. Let n be a nonnegative integer and $n_1,n_2,\dots,n_k,t_1,t_2,\dots,t_k$ be positive integers such that*

(a) $n_j\geq t_j$ *for all* $1\leq j\leq k$,

(b) $n+\sum_{j=1}^{k}n_j\geq 2+\sum_{j=1}^{k}t_j$.

Let $\mathcal{F}$ be a family of holomorphic functions in a domain D such that for every pair of functions $f,g\in\mathcal{F}$, $f^n(z)(f^{n_1})^{(t_1)}(z)\dots(f^{n_k})^{(t_k)}(z)$ and $g^n(z)(g^{n_1})^{(t_1)}(z)\dots(g^{n_k})^{(t_k)}(z)$ share $\alpha(z)$ IM on D. Then $\mathcal{F}$ is normal in D.

The proof is similar to the proof of Theorem 5.1.

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