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More characterizations of Dedekind domains and V-rings

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Abstract: In this paper, cyclic c -injective modules are introduced and investigated. It is shown that a commutative Noetherian domain is Dedekind if and only if every simple module is cyclic c -injective. Finally, it is shown that injectivity, principal injectivity, mininjectivity, and simple injectivity are all equal to characterize right V-rings, right GV-rings, right pV-rings, and WV-rings.

Key words: c -Injective modules, Dedekind domains, right V-rings, right GV-rings, WV-rings

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary (unless otherwise stated).

Recall that an integral domain R that is not a field is called a Dedekind domain if every nonzero proper ideal factors into primes. This is equivalent to R being an integrally closed, Noetherian domain with Krull dimension one (i.e. every nonzero prime ideal is maximal). Mermut et al. [10] showed that a commutative Noetherian domain is Dedekind if and only if every simple module is c -injective. According to Baccella [1], a ring R is called a *right V-ring* (resp. *right GV-ring*) if every simple (resp. simple singular) right R -module is R_R -injective, i.e. injective. According to López-Permouth et al. [9], a ring R is called a *right pV-ring* provided every simple right R -module is p -injective (there, p -injectivity is exactly principal injectivity). It is shown that being a right or left pV-ring is a Morita invariant. Following Holston [7], a ring R is called a *WV-ring* if every simple right R -module is injective relative to proper cyclics, i.e. M is R/I -injective for every right ideal I of R satisfying $R/I \not\cong R$. We ask here:

Can we characterize Dedekind domains, right V-rings, right GV-rings, right pV-rings, and WV-rings by weaker forms of injectivity instead of the defined ones?

The answer is definitely yes as we shall see in the last section of this text. In fact, commutative Noetherian domains are characterized by cyclic c -injective modules. Right V-rings, right GV-rings, right pV-rings, and WV-rings are characterized by mininjective modules and simple injective modules.

Let M be a right R -module and $S = \text{End}(M_R)$ its endomorphism ring. M is called a *simple module* if M and 0 are the only submodules of M . M is called a *cyclic module* if $M = mR$ for some $m \in M$, i.e. M

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is generated by a single element. A submodule X of M (denoted by $X \hookrightarrow M$) is called a *cyclic submodule* (resp. *simple or minimal submodule*) if X is a cyclic module (resp. simple module). A submodule N of M is said to be *essential* in M (denoted by $N \overset{*}{\hookrightarrow} M$) if N has nonzero intersection with every nonzero submodule of M , i.e. $N \cap A \neq 0$ for every $0 \neq A \hookrightarrow M$. A submodule $X \hookrightarrow M$ is called a *closed* submodule if it has no proper essential extension in M , i.e. for every submodule B of M , $X \overset{*}{\hookrightarrow} B$ implies $X = B$. Note that if X is a closed submodule of Y and Y is a closed submodule of M , then X is closed in M (see [12, section 1.29]). A submodule K of M is called an *M -cyclic submodule* if $K = s(M)$ for some $s \in S$. A fact that cyclic submodules and M -cyclic submodules are distinguished as we will see in Example 1. However, in the case of rings, they meet each other.

We denote $\mathbf{r}_X(Y), \mathbf{l}_X(Y)$ for the right annihilator, the left annihilator of Y in X , respectively. We simply write $\mathbf{r}(Y), \mathbf{l}(Y)$ if X is a ring R and $Y \subset R$. According to [6, 12], a module M_R is called a *CS module* (or *extending module*) if every submodule of M is essential in a direct summand, or, equivalently, every closed submodule of M is a direct summand.

Let M and N be two right R -modules. The following notions of injectivity have been established and investigated extensively.

