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Uniqueness of entire graphs in Riemannian warped products

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Abstract: In this paper, by applying the generalized Omori–Yau maximum principle for complete spacelike hypersurfaces in warped product spaces, we obtain the sign relationship between the derivative of warping function and support function. Afterwards, by using this result and imposing suitable restrictions on the higher order mean curvatures, we establish uniqueness results for the entire graph in a Riemannian warped product space, which has a strictly monotone warping function. Furthermore, applications to such a space are given.

Key words: Warped product, monotonic function, complete spacelike hypersurface, r-th mean curvature

1. Introduction

In recent years, there has been steadily growing interest in the study of hypersurfaces immersed into a Riemannian product space $\mathbb{R} \times_f M^n$. A basic question on this topic is the problem of uniqueness of spacelike hypersurfaces with some suitable restriction on the mean curvature, more generally, on the higher order mean curvatures. Before giving details of our work we present a brief outline of some recent results related to it.

Some works have studied hypersurfaces with constant mean curvature (more generally, constant higher order mean curvatures) immersed in warped product spaces. In [15] Montiel studied the uniqueness of constant mean curvature compact hypersurfaces immersed in warped products of the type $\mathbb{R} \times_f M^n$ and $\mathbb{S}^1 \times_f M^n$, whose Ricci curvature Ric_M of the fiber M^n and the warping function f satisfy the following convergence condition $\mathrm{Ric}_M \geq (n-1)\sup_{\mathbb{R}}(f'^2 - f''f)\langle\ ,\ \rangle_M$. Later, in [2, 7] the authors extended the results of [15] to the complete noncompact hypersurface. More recently, in [3] Alías et al. generalized the result in [2] to constant higher order mean curvatures by using a suitable generalized version of the Omori–Yau maximum principle.

On the other hand, under suitable restrictions on the value of higher order mean curvatures including mean curvature, some authors obtained that the hypersurfaces are slices. In [9], Camargo et al. applied a technique of Yau [19] to obtain that the complete hypersurfaces immersed in pseudo-hyperbolic space $\mathbb{R} \times_{e^t} M^n$ are slices when the norm of the gradient of the height function is integrable and mean curvature (non-necessarily constant) satisfies $0 < H \leq 1$ or higher order mean curvatures satisfy $0 < H_k \leq H_{k+1}$. Afterwards, in [4, 6], Colares, de Lima, and Alías obtain an extension of the result of [7]; they get that the complete hypersurfaces immersed in a warped product space are slices when the Ricci curvatures and warping function f satisfy some suitable restriction. Furthermore, higher order mean curvatures H_k satisfy $0 < \frac{H_{k+1}}{H_k} \leq \frac{f'}{f}$ with f' and H_k are positive.

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Moreover, we have noticed that many works have approached problems in this branch that have similar requests, such as [5, 10, 13]; they all require higher order mean curvatures and the derivative of warping function be positive, which is not suitable for spaces such as $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$ and $\mathbb{R} \times_{\cosh t} \mathbb{S}^n$. At this point, it is worth considering the case when the warping function of the ambient space has a negative derivative because there is a close sign relationship among mean curvature H, support function $\langle N, \partial_t \rangle$, and derivative of warping function. Inspired by the work by Alías et al. in [3] we obtain Theorem 3.1 and Corollary 3.1 by using the Omori–Yau maximum principle. Then combining with a consequence of Stokes' theorem for complete manifolds in [12] of Caminha, which is our main analytical tool, we obtain Theorem 3.2 when there exist some suitable restrictions on the mean curvature.

Furthermore, we extend this result to the higher order mean curvatures, and obtain Theorems 4.1 and 4.2, which are rigidity theorems concerning entire graphs in a Riemannian warped product space $\mathbb{R} \times_f M^n$. We also give some applications related to the previous theorems in sections 3 and 4, such as Corollaries 3.3 and 3.4 and Corollary 4.3.

2. Preliminaries

Let M^n be a connected n-dimensional Riemannian manifold, and $f : \mathbb{R} \to \mathbb{R}^+$ is a positive smooth function. In the product differentiable manifold $\overline{M}^{n+1} = \mathbb{R} \times_f M^n$, let $\pi_{\mathbb{R}}$ and π_M denote the projections onto the fibers $\mathbb{R}$ and M, respectively. A particular class of Riemannian manifold is the one obtained by furnishing $\overline{M}$ with the metric

$$\langle v, w \rangle_p = \langle (\pi_{\mathbb{R}})_* v, (\pi_{\mathbb{R}})_* w \rangle_{\mathbb{R}} + (f \circ \pi_{\mathbb{R}})^2(p) (\pi_M)_* v, (\pi_M)_* w \rangle_M,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$, where $\langle \ , \ \rangle_{\mathbb{R}}$ and $\langle \ , \ \rangle_M$ stand for the metrics of $\mathbb{R}$ and M^n, respectively. Such a space is called a *warped product*; in what follows we shall denote it as $\overline{M}^{n+1} = \mathbb{R} \times_f M^n$. For simplicity of notation we will denote the warped metric as

$$\langle \ , \ \rangle = dt^2 + f^2(t) \langle \ , \ \rangle_M.$$

Under this condition, for a fixed $t_0 \in \mathbb{R}$, we say that $M^n_{t_0} = \{t_0\} \times M^n$ is a slice of $\overline{M}^{n+1}$.

