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Extensions of quasipolar rings

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Abstract: An associative ring with identity is called *quasipolar* provided that for each $a \in R$ there exists an idempotent $p \in R$ such that $p \in comm^2(a)$, $a + p \in U(R)$ and $ap \in R^{qnil}$. In this article, we introduce the notion of quasipolar general rings (with or without identity). Some properties of quasipolar general rings are investigated. We prove that a general ring I is quasipolar if and only if every element $a \in I$ can be written in the form $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, q is quasinilpotent, and $sq = qs = 0$. It is shown that every ideal of a quasipolar general ring is quasipolar. Particularly, we show that R is pseudopolar if and only if R is strongly π-rad clean and quasipolar.

Key words: Quasipolar general rings, strongly clean general rings, strongly π-regular general rings, (generalized) Drazin inverse, pseudopolar rings

1. Introduction

Throughout this paper, a ring means an associative ring with identity and a general ring means an associative ring with or without identity. For clarity, R and S will always denote rings, and I and A denote general rings. The notation $U(R)$ denotes the group of units of R, $J(I)$ denotes the Jacobson radical of I, and $Nil(I)$ denotes the set of all nilpotent elements of I. The *commutant* and *double commutant* of an element a in a ring R are defined by $comm_R(a) = \{x \in R \mid xa = ax\}$, $comm_R^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm_R(a)\}$, respectively. If there is no ambiguity, we simply use $comm(a)$ and $comm^2(a)$. Let $R^{qnil} = \{a \in R \mid 1 + ax \in U(R) \text{ for every } x \in comm(a)\}$. If $a \in R^{qnil}$, then a is said to be *quasinilpotent* [9]. Set $J^{\#}(R) = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}$. Clearly, $J(R) \subseteq J^{\#}(R) \subseteq R^{qnil}$.

An element $a \in R$ is called *quasipolar* provided that there exists an idempotent $p \in comm^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{qnil}$. A ring R is *quasipolar* in case every element in R is quasipolar. This concept ensues from Banach algebra. Indeed, for a Banach algebra R (see [8, page 251]),

$$a \in R^{qnil} \Leftrightarrow \lim_{n\to\infty} \| a^n \|^{\frac{1}{n}} = 0.$$

Quasipolar rings were studied in [6,8–12,21].

Ara [1] defined and investigated the notion of an exchange ring without identity. Chen and Chen [3] introduced the concept of strongly π-regular general rings. In [14], Nicholson and Zhou defined the notion of a clean general ring and they extended some of the basic results about clean rings to general rings. In [17], Wang

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and Chen defined the concept of a strongly clean general ring, and some properties about strongly clean rings were extended. These works motivate us to define quasipolar general rings. In this paper we see that every strongly π-regular general ring is a quasipolar general ring and any quasipolar general ring is a strongly clean general ring. We also see that every (two-sided) ideal of a quasipolar ring is a quasipolar general ring, but there exist quasipolar general rings that are not ideals of quasipolar rings (Example 3.3). In particular, we prove that $a \in R$ is strongly π-regular if and only if there exists a strongly regular element $s \in R$ and $n \in Nil(R)$ such that $a = s + n$ and $sn = ns = 0$ (Theorem 2.14), and $a \in R$ is quasipolar if and only if there exists a strongly regular element $s \in comm^2(a)$ and $q \in R^{qnil}$ such that $a = s + q$ and $sq = qs = 0$ (Corollary 2.17).

An element a of R is *(generalized) Drazin invertible* (see [6, 11, 12]) if there is an element $b \in R$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and ($a^2b - a \in R^{qnil}$) $a^2b - a \in Nil(R)$. Such a b, if it exists, is unique; it is called the *(generalized) Drazin inverse* of a. Koliha [11] showed that an element $a \in R$ is Drazin invertible if and only if a is strongly π-regular [11, Lemma 2.1]. Koliha and Patricio [12] proved that an element $a \in R$ is generalized Drazin invertible if and only if a is quasipolar [12, Theorem 4.2]. With this in mind, we show that, for a general ring I, $a \in I$ is quasipolar if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and $a^2b - a \in QN(I)$ (Theorem 2.8), and $a \in I$ is strongly π-regular if and only if there is an element $b \in I$ satisfying $ab^2 = b$, $b \in comm^2(a)$ and $a^2b - a \in Nil(I)$ (Theorem 2.10).

Finally, we characterize a pseudopolar element of a ring, and we address the relations among quasipolarity, strong π-rad cleanness, and pseudopolarity. It is shown that R is pseudopolar if and only if R is strongly π-rad clean and quasipolar (Theorem 4.4).