1. N is called an *M -injective module* if every homomorphism from any submodule of M to N extends to an element of $\text{Hom}_R(M, N)$. N is called a *quasi-injective module* if it is N -injective. N is called an *injective module* if it is M -injective for every right R -module M .
2. N is called an *M - c -injective module* if every homomorphism from any closed submodule of M to N extends to an element of $\text{Hom}_R(M, N)$. N is called *quasi- c -injective* if it is N - c -injective. N is called a *c -injective module* if it is M - c -injective for every right R -module M (see [5, 10]).
3. N is called an *M - cp -injective module* if every homomorphism from any closed and M -cyclic submodule of M to N extends to an element of $\text{Hom}_R(M, N)$. N is called a *cp -injective module* if it is M - cp -injective for every right R -module M (see [3]).
4. N is called a *principally M -injective module* if every homomorphism from any cyclic submodule of M to N extends to an element of $\text{Hom}_R(M, N)$. According to [11], N is called a *principally quasi-injective module* if N is principally N -injective.
5. N is called an *M -mininjective module* if every homomorphism from any minimal submodule of M to N extends to an element of $\text{Hom}_R(M, N)$. N is called a *quasi-mininjective module* if it is N -mininjective (see [12]). In [14], quasi-mininjective modules are referred to by the term *minimal quasi-injective modules*.
6. N is called a *simple M -injective module* if for every $A \hookrightarrow M$, any $f \in \text{Hom}_R(A, N)$ with simple image extends to an element of $\text{Hom}_R(M, N)$. N is called a *simple quasi-injective module* if it is simple N -injective (see [12]).

Motivated by the notions above, we investigate some weak forms of principal injectivity relative to the classes of cyclic submodules, closed submodules, and minimal submodules. Cyclic M - c -injective modules and cyclic quasi- c -injective modules are defined and studied in sections 2 and 3. Our main results that answer our question are shown in the last section as applications of cyclic c -injectivity, simple injectivity, and mininjectivity. Surprisingly, in spite of weak injectivity, characterizations of right V-rings, right GV-rings, right pV-rings, WV-rings, and commutative Noetherian domains that are Dedekind are obtained.

2. On cyclic $M - c$ -injective modules

Definition 1 Let M and N be two right R -modules. N is called a cyclic closed M -injective (or cyclic $M - c$ -injective) module if every homomorphism from any closed and cyclic submodule of M to N extends to an element of $\text{Hom}_R(M, N)$.

M is called a cyclic quasi- c -injective module if it is cyclic $M - c$ -injective. M is said to be cyclic c -injective if it is cyclic $N - c$ -injective for every right R -module N . In particular, a ring R is called a right cyclic c -injective ring if R_R is a cyclic quasi- c -injective module.

Note that, in general, cp -injective modules (see [3]) and cyclic c -injective modules are different, since cyclic submodules and M -cyclic submodules are distinguished. However, for a ring, being right cp -injective (by sense of [3]) and being right cyclic c -injective are the same. Therefore, in order to unify terminologies in the case of rings, we say right cyclic c -injective rings.

Let M and N be right R -modules. Then, we have

- (1) If N is M -injective, then N is $M - c$ -injective.
- (2) If N is $M - c$ -injective, then N is cyclic $M - c$ -injective and $M - cp$ -injective.
- (3) If M is principally quasi-injective or quasi- c -injective, then M is cyclic quasi- c -injective.

For a ring R , right (self) injectivity implies right principal injectivity and right c -injectivity, and right principal injectivity or right c -injectivity implies right cyclic c -injectivity.

Proposition 1 Let M_1 and M_2 be right R -modules. If $M = M_1 \oplus M_2$ is a cyclic quasi- c -injective module, then M_i is cyclic $M_j - c$ -injective, for $i, j = 1, 2$. In particular, if M^n is a cyclic quasi- c -injective for any integer $n \geq 2$, then M is cyclic quasi- c -injective.

An element m of a module $M = M_R$ is called a *closed element* if mR is a closed submodule of M . In particular, an element x of a ring R is said to be *right closed* (resp. *left closed*) if xR (resp. Rx) is a closed right ideal (resp. closed left ideal) of R .

According to [8], M is called a *P-extending module* provided every cyclic submodule is essential in a direct summand. By [13], *ECS modules* are those whose every closed submodule containing essentially a cyclic submodule is a direct summand. Following [4], for *CMS modules* all closed and M -cyclic submodules are direct summands. For *right C-regular ring* every closed principal right ideal is a direct summand. Next, we study a weak CS property.