We consider the entire graphs in a warped product $\overline{M} = \mathbb{R} \times_f M^n$, which are defined by

$$\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset \overline{M},$$

where u is a smooth function on M. Let $\Omega \subseteq M^n$ be a connected domain of a complete Riemannian manifold M^n. The metric induced on Ω from the metric on the ambient space via $\Sigma^n(u)$ is $\langle \ , \ \rangle = dt^2 + f^2(t) \langle \ , \ \rangle_M$. The graph $\Sigma^n(u)$ is said to be *entire* if $\Omega = M^n$. It can be easily seen from the metric induced on Ω of $\Sigma^n(u)$ that when the function $f(u)$ is bounded on M^n, the entire graph $\Sigma^n(u)$ is complete. In particular, this occurs when $\Sigma^n(u)$ lies in a slab of $\mathbb{R} \times_f M^n$.

In this setting, if we let A be the corresponding shape operator, then at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $0 \leq k \leq n$, let $S_k(p)$ denote the r-th elementary symmetric function on the eigenvalues of A_p. We get n smooth functions $S_k : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with the corresponding eigenvalues $\{\lambda_k\}$, then

$$S_k(p) = \sigma_k(\lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

The kth-mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = S_k = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1}$$

Thus $H_0 = 1$ and $H_1 = -\frac{1}{n} tr(A) = H$ is the mean curvature of Σ^n.

In what follows we will work with the so-called *Newton transformations* $P_k : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, which are defined from A by $P_0 = I$ (the identity of $\mathfrak{X}(\Sigma)$) and for $1 \le k \le n$,

$$P_k = S_k I - A \circ P_{k-1}.$$

Observe that the Newton transformations P_k are all self-adjoint operators that commute with the shape operator A. Even more, if $\{e_k\}$ is an orthonormal frame on $T_p\Sigma$ that is diagonalizable with A_p, $A_p(e_i) = \lambda_i(p)e_i$, then

$$(P_k)_p(e_i) = \mu_{i,k}(p)e_i, \tag{2}$$

where

$$\mu_{i,k} = \sum_{i_1 < \cdots < i_k, i_j \neq i} \lambda_{i_1} \cdots \lambda_{i_k}.$$

For each $k,\ 0 \le k \le n-1$, we have

$$\mathrm{tr}(P_k) = c_k H_k, \quad \mathrm{tr}(A \circ P_k) = c_k H_{k+1},$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$. Associated with each *Newton transformation* P_k, we consider the second order linear differential operator $L_k : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$, given by

$$L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f).$$

Here $\nabla^2 f : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f, and it is given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle, \quad X, Y \in \mathfrak{X}(\Sigma).$$

In particular, $L_0 = \Delta$ and if $\overline{M}$ has constant sectional curvature, Rosenberg proved in [18] that $L_k(f) = \mathrm{div}(P_k \nabla f)$, where div stands for the divergence on Σ.

Let $\psi : \Sigma^n \to \mathbb{R} \times_f M^n$ be a Riemannian immersion, with Σ oriented by unit vector field N. We will refer to the normal vector field N as *future-pointing Gauss map* of the hypersurface if N is in the same time-orientation as ∂_t. In what follows, we suppose *support function* $\langle N, \partial_t\rangle$ does not change sign on Σ^n. Let h denote the (vertical) height function naturally attached to Σ^n, namely, $h = (\pi_{\mathbb{R}})\,|_\Sigma$.

Let $\overline{\nabla}$ and ∇ denote gradients with respect to the metrics of $\mathbb{R}\times_f M^n$ and Σ^n, respectively. A simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $\mathbb{R}\times_f M^n$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = \langle\overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle = \partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^{\top} = \partial_t^{\top} = \partial_t - \langle N, \partial_t\rangle N.$$

In particular, we get

$$|\nabla h|^2 = 1 - \langle N, \partial_t\rangle^2, \tag{3}$$

where $|\ |$ denotes the norm of a vector field on Σ^n. Moreover, observe that for a graph $\Sigma^n(u)$, its height function h is nothing but the function u seen as a function on $\Sigma^n(u)$.

We will need the following result of Alías et al.

Lemma 2.1 ([3]) *Let $\psi:\Sigma^n\to\mathbb{R}\times_f M^n$ be a spacelike hypersurface, $h=(\pi_{\mathbb{R}})|_{\Sigma}:\Sigma^n\to\mathbb{R}$ the height function of Σ^n, and $g:\mathbb{R}\to\mathbb{R}$ be any primitive of the warping function f; then*

$$\operatorname{div}(P_k\nabla g(h)) = \langle \operatorname{div}P_k, \nabla g(h)\rangle + L_k(g(h)), \tag{4}$$

where

$$L_k(g(h)) = c_k f(h)(\frac{f'}{f}(h)H_k + \langle N, \partial_t\rangle H_{k+1}).$$

In particular, when $k=0$, $\operatorname{div}(\nabla g(h)) = nf(h)(\frac{f'}{f}(h) + \langle N, \partial_t\rangle H)$.

