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ZOHREH HABIBI

MARYAM JAHANGIRI

KHADIJEH AHMADI AMOLI

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## On the attached prime ideals of local cohomology modules defined by a pair of ideals

Zohreh HABIBI<sup>1,\*</sup>, Maryam JAHANGIRI<sup>2</sup>, Khadijeh AHMADI AMOLI<sup>1</sup>

<sup>1</sup>Payame Noor University, Tehran, Iran

<sup>2</sup>Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

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**Abstract:** Let  $I$  and  $J$  be two ideals of a commutative Noetherian ring  $R$  and  $M$  be an  $R$ -module of dimension  $d$ . For each  $i \in \mathbb{N}_0$  let  $H_{I,J}^i(-)$  denote the  $i$ -th right derived functor of  $\Gamma_{I,J}(-)$ , where  $\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}$ . If  $R$  is a complete local ring and  $M$  is finite, then attached prime ideals of  $H_{I,J}^{d-1}(M)$  are computed by means of the concept of co-localization. Moreover, we illustrate the attached prime ideals of  $H_{I,J}^t(M)$  on a nonlocal ring  $R$ , for  $t = \dim M$  and  $t = \text{cd}(I, J, M)$ , where  $\text{cd}(I, J, M)$  is the last nonvanishing level of  $H_{I,J}^i(M)$ .

**Key words:** Local cohomology modules with respect to a pair of ideals, attached prime ideals, co-localization

### 1. Introduction

Throughout this paper,  $R$  denotes a commutative Noetherian ring and  $M$  an  $R$ -module and  $I$  and  $J$  stand for two ideals of  $R$ . For all  $i \in \mathbb{N}_0$  the  $i$ -th local cohomology functor with respect to  $(I, J)$ , denoted by  $H_{I,J}^i(-)$ , is defined by Takahashi et al. in [11] as the  $i$ -th right derived functor of the  $(I, J)$ -torsion functor  $\Gamma_{I,J}(-)$ , where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor  $H_I^i(-)$  when  $J = 0$  (see [3]).

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules  $H_I^i(M)$  ([10]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [4, 5, 11].

The second section of this paper is devoted to studying the attached prime ideals of local cohomology modules with respect to a pair of ideals by means of co-localization. The concept of co-localization was introduced by Richardson in [9].

Let  $(R, \mathfrak{m})$  be local and  $M$  be a finite  $R$ -module of dimension  $d$ . If  $c$  is a nonnegative integer such that  $H_{I,J}^i(R) = 0$  for all  $i > c$  and  $H_{I,J}^c(R)$  is representable, then we illustrate the attached prime ideals of  ${}^p H_{I,J}^c(M)$  (see Theorem 2.6). In addition, if  $R$  is complete, then we make use of Theorem 2.6 to prove that in a special case

$$\text{Att}(H_{I,J}^{d-1}(M)) \subseteq T \cup \text{Assh}(M) \text{ and } T \subseteq \text{Att}(H_{I,J}^{d-1}(M)),$$

\*Correspondence: z.habibi@pnu.ac.ir

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where

$$T = \{\mathfrak{p} \in \text{Supp}(M) : \dim M/\mathfrak{p}M = d - 1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I + \mathfrak{p}} = \mathfrak{m}\},$$

(see Theorem 2.8).

In [4, Theorem 2.1] the set of attached prime ideals of  $H_{I,J}^{\dim M}(M)$  was computed on a local ring. We generalize this theorem to the nonlocal case (see Proposition 2.13). The authors in [6, 2.4] specified a subset of attached prime ideals of ordinary top local cohomology module  $H_I^{\text{cd}(I,M)}(M)$ . We improve it for  $H_{I,J}^{\text{cd}(I,J,M)}(M)$  over a not necessarily local ring, where  $\text{cd}(I, J, M) = \sup\{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$  with the convention that  $\text{cd}(I, M) = \text{cd}(I, 0, M)$  (see Theorem 2.15).

## 2. Attached prime ideals

In this section we study the set of attached prime ideals of local cohomology modules with respect to a pair of ideals.