Definition 2 A right R -module M is called a *CCS module* if for every closed element $m \in M$, mR is a direct summand. In particular, a ring R is called a *right CCS ring* if every closed, principal right ideal is a direct summand.

Example 1 We consider Z -modules, $M = Z \oplus Z$, and $A = 2Z \oplus 3Z$. Then it is easy to check that A is an M -cyclic submodule of M but is not cyclic. On the other hand, $(2, 3)Z$ is not M -cyclic.

Considering Z -module $M = Q \oplus Z/Zp$, where Q is the set of all rational numbers, Z is the set of all integers, and p is a prime number. Note that Z/Zp is a simple Z -module, Q is a uniform Z -module, and Q has no closed proper submodule. Therefore, the only closed and cyclic submodules of M are 0 , Q , and Z/Zp . Thus M is CCS and so is cyclic quasi- c -injective (by Proposition 2 in the following) but not principally quasi-injective.

Left CCS rings are defined analogously. We have the relation of extending properties on a module: being CS \Rightarrow being ECS \Rightarrow being P-extending \Rightarrow being CCS. Every CCS module is cyclic quasi- c -injective (by the following result). By Example 1, we see that CCS modules and CMS modules are distinguished. However, for a ring R , R_R -cyclic submodules and cyclic submodules of R_R coincide (in fact, both are principal (or cyclic) right ideal of R). Thus, saying that R is right CCS, or R is right C -regular [4], or R_R is CMS [4] is the same. Therefore, we say right CCS rings in unity.

Proposition 2 *The following conditions are equivalent for a right R -module M :*

- (1) M is a CCS module;
- (2) Every right R -module is cyclic $M - c$ -injective;
- (3) Every cyclic right R -module is cyclic $M - c$ -injective;
- (4) Every closed and cyclic submodule of M is cyclic $M - c$ -injective.

Proof The proof is routine. □

We say that a right R -module M has the *CC property* if every closed submodule of M is cyclic.

Proposition 3 *The following statements are equivalent for a right R -module M with the CC property:*

- (1) M is a CS module;
- (2) Every right R -module is $M - c$ -injective;
- (3) Every right R -module is cyclic $M - c$ -injective;
- (4) M is a CCS module.

Proof The implications (1) \Rightarrow (2) \Rightarrow (3) are routine.

(3) \Rightarrow (4) Let A be an arbitrary closed and cyclic submodule of M . Then A is cyclic $M - c$ -injective and so is a direct summand of M . This implies that M is a CCS module.

(4) \Rightarrow (1) Let A be an arbitrary closed submodule of M . Since M has the CC property, A is cyclic. Thus A is a direct summand of M , since M is CCS. This shows that M is a CS module, completing the proof. □

Example 2 *Let Z be the set of all integers. Consider Z -module $M = (Z/Z2) \oplus (Z/Z8)$. We observe that all closed submodules of M are cyclic. However, there are cyclic submodules that are not closed. It is easy to check that M has the CC property but it is not a CCS module and so it is not extending by Proposition 3 (in fact, $((1 + Z2) \oplus (2 + Z8))Z$ is a cyclic closed submodule of M but not a direct summand). By this example, we also see that a direct sum of two CCS (even extending) modules may not be a CCS module.*

Theorem 1 *Let M be a projective right R -module. Then the following conditions are equivalent:*

- (1) Every homomorphic image of a cyclic $M - c$ -injective module is cyclic $M - c$ -injective;
- (2) Every homomorphic image of an $M - c$ -injective module is cyclic $M - c$ -injective;
- (3) Every homomorphic image of an M -injective module is cyclic $M - c$ -injective;
- (4) Every homomorphic image of an injective module is cyclic $M - c$ -injective;
- (5) Every cyclic and closed submodule of M is projective.
- (6) For every closed element $m \in M$, $\mathfrak{r}_R(m)$ is a direct summand of R .

Proof The implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) are obvious.