When $k=1$, denote N^* as the projection of N onto the fiber M^n, Ric_M is the Ricci curvature of fiber M^n and $N^* = N - \langle N, \partial_t\rangle\partial_t$; then from Corollary 7.43 of [17] and [3] we have

$$\begin{aligned}\operatorname{div}(P_1\nabla g(h)) \quad &= \quad -f(h)\langle N, \partial_t\rangle(\mathrm{Ric}_M(N^*, N^*) - (n-1)(f'^2 - f''f)(h)\langle N^*, N^*\rangle_M) + \\ &\quad + n(n-1)f(h)H(\frac{f'}{f}(h) + \langle N, \partial_t\rangle\frac{H_2}{H}).\end{aligned}$$

When $k\geq 2$, if the sectional curvature of fiber M^n is constant κ, then

$$\begin{aligned}\operatorname{div}(P_k\nabla g(h)) \quad &= \quad -(n-k)\langle N, \partial_t\rangle\frac{\kappa - (f'^2 - f''f)}{f}\langle P_{k-1}\nabla h, \nabla h\rangle + \\ &\quad + c_k f(u)H_k(\frac{f'}{f} + \langle N, \partial_t\rangle\frac{H_{k+1}}{H_k}).\end{aligned}$$

Now we quote some useful lemmas in which geometric conditions are given in order to guarantee the sign of H_k and P_k when $k\geq 1$.

Lemma 2.2 ([14]) *Let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a Riemannian immersion in a Riemannian manifold $\overline{M}^{n+1}$. If $H_2>0$ on Σ^n, then P_1 is positive definite for an appropriate choice of the Gauss map N.*

Lemma 2.3 ([8]) *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion in a Riemannian manifold $\overline{M}^{n+1}$. If Σ^n has an elliptic point with respect to an appropriate choice of the Gauss map N and $H_{r+1} > 0$ on Σ^n for some $2 \leq r \leq n-1$, then P_k is positive definite and H_k is positive for all $1 \leq k \leq r$.*

Lemma 2.4 ([4]) *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a Riemannian immersion in a Riemannian manifold $\overline{M}^{n+1}$. If $f(h)$ attains a local maximum at some $p \in \Sigma^n$, such that $f'(h(p)) \neq 0$, then p is an elliptic point for Σ^n.*

Recall that by an elliptic point in a spacelike hypersurface we mean a point $p_0 \in \Sigma$ where all principal curvatures $\lambda_i(p_0)$ have the same sign. There is also a Lorentzian version in [1] for Lemma 2.4.

In order to prove our rigidity results, we will use the following result due to Yau. In [19] Yau has the Stokes' Theorem on an n-dimensional, complete noncompact Riemannian manifold. Then in [11] Caminha et al. obtained a suitable consequence of Yau's result. We state it as follows. Let $\mathcal{L}^1(\Sigma)$ be the space of Lebesgue integrable functions on Σ.

Lemma 2.5 ([11]) *Let X be a smooth vector field on the n-dimensional complete noncompact oriented Riemannian manifold Σ^n, such that $\mathrm{div}X$ does not change sign on Σ^n. If $|X| \in \mathcal{L}^1(\Sigma)$, then $\mathrm{div}X = 0$.*

3. Sign relationship and uniqueness theorem in Riemannian warped products

In this section, we will apply the results that we have discussed in the previous section to study the rigidity of spacelike hypersurfaces in Riemannian warped products $\mathbb{R} \times_f M^n$, where M^n is a complete Riemannian manifold. In order to prove our results, we need the following lemma, which is the well-known generalized maximum principal due to [16] of Omori and [19] of Yau.

Lemma 3.1 *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below on Σ and $u : \Sigma^n \to \mathbb{R}$ be a smooth function that is bounded from below on Σ^n. Then there is a sequence of points $p_k \in \Sigma^n$ such that*

$$\lim_k u(p_k) = \inf u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \lim_k \triangle u(p_k) \geq 0.$$

Equivalently, for any smooth function $u : \Sigma^n \to \mathbb{R}$ that is bounded from above on Σ^n, there is a sequence of points $p_k \in \Sigma^n$ such that

$$\lim_k u(p_k) = \sup u, \quad \lim_k |\nabla u(p_k)| = 0 \quad \text{and} \quad \lim_k \triangle u(p_k) \leq 0.$$

As an application of Lemma 3.1, we will prove the following result, which obtains a sign relationship among mean curvature H, support function $\langle N, \partial_t \rangle$, and the derivative of warping function f.