**Remark 2.1** *Following [9], for a multiplicatively closed subset  $S$  of the local ring  $(R, \mathfrak{m})$ , the co-localization of the  $R$ -module  $M$  relative to  $S$  is defined to be the  $S^{-1}R$ -module  $S_{-1}(M) := D_{S^{-1}R}(S^{-1}D_R(M))$ , where  $D_R(-)$  is the Matlis dual functor  $\text{Hom}_R(-, E_R(R/\mathfrak{m}))$ . If  $S = R \setminus \mathfrak{p}$  for some  $\mathfrak{p} \in \text{Spec}(R)$  we write  ${}^{\mathfrak{p}}M$  for  $S_{-1}(M)$ .*

*Richardson in [9, 2.2] proved that if  $M$  is a representable  $R$ -module, then so is  $S_{-1}(M)$  and  $\text{Att}(S_{-1}M) = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in \text{Att}(M), \mathfrak{p} \cap S = \emptyset\}$ . Therefore, in order to get some results about attached prime ideals of a module, it is convenient to study the attached prime ideals of the co-localization of it.*

**Lemma 2.2** *Let  $(R, \mathfrak{m})$  be a local ring,  $\mathfrak{a}$  be an ideal of  $R$ , and  $\mathfrak{p} \in \text{Spec}(R)$  with  $\mathfrak{a} \subseteq \mathfrak{p}$ . Let  $R' = R/\mathfrak{a}$  and  $\mathfrak{p}' = \mathfrak{p}/\mathfrak{a}$ . Then for any  $R'$ -module  $X$  and  $R'_{\mathfrak{p}'}$ -module  $Y$ , the following isomorphisms hold:*

- (i)  $D_R(X) \cong D_{R'}(X)$  as  $R$ -modules.
- (ii)  $D_R(X)_{\mathfrak{p}} \cong D_{R'}(X)_{\mathfrak{p}'}$  as  $R_{\mathfrak{p}}$ -modules.
- (iii)  $D_{R_{\mathfrak{p}}}(Y) \cong D_{R'_{\mathfrak{p}'}}(Y)$  as  $R_{\mathfrak{p}}$ -modules.

**Proof** (i) By using [3, Lemma 10.1.15], we get

$$\begin{aligned} \text{Hom}_R(X, E_R(R/\mathfrak{m})) &\cong \text{Hom}_R(X \otimes_{R'} R', E_R(R/\mathfrak{m})) \\ &\cong \text{Hom}_{R'}(X, \text{Hom}_R(R', E_R(R/\mathfrak{m}))) \\ &\cong \text{Hom}_{R'}(X, (0 :_{E_R(R/\mathfrak{m})} \mathfrak{a})) \\ &\cong \text{Hom}_{R'}(X, E_{R'}(R/\mathfrak{m})). \end{aligned}$$

(ii) It is clear, by (i).

(iii) It is enough to apply (i), by substituting  $R$  and  $X$  with  $R_{\mathfrak{p}}$  and  $Y$ , respectively.

□

The following lemmas will be used in the rest of the paper.

**Lemma 2.3** *Let  $T$  be an  $R$ -linear covariant right exact functor on the category of  $R$ -modules and  $R$ -homomorphisms. Then there is a natural equivalence of functors,  $T(-) \cong T(R) \otimes_R -$  on the category of finite  $R$ -modules.*

**Proof** See [3, 6.1.8]. □

**Lemma 2.4** *Let  $(R, \mathfrak{m})$  be a local ring and  $M$  be a nonzero finite  $R$ -module of dimension  $d$ . Then the following statements are equivalent:*

- (i)  $H_{I,J}^d(M) = 0$ ,
- (ii)  $\dim \hat{R}/(I\hat{R} + \mathfrak{p}) > 0$ , for any prime ideal  $\mathfrak{p}$  of  $\text{Supp}_R(\hat{R} \otimes_R M/JM)$  satisfying  $\dim \hat{R}/\mathfrak{p} = d$ , where  $\hat{R}$  denotes the  $\mathfrak{m}$ -adic completion of  $R$ .