(4) \Rightarrow (5) Let X be a closed and cyclic submodule of M . We assume that N and K are two right R -modules, N is injective, and $\phi : N \rightarrow K$ is an epimorphism. By [2, Proposition 5.1], in order to prove that X is projective, it is sufficient to show that every homomorphism $\psi : X \rightarrow K$ lifts to a homomorphism from X to N .

By (4), K is cyclic $M - c$ -injective; thus ψ extends to a homomorphism $\alpha : M \rightarrow K$. Since M is a projective module, there exists $\beta : M \rightarrow N$ such that $\phi\beta = \alpha$. Then we see that β lifts ψ to N , implying that X is projective.

$$\begin{array}{ccccccc} 0 & \rightarrow & X & \hookrightarrow & M & & \\ & & & & \psi \searrow & \downarrow & \alpha \\ & & & & N & \xrightarrow{\phi} & K \rightarrow 0 \end{array}$$

(5) \Rightarrow (1) Assuming that N is cyclic $M - c$ -injective and X is projective for any closed and cyclic submodule X of M . Then any homomorphism $\psi : X \rightarrow K$ lifts to a homomorphism $\gamma : X \rightarrow N$ and γ extends to a homomorphism $\lambda : M \rightarrow N$. Thus, $\phi\lambda : M \rightarrow K$ is an extension of ψ , showing that K is cyclic $M - c$ -injective.

(5) \Leftrightarrow (6) It is clear, finishing the proof. □

Let M and N be right R -modules. It is obvious to say that if every quotient of N is cyclic $M - c$ -injective and then so is N . However, the converse need not be true. Theorem 1 gives us a necessary and sufficient condition for a factor module of a cyclic $M - c$ -injective module to be cyclic $M - c$ -injective.

Corollary 1 *Let $M = M_R$ be a projective module. If $N = N_R$ is a cyclic $M - c$ -injective module, then every quotient of N is again cyclic $M - c$ -injective if and only if every closed and cyclic submodule of M is projective.*

3. On cyclic quasi- c -injective modules

In this section, properties of cyclic c -injectivity that are similar to principal injectivity in [12] are proved. By Theorem 1, we have the following corollaries.

Corollary 2 *Let M be a projective right R -module that is cyclic quasi- c -injective. Then the following statements are equivalent:*

- (1) Every factor of M is cyclic $M - c$ -injective;
- (2) $f(M)$ is cyclic $M - c$ -injective for every endomorphism f of M ;
- (3) mR is projective for every closed element $m \in M$.

Corollary 3 *For a right cyclic c -injective ring R , the following conditions are equivalent:*

- (1) xR is cyclic $R_R - c$ -injective for every $x \in R$;
- (2) R/I is cyclic $R_R - c$ -injective for every right ideal I ;
- (3) aR is projective for every right closed element $a \in R$.

Example 3 *Let F be a field with only two elements and R be a F -algebra having basis $\{e_1, e_2, e_3, n_1, n_2, n_3, n_4\}$ with the following multiplication table:*

	e_1	e_2	e_3	n_1	n_2	n_3	n_4
e_1	e_1	0	0	0	0	0	0
e_2	0	e_2	0	n_1	0	0	n_4
e_3	0	0	e_3	0	0	0	0
n_1	n_1	0	0	0	0	0	0
n_2	n_2	0	0	0	0	0	0
n_3	0	0	n_3	0	0	0	0
n_4	0	0	n_4	0	0	0	0

Considering the right R -module $M = e_2R$. Since e_2R is a direct summand of R , M is projective. Closed and cyclic submodules of M are precisely n_1R and n_4R . We observe that M is cyclic quasi- c -injective (but not CCS) and that $\mathbf{r}_R(n_1)$ and $\mathbf{r}_R(n_4)$ are direct summands of R . Thus, Corollary 2 is applied.

Now we are going to prove a characterization of cyclic quasi- c -injective modules.