Theorem 3.1 *Let $\varphi : \Sigma^n \to \mathbb{R} \times_f M^n$ be a complete spacelike hypersurface with non-vanishing mean curvature that is contained in a slab. Choose on Σ some orientation of N such that $H > 0$. Suppose that the support function $\langle N, \partial_t \rangle$ does not change sign and the Omori–Yau maximum principle for Laplacian holds on Σ. Assume that one of the following conditions holds:*
(C_1) the warping function is strictly monotonic.
(C_2) $\log f$ is convex.
Then we have $f'\langle N, \partial_t \rangle < 0$. On the other hand, if $H < 0$, then $f'\langle N, \partial_t \rangle > 0$

Proof (C_1) Since the hypersurface is contained in a bounded slab we have h is bounded and we apply the Omori–Yau maximum principle to the Laplacian to assure the existence of sequences $\{p_j\}, \{q_j\} \subset \Sigma$ such that

$$\lim_j h(p_j) = \sup h = h^*, \quad \lim_j |\nabla h(p_j)| = 0, \quad \lim_j \triangle h(p_j) \leq 0;$$

$$\lim_j h(q_j) = \inf h = h_*, \quad \lim_j |\nabla h(q_j)| = 0, \quad \lim_j \triangle h(q_j) \geq 0.$$

Since $\triangle h = \frac{f'}{f}(h)(n - |\nabla h|^2) + nH_1\langle N, \partial_t \rangle$, then we have

$$\lim_j \triangle h(p_j) = \lim_j (\frac{f'}{f}(h(p_j))(n - |\nabla h(p_j)|^2) + nH_1(p_j)\langle N, \partial_t\rangle(p_j)) \leq 0. \tag{5}$$

$$\lim_j \triangle h(q_j) = \lim_j (\frac{f'}{f}(h(q_j))(n - |\nabla h(q_j)|^2) + nH_1(q_j)\langle N, \partial_t\rangle(q_j)) \geq 0. \tag{6}$$

Since f is strictly monotonic, we have $f' > 0$ or $f' < 0$. If $f' > 0$, then from (3.1) making $j \to \infty$ we obtain

$$\frac{f'}{f}(h^*) \leq -\lim_k H_1(p_j)\langle N, \partial_t\rangle(p_j).$$

Moreover, from $f'(h^*) \geq 0$ and $H_1 > 0$, we obtain $\langle N, \partial_t\rangle < 0$.

On the other hand, if $f' < 0$, then from (3.2) we have

$$\frac{f'}{f}(h_*) \geq -\lim_k H_1(q_j)\langle N, \partial_t\rangle(q_j).$$

From $f'(h_*) \leq 0$ and $H_1 > 0$, we have $\langle N, \partial_t\rangle > 0$.

(C_2) From the hypothesis $\log f$ is convex we have that

$$\frac{f'}{f}(h_*) \leq \frac{f'}{f}(h) \leq \frac{f'}{f}(h^*).$$

Then, taking into account that $\langle N, \partial_t\rangle$ does not change sign, thus if $\langle N, \partial_t\rangle > 0$, from the inequality (3.1) we have

$$\frac{f'}{f}(h) \leq \frac{f'}{f}(h^*) \leq -\lim_k H_1(p_j)\langle N, \partial_t\rangle(p_j) < 0.$$

If $\langle N, \partial_t\rangle < 0$, from the inequality (3.2) we have

$$\frac{f'}{f}(h) \geq \frac{f'}{f}(h_*) \geq -\lim_k H_1(q_j)\langle N, \partial_t\rangle(q_j) > 0.$$

Thus, we conclude the result.

From the proof we also have if $H_1 < 0$ and $\langle N, \partial_t\rangle$ does not change sign, it is easy to get that $f'\langle N, \partial_t\rangle > 0$. □

By Lemma 2.3 we generalized Theorem 3.1 to the case of higher order mean curvatures, which is presented below.

Corollary 3.1 *Let $\varphi : \Sigma^n \to \mathbb{R} \times_f M^n$ be a complete spacelike hypersurface with non-vanishing mean curvature that is contained in a slab of $\mathbb{R} \times_f M^n$. There exists an elliptic point on Σ^n and the Omori–Yau maximum principle for Laplacian holds on Σ, Assume that the warping function satisfies C_1 or C_2 and the support function $\langle N, \partial_t\rangle$ does not change sign; then for $2 \leq k \leq n-1$ we have*
(i) if $H_{k+1}H_k > 0$, then $f'\langle N, \partial_t\rangle < 0$,
(ii) if $H_{k+1}H_k < 0$, then $f'\langle N, \partial_t\rangle > 0$.

Proof (i) Since there exists an elliptic point on Σ and $H_{k+1}H_k > 0$, we have that N is the right orientation such that both H_k and H_{k+1} are positive. Now using Lemma 2.3 we have $H_1 > 0$; thus by Theorem 3.1 we have $f'\langle N, \partial_t\rangle < 0$.

(ii) From the hypothesis we have one of H_k and H_{k+1} is negative. Since there exists an elliptic point then from (2.1) and the definition of elliptic point we have $H_j < 0$ with respect to this orientation, where $1 \leq j = 2t+1 \leq k+1$; thus $H_1 < 0$. Now from Theorem 3.1 it is easy to get the result. □

Before stating our main results, we will introduce a sufficient condition that guarantees the Omori–Yau maximum principle holds on Σ for the Laplacian.