**Proof** See [4, Theorem 2.4]. □

**Lemma 2.5** *Let  $M$  and  $N$  be two  $R$ -modules such that  $M$  has a secondary representation. Then  $M \otimes_R N$  has a secondary representation and  $\text{Att}_R(M \otimes_R N) \subseteq \text{Att}_R(M) \cap \text{Supp}_R(N)$ .*

**Proof** See [12, 3.1]. □

In [8, 2.1 and 2.2] the following theorems have been proved for the attached prime ideals of  $H_I^d(R)$  and  $H_I^{d-1}(R)$ , where  $d = \dim R$ . Here we generalize these theorems for the local cohomology modules of  $M$  with respect to a pair of ideals when  $M$  is a finite  $R$ -module with  $\dim M = d$ .

**Theorem 2.6** *Let  $(R, \mathfrak{m})$  be a local ring,  $M$  be a finite  $R$ -module, and  $\mathfrak{p} \in \text{Spec}(R)$ . Assume that  $c = \text{cd}(I, J, R)$  and  $H_{I,J}^c(R)$  is representable. Then*

1.  $\text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^c(M)) \subseteq \{\mathfrak{q}R_{\mathfrak{p}} : \dim M/\mathfrak{q}M \geq c, \mathfrak{q} \subseteq \mathfrak{p}, \text{ and } \mathfrak{q} \in \text{Spec}(R)\}$ .
2. If  $R$  is complete, then

$$\text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^{\dim M}(M)) = \{ \mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \text{Supp}(M), \dim M/\mathfrak{q}M = \dim M, J \subseteq \mathfrak{q} \subseteq \mathfrak{p}, \text{ and } \sqrt{I + \mathfrak{q}} = \mathfrak{m} \}.$$

**Proof**

(1) Let  $\mathfrak{q}R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^c(M))$ . By Lemma 2.5 and Remark 2.1, we have  $H_{I,J}^c(M)$  is representable and  $\text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^c(M)) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \text{Att}(H_{I,J}^c(M)) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$ . Moreover, using Lemma 2.3 and [1, 2.11]

$$\text{Att}(H_{I,J}^c(M/\mathfrak{q}M)) = \text{Att}(H_{I,J}^c(M)) \cap \text{Supp}(R/\mathfrak{q}).$$

This implies that  $H_{I,J}^c(M/\mathfrak{q}M) \neq 0$  and consequently  $\dim M/\mathfrak{q}M \geq c$ .

(2) Let  $\mathfrak{p} \in \text{Supp}(M)$ . Put  $d := \dim M$ ,  $\bar{R} = R/\text{Ann}_R M$ , and

$$T := \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \text{Supp}(M), \dim M/\mathfrak{q}M = d, J \subseteq \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \sqrt{I + \mathfrak{q}} = \mathfrak{m}\}.$$

Since  $\dim_{\bar{R}} M = \dim_R M$ , [11, 2.7] and Lemma 2.2 imply that  $\bar{\mathfrak{p}}H_{\bar{I}, \bar{J}}^d(M) \cong {}^{\mathfrak{p}}H_{I,J}^d(M)$ , as  $R_{\mathfrak{p}}$ -modules.

Therefore, by [3, 8.2.5],  $\mathfrak{q} \in \text{Att}_{\bar{R}_{\bar{\mathfrak{p}}}}(\bar{\mathfrak{p}}H_{\bar{I}, \bar{J}}^d(M))$  if and only if

$$\mathfrak{q} \cap R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}(\bar{\mathfrak{p}}H_{\bar{I}, \bar{J}}^d(M)) = \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^d(M)).$$

Now, without loss of generality, we may assume that  $M$  is faithful and  $\dim R = d$ . If  $H_{I,J}^d(M) = 0$ , then  $\text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^d(M)) = \emptyset$ . Assume that  $T \neq \emptyset$  and  $\mathfrak{q}R_{\mathfrak{p}} \in T$ . Since  $\dim M/\mathfrak{q}M = \dim R$ , we have  $\dim R/\mathfrak{q} = d$ . On the other hand,  $\mathfrak{q} \in \text{Supp}(M/JM)$ . Thus, by Lemma 2.4,  $\dim R/(I + \mathfrak{q}) > 0$ , which contradicts  $\sqrt{I + \mathfrak{q}} = \mathfrak{m}$ . Hence  $T = \emptyset$ .

Now we assume that  $H_{I,J}^d(M) \neq 0$ .