Theorem 2 Let M be a right R -module and its endomorphism ring $S = \text{End}(M_R)$. Then the following conditions are equivalent:

- (1) M is a cyclic quasi- c -injective module;
- (2) $\mathbf{l}_M \mathbf{r}_R(m) = Sm = \{f(m) \mid f \in S\}$ for every closed element $m \in M$;
- (3) $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(n)$ implies $Sn \subseteq Sm$, for every closed element $m \in M$ and for any $n \in M$;

Proof Let m be an arbitrary closed element of M .

(1) \Rightarrow (2) For every $f \in S$, we have $f(m)\mathbf{r}_R(m) = 0$ and so $Sm \subseteq \mathbf{l}_M \mathbf{r}_R(m)$. For $x \in \mathbf{l}_M \mathbf{r}_R(m)$, it is clear that $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(x)$; thus the mapping $\beta : mR \rightarrow M, ma \mapsto xa, a \in R$, is well defined. It is easy to see that β is a homomorphism. Since M is a cyclic quasi- c -injective module, β extends to an endomorphism α of M ; hence $x = \beta(m) = \alpha(m) \in Sm$. This shows that $\mathbf{l}_M \mathbf{r}_R(m) \subseteq Sm$; thus $\mathbf{l}_M \mathbf{r}_R(m) = Sm$.

(2) \Rightarrow (3) For every $n \in M$, if $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(n)$, then $n \in \mathbf{l}_M \mathbf{r}_R(m)$. By (2), $Sm = \mathbf{l}_M \mathbf{r}_R(m)$, and so $n = \beta(m)$ for some $\beta \in S$. Thus $Sn = S\beta(m) \subseteq Sm$.

(3) \Rightarrow (1) We will show that for every closed element $m \in M$, if $\gamma : mR \rightarrow M$ is an R -homomorphism, then $\gamma(m) \in Sm$. Assuming that $\gamma(m) = x$, for some $x \in M$, we have $\mathbf{r}_R(m) \subseteq \mathbf{r}_R(x)$, and so $Sx \subseteq Sm$ by (3). This implies $x = f(m)$ for some $f \in S$, showing $\gamma(m) \in Sm$. Therefore, M is cyclic quasi- c -injective. \square

We obtain a characterization of right cyclic c -injective rings by the following corollary.

Corollary 4 (also see [3, Corollary 3.9])

The following conditions are equivalent for a ring R :

- (1) R is right cyclic c -injective;
- (2) $\mathbf{l}_R(a) = Ra$ for every right closed element $a \in R$;
- (3) $\mathbf{r}(a) \subseteq \mathbf{r}(b)$ implies $Rb \subseteq Ra$, for every right closed element $a \in R$ and for every $b \in R$;
- (4) For every right closed element $a \in R$, $\mathbf{l}(kR \cap \mathbf{r}(a)) = \mathbf{l}(k) + Ra$, for any $k \in R$ such that akR is closed.

Proof (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) This follows from Theorem 2.

(3) \Rightarrow (4) Let k be any element of R such that akR is closed in R_R . If $x \in \mathbf{l}(kR \cap \mathbf{r}(a))$ then $\mathbf{r}(ak) \subseteq \mathbf{r}(xk)$, and so $xk = yak, y \in R$, by (3). Thus, $x - ya \in \mathbf{l}(k)$; hence $x = z + ya, z \in \mathbf{l}(k)$. This implies that $\mathbf{l}(kR \cap \mathbf{r}(a)) \subseteq \mathbf{l}(k) + Ra$. The other inclusion always holds.

(4) \Rightarrow (2) It is routine by taking $k = 1$. \square

Proposition 4 For a cyclic quasi- c -injective module $M = M_R$, and for every closed element $m \in M$, if $mR \cong A$ and A is a direct summand and then so is mR .

In particular, if R is right cyclic c -injective, then for every closed element $a \in R$, $aR \cong eR$, $e = e^2 \in R$ implies $aR = bR$, for some $b = b^2 \in R$.