Lemma 3.2 *Let $\varphi : \Sigma^n \to I \times_f M^n$ be an immersed hypersurface and M^n be an n-dimensional complete Riemannian manifold with sectional curvature bounded from below. Assume that Σ is contained in a slab and the mean curvature is bounded; then the Omori–Yau maximum principle holds on Σ for the Laplacian.*

Now, taking into account the results above, we deduce the following result. We denote Du to be the gradient of u as a function on M^n, while $\nabla u = \nabla h$ is the gradient of the height function on $\Sigma^n(u)$.

Theorem 3.2 *Let $\overline{M}^{n+1} = \mathbb{R} \times_f M^n$ be a Riemannian warped product space and $\Sigma^n(u)$ be an entire graph that lies in a slab $[t_1, t_2] \times M^n$, where the sectional curvature of fiber M^n is bounded from below. Suppose the warping function f satisfies C_1 or C_2 on $[t_1, t_2]$ and $|Du| \in \mathcal{L}^1(M^n)$. Assume that either*

(i) $0 < H \leq \frac{|f'|}{f}(u)$, or

(ii) $-\frac{|f'|}{f}(u) \leq H < 0$;

then $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

Proof (i) Since the sectional curvature of fiber M^n is bounded from below, $\Sigma^n(u)$ lies in a slab $[t_1, t_2] \times M^n$ and $|Du| \in \mathcal{L}^1(M^n)$; then from Lemma 3.2 we have that the Omori–Yau maximum principle for Laplacian holds on Σ^n. Combining the assumption that the warped product function satisfies condition C_1 or C_2 on $[t_1, t_2]$, we obtain that Theorem 3.1 holds true. Let $u = h$ be the height function on $\Sigma^n(u)$. Since $N = \langle N, \partial_t\rangle\partial_t + N^*$, where N^* denotes the projection of N on the fiber M^n, we have from (2.4)

$$|\nabla u|^2 = \langle N^*, N^*\rangle = f^2(u)\langle N^*, N^*\rangle_M = \frac{|Du|_M^2}{f^2(u) + |Du|_M^2}.$$

Furthermore, we also have $\mathrm{d}\Sigma = \sqrt{|G|}\mathrm{d}M$, where $\mathrm{d}\Sigma$ and $\mathrm{d}M$ stand for the Riemannian volume elements of $(\Sigma^n(u), \langle\ ,\ \rangle)$ and $(M^n, \langle\ ,\ \rangle)$, respectively, $|G| = f^{2n-2}(u)(f^2(u) + |Du|_M^2)$, and $|\nabla u|\mathrm{d}\Sigma = f^{n-1}(u)|Du|_M\mathrm{d}M$. Since $\Sigma^n(u)$ lies in a slab of $\mathbb{R} \times_f M^n$, we have $|\nabla g(u)| \leq f(u)|\nabla u| \in \mathcal{L}^1(\Sigma^n(u))$.

Now from Lemma 2.1 we have

$$\mathrm{div}(\nabla g(u)) = nf(u)(\frac{f'}{f}(u) + \langle N, \partial_t\rangle H). \tag{7}$$

Moreover, from the hypothesis we have $H > 0$; then by Theorem 3.1 we get $f'\langle N, \partial_t\rangle < 0$.

First, assume that $f' > 0$; then we have $\langle N, \partial_t\rangle < 0$ and $\frac{f'}{f}(u) + \langle N, \partial_t\rangle H \geq \frac{f'}{f}(u) - H \geq 0$; thus we have $\mathrm{div}(\nabla g(u)) \geq 0$ does not change sign. Since $\Sigma^n(u)$ is complete and $|\nabla g(u)| \in \mathcal{L}^1(\Sigma^n)$, we can apply

Lemma 2.5 to get $\operatorname{div}(\nabla g(u)) = 0$. Taking into account the hypothesis of (i), we obtain $\langle N, \partial_t \rangle \equiv -1$, which concludes that $|\nabla u| \equiv 0$ on $\Sigma^n(u)$. Therefore, $\Sigma^n(u)$ is a slice.

On the other hand, if $f' < 0$, $\langle N, \partial_t \rangle > 0$, then $\frac{f'}{f}(u) + \langle N, \partial_t \rangle H \leq \frac{f'}{f}(u) + H \leq 0$; consequently, $\operatorname{div}(\nabla g(u)) \leq 0$. Now, taking into account $|\nabla g(u)| \in \mathcal{L}^1(M^n)$, we can apply Lemma 2.5 to get that $\operatorname{div}(\nabla g(u)) = 0$ and from (3.3) we conclude that $\langle N, \partial_t \rangle \equiv 1$; thus $|\nabla u| \equiv 0$ on $\Sigma^n(u)$. Then $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

(ii) From the hypothesis we have $H < 0$, and by Theorem 3.1 we have $f' \langle N, \partial_t \rangle > 0$.