$\supseteq$ : Let  $\mathfrak{q}R_{\mathfrak{p}} \in T$ . Since  $H_{I,J}^d(M)$  is an Artinian  $R$ -module (see [5, 2.1]), by Remark 2.1, it is enough to show that  $\mathfrak{q} \in \text{Att}(H_{I,J}^d(M))$ . As  $M/\mathfrak{q}M$  is  $J$ -torsion with dimension  $d$  and  $\sqrt{I + \mathfrak{q}} = \mathfrak{m}$ , by [3, 4.2.1 and 6.1.4],

$$H_{I,J}^d(M/\mathfrak{q}M) \cong H_I^d(M/\mathfrak{q}M) \cong H_{I(R/\mathfrak{q})}^d(M/\mathfrak{q}M) \cong H_{\mathfrak{m}/\mathfrak{q}}^d(M/\mathfrak{q}M) \neq 0.$$

Hence Lemma 2.3 and [1, 2.11] imply that  $\emptyset \neq \text{Att}(H_{I,J}^d(M/\mathfrak{q}M)) = \text{Att}(H_{I,J}^d(M)) \cap \text{Supp}(R/\mathfrak{q})$ . Let  $\mathfrak{q}_0 \in \text{Att}(H_{I,J}^d(M))$  be such that  $\mathfrak{q} \subset \mathfrak{q}_0$ . Thus  $\dim M/\mathfrak{q}_0M < d$ . On the other hand, by Remark 2.1,  $\mathfrak{q}_0R_{\mathfrak{q}_0} \in \text{Att}_{R_{\mathfrak{q}_0}}({}^{\mathfrak{q}_0}H_{I,J}^d(M))$  and this implies that  $\dim M/\mathfrak{q}_0M \geq d$ , which is a contradiction. Therefore,  $\mathfrak{q} = \mathfrak{q}_0$ .

$\subseteq$ : Let  $\mathfrak{q}R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^d(M))$ . As we have seen in the proof of part (1),  $\dim M/\mathfrak{q}M = d$  and  $\mathfrak{q} \subseteq \mathfrak{p}$ . Thus, by [11, 2.7],

$$H_{IR/\mathfrak{q},JR/\mathfrak{q}}^d(M/\mathfrak{q}M) \cong H_{I,J}^d(M/\mathfrak{q}M) \neq 0.$$

Now, by Lemma 2.4, there exists  $\mathfrak{r}/\mathfrak{q} \in \text{Supp}(R/\mathfrak{q} \otimes_{R/\mathfrak{q}} \frac{M/\mathfrak{q}M}{(JR/\mathfrak{q})(M/\mathfrak{q}M)})$  such that  $\dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = d$  and  $\dim \frac{R/\mathfrak{q}}{IR/\mathfrak{q} + \mathfrak{r}/\mathfrak{q}} = 0$ . Since  $\mathfrak{q}R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^d(M))$ , we have  $\mathfrak{q} \in \text{Att}(H_{I,J}^d(M))$  and so  $\mathfrak{q} \in \text{Supp}(M) \cap V(J)$ . Hence  $\mathfrak{q}/\mathfrak{q} \in \text{Supp}_{R/\mathfrak{q}}(M/\mathfrak{q}M)$  and then

$$\dim R/\mathfrak{q} = \dim M/\mathfrak{q}M = d = \dim \frac{R/\mathfrak{q}}{\mathfrak{r}/\mathfrak{q}} = \dim R/\mathfrak{r}.$$

Therefore,  $\dim R/\mathfrak{q} = \dim R/\mathfrak{r}$ , which shows that  $\mathfrak{q} = \mathfrak{r}$ . Thus  $\sqrt{I + \mathfrak{q}} = \mathfrak{m}$ . □

**Remark 2.7** *The inclusion in Theorem 2.6 (1) is not an equality in general. Let the assumption be as in Theorem 2.6. Assume that  $H_{I,J}^d(M) = 0$ ,  $\mathfrak{p} \in \text{Min}(M)$ , and  $\dim M/\mathfrak{p}M = d$ . Then  $\text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^d(M)) = \emptyset$ . However,*

$$\{\mathfrak{q}R_{\mathfrak{p}} : \dim M/\mathfrak{q}M = d, \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \mathfrak{q} \in \text{Supp}(M)\} = \{\mathfrak{p}R_{\mathfrak{p}}\}.$$