2. Quasipolar general rings

Let I be a general ring with $p, q \in I$. We write $p * q = p + q - pq$. Let

$$Q(I) = \{q \in I \mid p * q = 0 = q * p \text{ for some } p \in I\}.$$

Note that $J(I) \subseteq Q(I)$. We define a set

$$QN(I) = \{q \in I \mid qx \in Q(I) \text{ for every } x \in comm(q)\}.$$

Clearly, $J(I) \subseteq Q(I)$ and $Nil(I) \subseteq QN(I)$. If R has an identity, then we have $Q(R) = \{q \in R \mid 1 - q \in U(R)\}$ and $QN(R) = R^{qnil}$. Further, if $a \in QN(I)$, then a is also said to be *quasinilpotent*.

Lemma 2.1 *The following conditions are equivalent for a ring R:*

(1) *R is quasipolar.*

(2) *For each $a \in R$, there exists $p^2 = p \in comm^2(a)$ such that $a + p \in Q(R)$ and $a - ap \in QN(R)$.*

Proof (1) $\Rightarrow$ (2) Let $a \in R$. Since R is quasipolar, there exists an idempotent $1 - p \in R$ such that $1 - p \in comm^2(a)$, $-a + 1 - p = u \in U(R)$, and $a(1 - p) = a - ap \in R^{qnil}$. Then $a + p = q$, $p \in comm^2(a)$ where $q = 1 - u$ and $q * r = 0 = r * q$ with $r = 1 - u^{-1}$. As $R^{qnil} = QN(R)$, $a - ap \in QN(R)$.

(2) $\Rightarrow$ (1) If $-a + p = q$ where $p^2 = p \in comm^2(a)$, $q \in Q(R)$, and $a - ap \in QN(R)$, then $a + 1 - p = 1 - q$ where $(1 - p)^2 = 1 - p \in comm^2(a)$, $1 - q \in U(R)$ and $a(1 - p) \in R^{qnil}$. □

Definition 2.2 An element a in a general ring I is called a *quasipolar element* if there exists $p^2 = p \in comm^2(a)$ such that $a + p \in Q(I)$ and $a - ap \in QN(I)$, and I is called a *quasipolar general ring* if every element is quasipolar.

Remark 2.3 If I is isomorphic to a general ring K by f, then $a \in I$ is quasipolar if and only if $f(a)$ is quasipolar in K.

Example 2.4 Idempotents, nilpotents, quasinilpotents, and quasiregular elements are all quasipolar.

Recall that an element a in a general ring I is called a *strongly clean element* if it is the sum of an idempotent and an element of $Q(I)$ that commute, and I is called a *strongly clean general ring* if every element is strongly clean [17]. Hence, by Definition 2.2, quasipolar elements (general rings) are strongly clean.

We need the following useful lemma.

Lemma 2.5 *Let a, b, c be elements of a general ring I. If $a \in Q(I) \cap comm(b)$ and $a * c = 0 = c * a$, then $c \in comm(b)$.*

Proof Let $a*c = 0 = c*a$ and $ba = ab$. Then $a+c = ac = ca$. This implies that $ba+bc-bca = 0 = ab+cb-cab$, and so

$$bc - bca = cb - cab. \tag{2.1}$$

Multiplying (2.1) by c from the right yields

$$bcc - bcac = cbc - cabc.$$

This gives $bca = cba = cab$ because $c - ac = -a$. This shows that $bc = cb$ and so $c \in comm(b)$. □

Lemma 2.6 *Let I be a general ring. If $a * b = 0$ and $c * a = 0$, then $b = c$.*

Proof Suppose that $a * b = 0$ and $c * a$ for $a, b, c \in I$. This gives $b = 0 * b = (c * a) * b = c * (a * b) = c * 0 = c$, as desired. □

Lemma 2.7 *Let I be a general ring and assume that $a \in I$ is quasinilpotent. Then $a, -a \in Q(I)$ and $-a \in I$ is quasinilpotent. Further, $QN(I) \subseteq Q(I)$.*

Proof Since $a \in QN(I)$ and $a \in comm(a)$, we get $a^2 \in Q(I)$. That is, there exists $b \in R$ such that $a^2 * b = a^2 + b - a^2 b = 0 = b + a^2 - ba^2 = b * a^2$. This implies that $0 = a^2 * b = [a * (-a)] * b = a * [(-a) * b]$ and $0 = b * a^2 = b * [(-a) * a] = [b * (-a)] * a$, and so we have $a \in Q(I)$ by Lemma 2.6. Similarly, it can be shown that $-a \in Q(I)$. On the other hand, we check easily that $-a \in QN(I)$. If $a \in QN(I)$, then $a \in Q(I)$. Hence, $QN(I) \subseteq Q(I)$. The proof is completed. □

The next result was proved in [12, Theorem 4.2] for a in any ring R.