If $f' > 0$, then we have $\langle N, \partial_t \rangle > 0$ and $\frac{f'}{f}(u) + \langle N, \partial_t \rangle H \geq \frac{f'}{f}(u) + H \geq 0$; thus $\operatorname{div}(\nabla g(u)) \geq 0$.

If $f' < 0$, then we have $\langle N, \partial_t \rangle < 0$ and $\frac{f'}{f}(u) + \langle N, \partial_t \rangle H \leq \frac{f'}{f}(u) - H \leq 0$; thus $\operatorname{div}(\nabla g(u)) \leq 0$.

Now in the same argument as (i), we conclude that $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$ with $t_0 \in [t_1, t_2]$. □

As a direct consequence of Theorem 3.2 we have the following corollary.

Corollary 3.2 *Let $\overline{M}^{n+1} = \mathbb{R} \times_f M^n$ be a Riemannian warped product space and $\Sigma^n(u)$ be an entire graph that lies in a slab $[t_1, t_2] \times M^n$, where the sectional curvature of fiber M^n is bounded from below. Suppose that the warping function f satisfies C_1 or C_2 on $[t_1, t_2]$ and $|Du| \in \mathcal{L}^1(M^n)$; then if the nonvanishing mean curvature satisfies*

$$-\frac{|f'|}{f}(u) \leq H \leq \frac{|f'|}{f}(u),$$

then $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

Now let us consider the ambient space is the warped product space $\mathbb{R} \times_{e^t} \mathbb{R}^n$, which can be considered as $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$; the slice M_t^n is isometric to $\mathbb{R}^n$ and is called the horosphere of $\mathbb{H}^{n+1}$. It is easy to show that warping function $f = e^t$ satisfies conditions both C_1 and C_2, and from Theorem 3.2 we have the following.

Corollary 3.3 *Let $\mathbb{H}^{n+1} = \mathbb{R} \times_{e^t} \mathbb{R}^n$ be the hyperbolic space and $\Sigma^n(u)$ be an entire graph that lies between two horospheres. Assume that the nonvanishing mean curvature H satisfies*

$$-1 \leq H \leq 1;$$

then if $|Du| \in \mathcal{L}^1(\mathbb{R}^n)$, the hypersurface $\Sigma^n(u)$ is a horosphere.

To close this section, we establish the following rigidity result when the ambient space $\overline{M}^{n+1} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$. The warping function $\cos t$ satisfies condition C_1 on $(-\frac{\pi}{2}, 0)$ or $(0, \frac{\pi}{2})$. We obtain the following result.

Corollary 3.4 *Let $\varphi : \Sigma^n(u) \to (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n$ be an embedding, where $\Sigma^n(u)$ is an entire graph that lies in a slab of $[t_1, t_2] \times \mathbb{H}^n$ with $-\frac{\pi}{2} < t_2 < 0$ or $0 < t_1 < \frac{\pi}{2}$. Assume that the nonvanishing mean curvature H satisfies*

$$-|\tan t| \leq H \leq |\tan t|;$$

then if $|Du| \in \mathcal{L}^1(\mathbb{H}^n)$, the hypersurface $\Sigma^n(u)$ is isometric to $\mathbb{H}^n$.

4. Extensions to the k-th mean curvatures

In this section, we extend the result of Theorem 3.2 to higher order mean curvatures. Under this condition, we need to consider the warped product spaces $\mathbb{R}\times_f M^n$ satisfy the following convergence condition:

$$\mathrm{Ric}_M \geq (n-1)\sup_{\mathbb{R}}(f'^2 - f''f)\langle\ ,\ \rangle_M, \tag{8}$$

where Ric_M is the Ricci tensor of the fiber M^n.

Theorem 4.1 *Let $\overline{M}=\mathbb{R}\times_f M^n$ be a Riemannian warped product space and $\Sigma^n(u)$ be an entire graph that lies in a slab $[t_1,t_2]\times M^n$, and the sectional curvature of fiber M^n satisfies the convergence condition (4.1). Let the mean curvature H be bounded, $H_2>0$, and the warping function f satisfies C_1 or C_2 on $[t_1,t_2]$. Assume that one of the following conditions holds:*

(i) $0<\frac{H_2}{H}\leq\frac{|f'|}{f}(u)$.

(ii) $-\frac{|f'|}{f}(u)\leq\frac{H_2}{H}<0$.

Then if $|Du|\in\mathcal{L}^1(M^n)$, $\Sigma^n(u)$ is a slice $\{t_0\}\times M^n$, where $t_0\in[t_1,t_2]$.

Proof (i) In a similar way as in Theorem 3.2, we have that the Omori–Yau maximum principle for Laplacian holds on Σ^n; thus we get Theorem 3.1 hold true on $\Sigma^n(u)$. From the definition of P_1 we have that it is bounded on $\Sigma^n(u)$ whenever $|A|$ is itself bounded on $\Sigma^n(u)$. Since H is bounded, $H_2>0$, and $|A|^2=n^2H^2-n(n-1)H_2$ we obtain that there exists a constant c, such that $|P_1\nabla u|\leq|P_1||\nabla g(u)|\leq cf(u)|\nabla u|\in\mathcal{L}^1(\Sigma^n(u))$.