**Theorem 2.8** *Let  $(R, \mathfrak{m})$  be a complete local ring and  $M$  be a finite  $R$ -module with dimension  $d$ . Assume that  $H_{I,J}^i(R) = 0$  for all  $i > d - 1$  and  $H_{I,J}^{d-1}(R)$  is representable. Then*

1.

$$\begin{aligned} \text{Att}_R(H_{I,J}^{d-1}(M)) \subseteq & \{ \mathfrak{p} \in \text{Supp}(M) : \dim M/\mathfrak{p}M = d - 1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I + \mathfrak{p}} = \mathfrak{m} \} \\ & \cup \text{Assh}(M). \end{aligned}$$

2.

$$\{\mathfrak{p} \in \text{Supp}(M) : \dim M/\mathfrak{p}M = d - 1, J \subseteq \mathfrak{p} \text{ and } \sqrt{I + \mathfrak{p}} = \mathfrak{m}\} \subseteq \text{Att}(H_{I,J}^{d-1}(M)).$$

**Proof** (1) In the case where  $H_{I,J}^{d-1}(M) = 0$  there is nothing to say. Therefore, we assume that  $H_{I,J}^{d-1}(M) \neq 0$ . Note that, by [11, 4.8] and Lemma 2.5,  $H_{I,J}^{d-1}(M)$  is representable and  $\text{Att}(H_{I,J}^{d-1}(M)) \subseteq \text{Supp}(M)$ . Now let  $\mathfrak{p} \in \text{Att}(H_{I,J}^{d-1}(M))$ . Since  $\mathfrak{p}R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^{d-1}(M))$ , by Theorem 2.6 (1),  $\dim M/\mathfrak{p}M \geq d - 1$ .

If  $\dim M/\mathfrak{p}M = d$ , then  $\dim R/\mathfrak{p} = d$  and so  $\mathfrak{p} \in \text{Assh}(M)$ .

Now assume that  $\dim M/\mathfrak{p}M = d - 1$ . Since  $\mathfrak{p} \in \text{Att}(H_{I,J}^{d-1}(M))$ ,  $H_{IR/\mathfrak{p}, JR/\mathfrak{p}}^{d-1}(M/\mathfrak{p}M) \cong H_{I,J}^{d-1}(M/\mathfrak{p}M) \neq 0$ . Thus, by Lemma 2.4, there exists  $\mathfrak{r}/\mathfrak{p} \in \text{Supp}(\frac{M/\mathfrak{p}M}{(JR/\mathfrak{p})(M/\mathfrak{p}M)})$  such that  $\dim \frac{R}{\mathfrak{r}} = d - 1$  and  $\dim \frac{R}{I+\mathfrak{r}} = 0$ . Hence  $\mathfrak{r} = \mathfrak{p}, J \subseteq \mathfrak{p}$ , and  $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$ .

(2) Let  $\mathfrak{p} \in \text{Supp}(M)$ ,  $J \subseteq \mathfrak{p}$ ,  $\dim M/\mathfrak{p}M = d - 1$ , and  $\sqrt{I + \mathfrak{p}} = \mathfrak{m}$ . Then, by Lemma 2.5 and Theorem 2.6 (2),  $H_{I,J}^{d-1}(M)$  is representable,  $\mathfrak{p}R_{\mathfrak{p}} \in \text{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{I,J}^{d-1}(M/\mathfrak{p}M))$ , and so  $\mathfrak{p} \in \text{Att}(H_{I,J}^{d-1}(M/\mathfrak{p}M))$ . Now the proof is completed by considering the epimorphism  $H_{I,J}^{d-1}(M) \rightarrow H_{I,J}^{d-1}(M/\mathfrak{p}M)$ . □

In the rest of the paper, following [11], we use the notation

$$W(I, J) := \{\mathfrak{p} \in \text{Spec}(R) : I^n \subseteq \mathfrak{p} + J \text{ for an integer } n \geq 1\}$$

and

$$\widetilde{W}(I, J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R; I^n \subseteq \mathfrak{a} + J \text{ for an integer } n \geq 1\}.$$

The following lemma can be proved using [11, 3.2].