Proof Observe that A is cyclic $M - c$ -injective. Since $mR \cong A$, mR is cyclic $M - c$ -injective. Thus mR is a direct summand of M . The case of the ring is clear. \square

We will conclude this section by giving conditions in which the endomorphism ring of a module, in particular, a self-generator, a self-cogenerator, is right (left) cyclic c -injective. By analogous way, a ring R is *left cyclic c -injective* if and only if for every left closed element $s \in R$ (i.e. Rs is closed in ${}_R R$) and any R -homomorphism $\alpha : Rs \rightarrow R$ extends to a right multiplication, $\alpha = \cdot t$, for some $t \in R$. It is routine to obtain the following lemma, which is similar to Corollary 4.

Lemma 1 The following conditions are equivalent for a ring R :

- (1) R is left cyclic c -injective;
- (2) $\mathbf{rl}(a) = aR$ for every left closed element $a \in R$;
- (3) $\mathbf{l}(a) \subseteq \mathbf{l}(b)$ implies $bR \subseteq aR$, for every left closed element $a \in R$ and for every $b \in R$;

Theorem 3 Let M be a right R -module and $S = \text{End}(M_R)$.

(1) We suppose that M generates $\ker(f)$ for every right closed element $f \in S$. Then, S is right cyclic c -injective if and only if $\ker(f) \subseteq \ker(g)$ implies $g \in Sf$, where g is an arbitrary element of S .

(2) Assuming that M cogenerates $M/f(M)$ for every left closed element $f \in S$. Then S is left cyclic c -injective if and only if $g(M) \subseteq f(M)$ implies $g \in fS$, where g is an arbitrary element of S .

Proof (1) Let f be an arbitrary right closed element of S . We assume that S is right cyclic c -injective and $\ker(f) \subseteq \ker(g)$, where $g \in S$. For every $s \in S$, $fs = 0$ implies $sM \subseteq \ker(f) \subseteq \ker(g)$; thus $gsM = 0$. This means that $gs = 0$, and hence $\mathbf{r}(f) \subseteq \mathbf{r}(g)$. By Corollary 4, $Sg \subseteq Sf$ and thus $g = sf \in Sf$ for some $s \in S$.

Conversely, for $h \in \mathbf{lr}(f)$, we will show that $\ker(f) \subseteq \ker(h)$. For any $x \in \ker(f)$, since $\ker(f)$ is generated by M , we have $x = \sum_I \gamma_i(m_i)$, $m_i \in M$, $\gamma_i : M \rightarrow \ker(f)$, for some finite index set I . Then $f\gamma_i = 0$ for each $i \in I$, and so $\gamma_i \in \mathbf{r}(f)$. Therefore, $\mathbf{lr}(f) \subseteq \mathbf{l}(\gamma_i)$, $i \in I$, and thus $h\gamma_i = 0$. As a consequence, $h(x) = 0$, so $x \in \ker(h)$ and hence $\ker(f) \subseteq \ker(h)$. By the hypothesis, $h \in Sf$ and hence $\mathbf{lr}(f) \subseteq Sf$. This implies $\mathbf{lr}(f) = Sf$, since the other inclusion always holds. By Corollary 4, S is a right cyclic c -injective ring.

(2) Let f be an arbitrary left closed element of S . We suppose that S is left cyclic c -injective and $g(M) \subseteq f(M)$, where $g \in S$. For every $s \in S$, $sf = 0$ implies $sf(M) = 0$; thus $sg(M) = 0$. This means that $sg = 0$ and hence $\mathbf{l}(f) \subseteq \mathbf{l}(g)$. By Lemma 1, $gS \subseteq fS$ and thus $g = fs \in fS$ for some $s \in S$.