From Lemma 2.1 we have

$$\begin{aligned}\mathrm{div}(P_1(\nabla g(u))) &= -f(u)\langle N,\partial_t\rangle(\mathrm{Ric}_M(N^*,N^*)-(n-1)(f'^2-f''f)(u)\langle N^*,N^*\rangle_M)+\\ &\quad + n(n-1)f(u)H(\frac{f'}{f}(u)+\langle N,\partial_t\rangle\frac{H_2}{H}),\end{aligned}$$

where N^* denotes the projection of N onto the fiber M^n and $N^*=N-\langle N,\partial_t\rangle\partial_t$.

Using the hypothesis we obtain that $H_1>0$; then by Theorem 3.1 we have $f'\langle N,\partial_t\rangle<0$.

If $f'>0$, then $\langle N,\partial_t\rangle<0$, and

$$\frac{f'}{f}(u)+\langle N,\partial_t\rangle\frac{H_2}{H}\geq\frac{f'}{f}(u)-\frac{H_2}{H}\geq 0.$$

Now, using the hypothesis that Ric_M satisfies the convergence condition (4.1), we obtain that $\mathrm{div}(P_1(\nabla g(u)))\geq 0$.

On the other hand, if $f'<0$, then $\langle N,\partial_t\rangle>0$, and

$$\frac{f'}{f}(u)+\langle N,\partial_t\rangle\frac{H_2}{H}\leq\frac{f'}{f}(u)+\frac{H_2}{H}\leq 0.$$

Moreover, from Ric_M satisfies the convergence condition (4.1), we also obtain that $\mathrm{div}(P_1(\nabla g(u)))$ does not change sign on $\Sigma^n(u)$. Therefore, by using the same argument of the proof of Theorem 3.2, we can apply Lemma 2.5 to conclude that $\Sigma^n(u)$ is a slice $\{t_0\}\times M^n$, where $t_0\in[t_1,t_2]$.

(ii) Using the hypothesis we obtain that $H_1<0$; thus $f\langle N,\partial_t\rangle>0$.

If $f'>0$, $\langle N,\partial_t\rangle>0$, and

$$H(\frac{f'}{f}(u)+\langle N,\partial_t\rangle\frac{H_2}{H})\leq H(\frac{f'}{f}(u)+\frac{H_2}{H})\leq 0,$$

together with Ric_M satisfies the convergence condition (4.1), we obtain $\mathrm{div}(P_1(\nabla g(u))) \le 0$.

If $f' < 0$, then $\langle N, \partial_t \rangle < 0$, and

$$H(\frac{f'}{f}(u) + \langle N, \partial_t \rangle \frac{H_2}{H}) \ge H(\frac{f'}{f}(u) - \frac{H_2}{H}) \ge 0;$$

thus, using the hypothesis (4.1), we also obtain that $\mathrm{div}(P_1(\nabla g(u))) \ge 0$.

Now as the proof of case (i) we obtain $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$ with $t_0 \in [t_1, t_2]$. □

From the proof of Theorem 4.1 we get the following corollary.

Corollary 4.1 *Let $\overline{M} = \mathbb{R} \times_f M^n$ be a Riemannian warped product space and $\Sigma^n(u)$ be an entire graph that lies in a slab $[t_1, t_2] \times M^n$, where the sectional curvature of the fiber M^n satisfies the convergence condition (4.1). Suppose that nonvanishing H is bounded, $H_2 > 0$, and the warping function f satisfies C_1 or C_2 on $[t_1, t_2]$. Then if $|Du| \in \mathcal{L}^1(M^n)$ and*

$$-\frac{|f'|}{f}(u) \le \frac{H_2}{H} \le \frac{|f'|}{f}(u),$$

$\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

Motivated by the previous results, in the next result we generalize Theorem 4.1 to the case of higher order mean curvatures.

Theorem 4.2 *Let $\overline{M} = \mathbb{R} \times_f M^n$ be a Riemannian warped product space, where the sectional curvature of fiber M^n is constant κ and satisfies the convergence condition (4.1). $\Sigma^n(u)$ is an entire graph that lies in a slab $[t_1, t_2] \times M^n$. Suppose that H is bounded and the warping function f satisfies C_1 or C_2 on $[t_1, t_2]$. Assume that $H_k > 0$ and one of the following conditions is satisfied:*

(i) $0 < \frac{H_{k+1}}{H_k} \le \frac{|f'|}{f}(u)$.

(ii) $-\frac{|f'|}{f}(u) \le \frac{H_{k+1}}{H_k} < 0$.

Then if $f(u)$ has a local maximum on $\Sigma^n(u)$ and $|Du| \in \mathcal{L}^1(M^n)$, $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

Proof (i) In a similar way as in Theorem 3.2, we get that Corollary 3.1 hold true on $\Sigma^n(u)$. Since $f(u)$ has a local maximum on $\Sigma^n(u)$, then by Lemma 2.4 we have that there exists an elliptic point on Σ^n. From the definition of elliptic point and (2.1) we get that when $H_k H_{k+1} > 0$, we must have both H_{k+1} and H_k positive. Furthermore, P_j is positive definite for all $j \in \{1, \cdots, k\}$. Moreover, $H_2 > 0$ on Σ^n.