**Lemma 2.9** For any nonnegative integer  $i$  and  $R$ -module  $M$ ,

- (i)  $\text{Supp}(H_{I,J}^i(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Supp}(H_{\mathfrak{a}}^i(M))$ .
- (ii)  $\text{Supp}(H_{I,J}^i(M)) \subseteq \text{Supp}(M) \cap W(I, J)$ .

**Proof**

- (ii) By (i),  $\text{Supp}(H_{I,J}^i(M)) \subseteq \text{Supp}(M) \cap V(I) \subseteq \text{Supp}(M) \cap W(I, J)$ . □

**Corollary 2.10** Let  $M$  be an  $R$ -module and  $c = \text{cd}(I, J, R)$ . Assume that  $M$  is representable or  $H_{I,J}^c(R)$  is finite. Then

$$\text{Att}(H_{I,J}^c(M)) \subseteq \text{Att}(M) \cap W(I, J).$$

**Proof**

By [11, 4.8], [1, 2.11], Lemma 2.5, and Lemma 2.9(ii), we have

$$\begin{aligned} \text{Att}(H_{I,J}^c(M)) &= \text{Att}(M \otimes H_{I,J}^c(R)) \subseteq \text{Att}(M) \cap \text{Supp}(H_{I,J}^c(R)) \\ &\subseteq \text{Att}(M) \cap W(I, J). \end{aligned}$$

□

Applying the set of attached prime ideals of the top local cohomology module in [4, Theorem 2.1], we obtain another presentation for it.

**Proposition 2.11** *Let  $(R, \mathfrak{m})$  be a local ring and  $\hat{R}$  denotes the  $\mathfrak{m}$ -adic completion of  $R$ . Suppose that  $M$  is a finite  $R$ -module of dimension  $d$ . Then*

$$\text{Att}_R(H_{I,J}^d(M)) = \{ \mathfrak{q} \cap R : \mathfrak{q} \in \text{Supp}_{\hat{R}}(\hat{R} \otimes_R M/JM), \dim(\hat{R}/\mathfrak{q}) = d, \\ \text{and } \dim \hat{R}/(I\hat{R} + \mathfrak{q}) = 0 \}.$$

**Proof** Denote the set on the right-hand side of the desired equality by  $T$ . It is clear that by Lemma 2.4,  $H_{I,J}^d(M) = 0$  if and only if  $T = \emptyset$ . Assume that  $H_{I,J}^d(M) \neq 0$  and  $\mathfrak{p} \in \text{Supp}(M/JM)$  with the property that  $\text{cd}(I, R/\mathfrak{p}) = d$ . Let  $\mathfrak{q} \in \text{Ass}(M/JM)$  be such that  $\mathfrak{q} \subseteq \mathfrak{p}$ . Then

$$d = \text{cd}(I, R/\mathfrak{p}) \leq \text{cd}(I, R/\mathfrak{q}) \leq \dim R/\mathfrak{q} \leq \dim M/JM \leq \dim M = d$$

implies that  $\mathfrak{p} = \mathfrak{q} \in \text{Ass}(M/JM)$  and  $\dim M/JM = d$ . Now the claim follows from [12, 3.10] and [4, Theorem 2.1]. □

The following lemma, which can be proved by using an argument similar to the proof of [11, 4.3], will be applied in the rest of the paper.

**Lemma 2.12** *Let  $M$  be a finite  $R$ -module. Suppose that  $J \subseteq J(R)$ , where  $J(R)$  denotes the Jacobson radical of  $R$ , and let  $d = \dim M/JM$ . Then  $H_{I,J}^i(M) = 0$  for all  $i > d$ .*

Using Lemma 2.12, we can compute  $\text{Att}(H_{I,J}^{\dim M}(M))$  in the nonlocal case as a generalization of [7, 2.5].