Theorem 2.8 *The following are equivalent for $a \in I$:*

(1) *a is quasipolar in I.*

(2) *There exists* $b \in comm^2(a)$ *such that* $ab^2 = b$ *and* $a^2b - a \in QN(I)$.

In this case, b *is unique.*

Proof (1) ⇒ (2) Write $a+p = q \in Q(I)$ where $p^2 = p \in comm^2(a)$ and $a - ap \in QN(I)$, say $q * r = r * q = 0$ where $r \in I$. Then $r + q = rq = qr$. In view of Lemma 2.5, $rp = pr$ because $q \in Q(I)$ and $q \in comm(p)$. Set $b = rp - p$. It is easy to verify that $p = ab$. Let $ax = xa$ for some $x \in I$. Since $p \in comm^2(a)$, we have $xp = px$ and so $xq = qx$. Moreover, as $r + q = rq = qr$, we see that

$$xr - xrq = rx - rxq. \tag{2.2}$$

Multiplying (2.2) by r from the right yields

$$xrr - xrqr = rxr - rxqr \quad \text{and so} \quad xrq = rxq = rqx.$$

This shows that $rx = xr$. That is, $r \in comm^2(a)$. Hence, we conclude that $b \in comm^2(a)$. Now we show that $ab^2 = b$ and $a^2b - a \in QN(I)$. We have

$$\begin{aligned} ab^2 &= (q-p)(rp-p)(rp-p) = (q-p)(r^2p - rp - rp + p) \\ &= qr^2p - qrp - qrp + qp - r^2p + rp + rp - p \\ &= qr^2p - rp - qp - rp - qp + qp - r^2p + rp + rp - p \\ &= qr^2p - r^2p - p - qp \\ &= (qr^2 - r^2 - p - q)p \\ &= (r^2 + rq - r^2 - p - q)p \\ &= (r-p)p \\ &= b. \end{aligned}$$

Moreover,

$$\begin{aligned} a^2b - a &= (q-p)(q-p)(rp-p) - (q-p) \\ &= (q^2 - qp - qp + p)(rp - p) - q + p \\ &= q^2rp - q^2p - qrp + qp - qrp + qp + rp - p - q + p \\ &= q^2rp - q^2p - rp - qp + qp - rp - qp + qp + rp - q \\ &= q^2rp - q^2p - rp - q \\ &= qpr + q^2p - q^2p - rp - q \\ &= rp + qp - rp - q \\ &= qp - q \\ &= ap - a \in QN(I). \end{aligned}$$

Thus (2) holds, as required.

(2) ⇒ (1) Set $p = ab$. Then $p \in comm^2(a)$, and $p^2 = abab = a^2b^2 = a(ab^2) = ab = p$. Since $a - ap = a - aab = a - a^2b$ and $a^2b - a \in QN(I)$, we have $a - ap \in QN(I)$. Now we show that $a+p = a+ab \in Q(I)$. We observe that $(a+ab)*(b+ab) = a+ab+b+ab-(a+ab)(b+ab) = a+ab+b+ab-ab-a^2b-b-ab = a-a^2b$. As $a - a^2b \in QN(I)$, $(a-a^2b)*x = x*(a-a^2b) = 0$ for some $x \in I$. This implies that $(a+ab)*(b+ab)*x = 0$ and $x*(b+ab)*(a+ab) = 0$. Further, $(b+ab)*x = 0*(b+ab)*x = \big(x*(b+ab)*(a+ab)\big)*(b+ab)*x = \big(x*(b+ab)\big)*\big((a+ab)*(b+ab)*x\big) = x*(b+ab)*0 = x*(b+ab)$. Then $(b+ab)*x*(a+ab) = x*(b+ab)*(a+ab) = 0$, so we have $a+ab = a+p = q \in Q(I)$. Hence, $a \in I$ is quasipolar. Moreover, as $q \in Q(I)$, there exists $r \in I$ such that $q*r = 0 = r*q$, and so $r+q = rq = qr$. As in the preceding discussion, we see that $r \in comm^2(a)$. Thus, $r*(q*(b+p)) = (r*q)*(b+p) = 0*(b+p) = b+p = r*(a-ap) = r+a-ap-ra+rap = r+q-p-pq+p-rq+rpq = rpq-pq = rp$. Therefore, $b = rp - p$.