From the definition, P_k is bounded on $\Sigma^n(u)$ whenever $|A|$ is itself bounded on $\Sigma^n(u)$. Since H is bounded and $H_2 > 0$, we obtain that there exists a constant c, such that $|P_k| < c$; thus,

$$|P_k \nabla u| \le |P_k||\nabla g(u)| \le c f(u) |\nabla u| \in \mathcal{L}^1(\Sigma^n(u)).$$

Consequently, from Lemma 2.1 we have

$$\begin{aligned} \mathrm{div}(P_k(\nabla g(u))) &= -(n-k)\langle N, \partial_t \rangle \frac{\kappa - (f'^2 - f'' f)}{f} \langle P_{k-1} \nabla u, \nabla u \rangle + \\ &\quad + c_k H_k f(u) (\frac{f'}{f}(u) + \langle N, \partial_t \rangle \frac{H_{k+1}}{H_k}). \end{aligned}$$

Moreover, from the existence of an elliptic point and the hypothesis, we have P_{k-1} positive definite and $f'\langle N, \partial_t\rangle < 0$.

Similarly as in the proof of Theorem 4.1 (i) and from the inequality (4.1) we also obtain that $\mathrm{div}(P_k(\nabla g(u)))$ does not change sign on $\Sigma^n(u)$. Therefore, by Lemma 2.5 we conclude that $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

(ii) Using the hypothesis we obtain $H_{k+1} < 0$; since there exists an elliptic point, k must be even; then P_{k-1} are negative definite. By Corollary 3.1 we have $f'(u)\langle N, \partial_t\rangle > 0$.
Hence, proceeding as the case (ii) of Theorem 4.1 we conclude that $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$ with $t_0 \in [t_1, t_2]$. □

Similarly to Corollary 4.1, we obtain the following result.

Corollary 4.2 *Let $\overline{M} = \mathbb{R} \times_f M^n$ be a Riemannian warped product space, where the sectional curvature of fiber M^n is constant κ and satisfies the convergence condition (4.1). $\Sigma^n(u)$ is an entire graph that lies in a slab $[t_1, t_2] \times M^n$. Suppose that the mean curvature H is bounded and the warping function f satisfies C_1 or C_2 on $[t_1, t_2]$. Assume that nonvanishing H_{k+1} and positive H_k satisfy*

$$-\frac{|f'|}{f}(u) \leq \frac{H_{k+1}}{H_k} \leq \frac{|f'|}{f}(u).$$

Then if $f(u)$ has a local maximum on $\Sigma^n(u)$ and $|Du| \in \mathcal{L}^1(M^n)$, $\Sigma^n(u)$ is a slice $\{t_0\} \times M^n$, where $t_0 \in [t_1, t_2]$.

Considering the warped product as $\mathbb{R} \times_{\cosh t} \mathbb{H}^n$, it is easy to know that warping function $f = \cosh t$ satisfies C_2. As an application of Theorems 4.1 and 4.2, we have the following.

Corollary 4.3 *Let $\overline{M}^{n+1} = \mathbb{R} \times_{\cosh t} \mathbb{H}^n$ be a warped product space and $\Sigma^n(u)$ be an entire graph that lies in a slab of $[t_1, t_2] \times M^n$. Assume that the mean curvature H is bounded, nonvanishing higher order mean curvature H_{k+1} and H_k $(1 \leq k \leq n-1)$ satisfy $H_k > 0$, and*

$$-|\tanh t| \leq \frac{H_{k+1}}{H_k} \leq |\tanh t|;$$

then if $|Du| \in \mathcal{L}^1(\mathbb{H}^n)$, the hypersurface $\Sigma^n(u)$ is isometric to $\mathbb{H}^n$.

When we consider the warped product space $\mathbb{R} \times_t \mathbb{S}^n$, it is easy to show that warping function $f = t$ satisfies C_1 but not C_2.

Corollary 4.4 *Let $\overline{M}^{n+1} = \mathbb{R} \times_t \mathbb{S}^n$ be a Riemannian warped space and $\varphi : \Sigma^n(u) \to \mathbb{R} \times_t \mathbb{S}^n$ be an embedding that lies in a slab of $\overline{M}$, such that $\Sigma^n(u) = \varphi(\Sigma)$. Assume that the mean curvature H is bounded, nonvanishing higher order mean curvature H_{k+1} and H_k for any $1 \leq k \leq n-1$ satisfy $H_k > 0$, and*

$$-\frac{1}{|\varphi(p)|} \leq \frac{H_{k+1}}{H_k} \leq \frac{1}{|\varphi(p)|};$$

then if $|Du| \in \mathcal{L}^1(\mathbb{R}^n)$, the hypersurface $\Sigma^n(u)$ is isometric to $\mathbb{S}^n$.

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