Conversely, if $h \in \mathbf{rl}(f)$, then $\mathbf{l}(f) \subseteq \mathbf{l}(h)$. We will show that $h(M) \subseteq f(M)$. In contrast, assume that there exists some $m_0 \in M$ such that $h(m_0) \notin f(M)$. Thus the homomorphism $\alpha : M \rightarrow M/f(M)$, $m \mapsto h(m) + f(M)$ is nonzero, because at least $\alpha(m_0) = h(m_0) + f(M) \neq 0$. Then there is an R -homomorphism $\gamma : M/f(M) \rightarrow M$ satisfying $\gamma\alpha(m_0) = \gamma(h(m_0) + f(M)) \neq 0$, since M cogenerates $M/f(M)$. We have $\beta f = 0$, where $\beta : M \rightarrow M$ is defined by $\beta(m) = \gamma(m + f(M))$, $m \in M$. Then $\beta h(m_0) = \gamma(h(m_0) + f(M)) \neq 0$, $\beta h \neq 0$, contradicting $\mathbf{l}(f) \subseteq \mathbf{l}(h)$. Therefore, $h(M) \subseteq f(M)$ must hold. Thus, by assumption, $h = fs \in fS$ and so $\mathbf{rl}(f) \subseteq fS$. This says that $\mathbf{rl}(f) = fS$, since the other inclusion always holds. Consequently, by Lemma 1, S is left cyclic c -injective. \square

4. Applications of weak forms of principal injectivity

This section aims to answer our question stated in the introductory part. We show that cyclic c -injectivity is useful for characterizing commutative Noetherian rings that are Dedekind domains; principal injectivity, simple injectivity, and mininjectivity help us to obtain right V-rings, right GV-rings, and WV-rings. The next theorems can be regarded as the most important ones in this paper. Throughout this section, R is an associative ring with identity and all modules are unitary, right over R .

Lemma 2 *The following conditions are equivalent for a simple module M_R :*

- (1) M is c -injective;
- (2) M is cyclic c -injective;
- (3) M is c -injective relative to minimal submodules, that is for every right R -module N , every homomorphism from any minimal, closed submodule of N to M extends to an element of $\text{Hom}_R(N, M)$.

Proof It is obvious for the implications (1) \Rightarrow (2) \Rightarrow (3).

(3) \Rightarrow (1) Let N be an arbitrary right R -module and A a closed submodule of N . For any homomorphism $f : A \rightarrow M$, since M is simple, $A/\ker(f)$ is a minimal submodule of $N/\ker(f)$. Moreover, $A/\ker(f)$ is closed in $N/\ker(f)$ (see [6]). We put $\bar{f} : A/\ker(f) \rightarrow M$, $\bar{f}(a + \ker(f)) = f(a)$, $a \in A$. Since M is c -injective relative to minimal submodules, \bar{f} extends to a homomorphism $\varphi : N/\ker(f) \rightarrow M$. Let $\pi : N \rightarrow N/\ker(f)$ be the canonical projection. Then we observe that $\alpha = \varphi\pi$ is an extension of f . Indeed, for every $a \in A$, $\alpha(a) = \varphi\pi(a) = \varphi(a + \ker(f)) = \bar{f}(a + \ker(f)) = f(a)$. This implies that M is N - c -injective. Consequently, M is c -injective, completing the proof. \square

Theorem 4 *If R is a commutative Noetherian domain, then the following statements are equivalent:*

- (1) R is Dedekind;
- (2) Every simple R -module is c -injective;
- (3) Every simple R -module is cyclic c -injective;
- (4) Every simple R -module is c -injective relative to minimal submodules;
- (5) Every simple R -module M is simple N -injective relative to closed submodules, for every R -module N , that is for any closed submodule A of N , every $f \in \text{Hom}_R(A, M)$ with a simple image extends to an element of $\text{Hom}_R(N, M)$.

Proof The implications (2) \Leftrightarrow (3) \Leftrightarrow (4) follow from Lemma 2.

(1) \Leftrightarrow (2) It is by [10, Corollary 5.5].

(2) \Rightarrow (5) It is straightforward.