**Proposition 2.13** *Let  $M$  be a finite  $R$ -module of dimension  $d$  and  $J \subseteq J(R)$ . Then  $H_{I,J}^d(M)$  is an Artinian  $R$ -module and*

$$\text{Att}(H_{I,J}^d(M)) = \text{Att}(H_I^d(M/JM)) \\ = \{ \mathfrak{p} \in \text{Ass}(M) \cap V(J) : \text{cd}(I, R/\mathfrak{p}) = d \}.$$

**Proof** In view of Lemma 2.12, [11, 4.3] holds for the nonlocal case. Now the assertion follows by applying the same method of the proofs of [5, 2.1] and [4, Theorem 2.1 and Proposition 2.1]. □

**Corollary 2.14** *Suppose that  $J \subseteq J(R)$  and  $M$  is a finite  $R$ -module such that  $\dim M = d$ . Then*

$$\text{Att}\left(\frac{H_{I,J}^d(M)}{JH_{I,J}^d(M)}\right) = \{ \mathfrak{p} \in \text{Supp}(M) \cap V(J) : \text{cd}(I, R/\mathfrak{p}) = d \}.$$

**Proof** Let  $\bar{R} = R/\text{Ann}_R M$ . Using [11, 2.7],  $H_{I,J}^d(M) \cong H_{I\bar{R}, J\bar{R}}^d(M)$  and also for a prime  $\mathfrak{p} \in \text{Supp}(M) \cap V(J)$ ,  $\text{cd}(I\bar{R}, \bar{R}/\mathfrak{p}) = \text{cd}(I, R/\mathfrak{p})$ . Thus we may assume that  $M$  is faithful and so  $\dim R = d$ . By virtue of Lemma 2.3,  $H_I^d(M/JM) \cong H_{I,J}^d(M/JM) \cong \frac{H_{I,J}^d(M)}{JH_{I,J}^d(M)}$ . Now the assertion follows by Proposition 2.13. □

The final result of this section is a generalization of [6, 2.4] in the nonlocal case for local cohomology modules with respect to a pair of ideals. Since this reference is unpublished, we provide the proof of [6, 2.4] here (with the needed changes) for the reader's convenience.

**Theorem 2.15** *Let  $J \subseteq J(R)$  and  $M$  be a finite  $R$ -module. Then*

$$\{\mathfrak{p} \in \text{Ass}(M) \cap V(J) : \text{cd}(I, R/\mathfrak{p}) = \dim R/\mathfrak{p} = \text{cd}(I, J, M)\} \subseteq \text{Att}(H_{I,J}^{\text{cd}(I,J,M)}(M)).$$

*Equality holds if  $\text{cd}(I, J, M) = \dim M$ .*

**Proof** Let  $T$  be the set on the left-hand side of the desired inclusion. Suppose that  $S := \{\mathfrak{p} \in \text{Ass}(M) \cap V(J) : \dim(R/\mathfrak{p}) = \text{cd}(I, J, M)\}$  is nonempty. By [2, page 263, Proposition 4], there exists a submodule  $N$  of  $M$  with  $\text{Ass}(N) = \text{Ass}(M) \setminus S$  and  $\text{Ass}(M/N) = S$ . Since  $\text{Supp}(N) \subseteq \text{Supp}(M)$ , by virtue of [5, 3.2], we have  $\text{cd}(I, J, N) \leq \text{cd}(I, J, M)$ , and so there is an exact sequence

$$H_{I,J}^{\text{cd}(I,J,M)}(N) \rightarrow H_{I,J}^{\text{cd}(I,J,M)}(M) \rightarrow H_{I,J}^{\text{cd}(I,J,M)}(M/N) \rightarrow 0.$$

It follows that  $\text{Att}(H_{I,J}^{\text{cd}(I,J,M)}(M/N)) \subseteq \text{Att}(H_{I,J}^{\text{cd}(I,J,M)}(M))$ . Therefore, it is sufficient to show that  $T = \text{Att}(H_{I,J}^{\text{cd}(I,J,M)}(M/N))$ . Note that  $\dim R/\mathfrak{p} = \text{cd}(I, J, M)$  for all  $\mathfrak{p} \in \text{Ass}(M/N)$ . Thus,  $\text{cd}(I, J, M) = \dim M/N$ , and hence  $H_{I,J}^{\text{cd}(I,J,M)}(M/N) = H_{I,J}^{\dim M/N}(M/N)$  is an Artinian  $R$ -module and  $\text{Att}(H_{I,J}^{\text{cd}(I,J,M)}(M/N)) = T$ , by Proposition 2.13.

For the special case,  $\text{cd}(I, J, M) = \dim M$ , again see Proposition 2.13.  $\square$

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