(5) \Rightarrow (4) For every R -module N and for any closed, minimal submodule A of N , every $f : A \rightarrow M$ has a simple image. Since M is simple N -injective relative to closed submodules, f extends to an element of $\text{Hom}_R(N, M)$. This implies that M is c -injective relative to minimal submodules, as required. The proof is now complete. \square

Theorem 5 *The following conditions are equivalent for a ring R :*

- (1) R is a right V-ring (resp. right GV-ring);

- (2) Every simple (resp. simple singular) module is injective;
 (3) Every simple (resp. simple singular) module is principally N -injective for every module N ;
 (4) Every simple (resp. simple singular) module is principally R/I -injective for every right ideal I of R ;
 (5) Every simple (resp. simple singular) module is N -mininjective for every module N ;
 (6) Every simple (resp. simple singular) module is R/I -mininjective for every right ideal I of R ;
 (7) Every simple (resp. simple singular) module is simple N -injective for every module N ;
 (8) Every simple (resp. simple singular) module is simple R/I -injective for every right ideal I of R ;
 (9) Every module is quasi-mininjective.

Proof We will prove for the case of being right V-rings. The other one is induced similarly.

(1) \Leftrightarrow (2) It is by the definition of right V-rings.

(1) \Leftrightarrow (9) It is by [14, Theorem 1.5].

(2) \Leftrightarrow (6) Let M be an arbitrary simple module. If M is injective, then it is obvious that M is R/I -mininjective for every right ideal I of R . Conversely, we assume that for every right ideal J of R , M is R/J -mininjective. For every right ideal I of R and for any homomorphism $f : I \rightarrow M$, we have $I/\ker(f) \cong M$ via the isomorphism $\bar{f}(a + \ker(f)) = f(a), \forall a \in I$. Moreover, $\ker(f)$ is a right ideal of R and $I/\ker(f)$ is a minimal submodule of $R/\ker(f)$. Since M is $R/\ker(f)$ -mininjective, \bar{f} extends to a homomorphism $\varphi : R/\ker(f) \rightarrow M$. Let $\pi : R \rightarrow R/\ker(f)$ be the canonical projection. Then $\alpha = \varphi\pi$ is an extension of f . Indeed, for every $a \in I, \alpha(a) = \varphi\pi(a) = \varphi(a + \ker(f)) = \bar{f}(a + \ker(f)) = f(a)$. This implies that M is R -injective and hence is injective, as required.

(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (6), (2) \Rightarrow (5) \Rightarrow (6), and (2) \Rightarrow (7) \Rightarrow (8) are clear.

(8) \Rightarrow (6) Let M be a simple module. For any right ideal I of R , we put $N = R/I$. Let A be an arbitrary minimal submodule of N . Then every homomorphic image of A is simple. Since M is simple N -injective, every homomorphism $f : A \rightarrow M$ extends to N . This shows that M is R/I -mininjective. \square

Theorem 6 The following conditions are equivalent for a ring R :

- (1) R is a WV-ring;
 (2) Every simple module is R/I -injective for every right ideal I of R satisfying $R/I \not\cong R$;
 (3) Every simple module is principally R/I -injective for every right ideal I of R satisfying $R/I \not\cong R$;
 (4) Every simple module is R/I -mininjective for every right ideal I of R satisfying $R/I \not\cong R$;
 (5) Every simple module is simple R/I -injective for every right ideal I of R satisfying $R/I \not\cong R$.

Proof (1) \Leftrightarrow (2), (2) \Rightarrow (3) \Rightarrow (4), and (2) \Rightarrow (5) It is obvious.

(4) \Rightarrow (2) It is similar to the proof of (2) \Leftrightarrow (6) of Theorem 5 that if a simple module is R/I -mininjective for every right ideal I of R satisfying $R/I \not\cong R$, then it is R/I -injective for every right ideal I of R satisfying $R/I \not\cong R$.

(5) \Rightarrow (2) It is elementary. \square

We see that our Theorem 5 agrees with Zhu and Tan [14, Theorem 1.5], in which right V-rings are characterized by minimal quasi-injective (or quasi-mininjective) modules. The proof of the following theorem is similar to such things of Theorem 5 and Theorem 6.

Theorem 7 *The following conditions are equivalent for a ring R :*

- (1) R is a right pV -ring;
- (2) Every simple module is principally R_R -injective;
- (3) Every simple module is simple R_R -injective;
- (4) Every simple module is $(R/I)_R$ -mininjective for every right ideal I of R .

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