To prove the uniqueness of b, assume that $c \in comm^2(a)$ so that $ac^2 = c$ and $a^2c - a \in QN(I)$. Then $ac - acab = ac - a^2cb = a^2c^2 - a^2cb = (a^2b - a)(b - c)$. Since $a^2b - a \in QN(I)$ and $b - c \in comm(a^2b - a)$, we have $ac - a^2cb \in Q(I)$. This gives that $ac = a^2cb$. Similarly, we show that $ab = a^2cb$, and so $ab = ac$. Thus, $b = rp - p = rab - ab = rac - ac = c$; that is, b is unique. Note that b is unique if and only if p is unique. We complete the proof. □

Corollary 2.9 *Let I be a general ring. If $a \in I$ is quasipolar, then $-a$ is quasipolar.*

Proof It is clear from Theorem 2.8. □

Recall that an element a in a general ring I is called *strongly π-regular* if there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $x \in comm(a)$ (see [2, 3, 17]). The next result is known if a is in a ring R (see [11, Lemma 2.1] and [12, Proposition 4.9]).

Theorem 2.10 *The following are equivalent for $a \in I$:*

(1) *a is strongly π-regular in I.*

(2) *There exists $p^2 = p \in comm^2(a)$ such that $a - ap \in Nil(I)$ and $a + p \in Q(I)$.*

(3) *There exists $p^2 = p \in comm(a)$ such that $a - ap \in Nil(I)$ and $a + p \in Q(I)$.*

(4) *There exists $b \in comm^2(a)$ such that $ab^2 = b$ and $a^2b - a \in Nil(I)$.*

(5) *There exists $b \in comm(a)$ such that $ab^2 = b$ and $a^2b - a \in Nil(I)$.*

Proof (1) ⇒ (2) Assume that $a \in I$ is strongly π-regular. Then there exist $n \in \mathbb{N}$ and $x \in I$ such that $a^n = a^{n+1}x$ and $ax = xa$. It is easy to check that $a^nx^n = x^na^n = p = p^2 \in I$. Since $a^n = a^nx^na^n$, we have $(a - ap)^n = 0$, and so $a - ap \in Nil(I)$.

Claim 1. $p \in comm^2(a)$.

Proof. Let $ay = ya$. This implies that $py - pyp = a^nx^ny - a^nx^nyp = a^nx^ny - x^nya^np = a^nx^ny - x^nya^n = a^nx^ny - a^nx^ny = 0$ because $ax = xa$ and $a^nx^n = x^na^n$, so $py = pyp$. Similarly, we see that $yp = pyp$. Then $py = yp$ and so $p \in comm^2(a)$.

The remaining proof is to show that $q = a + p$ is a quasiregular element of I. Set $t = a + a^2 + a^3 + \cdots + a^{n-1}$ and $r = tp - t + a^{n-1}x^np + p$. Hence,

$$\begin{aligned} q * r &= a + p + tp - t + a^{n-1}x^np + p - \\ &\quad atp - at - p - ap - a^{n-1}x^np - p \\ &= a + p + ap - a - a^np + a^np - p - ap \\ &= 0. \end{aligned}$$

Analogously, we have $r * q = 0$. Thus (2) holds.

(2) ⇒ (3) Clear by $comm^2(a) \subseteq comm(a)$.

(3) ⇒ (4) Assume that $a + p = q \in Q(I)$ where $p^2 = p \in comm(a)$ and $a - ap \in Nil(I)$, say $q * r = r * q = 0$ and $(a - ap)^k = a^k - a^kp = 0$ where $r \in I$ and $k \in \mathbb{N}$. By Lemma 2.5, $rp = pr$ because $q \in Q(I)$ and $q \in comm(p)$. Set $b = rp - p$ and let $ax = xa$ for some $x \in I$. Then we have $ab = p = ba$, and so $xp - pxp = xa^kb^k - pxa^kb^k = a^kxb^k - pa^kxb^k = (a^k - pa^k)xb^k = 0$. That is, $xp = pxp$. Analogously,

we see that $px = pxp$. This gives $xp = px$, so $p \in comm^2(a)$. Therefore, an argument similar to the proof of Theorem 2.8 shows that $b \in comm^2(a)$, $ab^2 = b$, and $a^2b - a = ap - a \in Nil(I)$.

(4) $\Rightarrow$ (5) It is obvious.

(5) $\Rightarrow$ (1) Let $ab = p$. Since $ab^2 = b$, we have $p = p^2$. As $a^2b - a \in Nil(I)$, there exists $k \in \mathbb{N}$ such that $(a^2b - a)^k = 0$. This implies that $(a^2b - a)^k = a^k p - a^k = 0$. Then $a^k = a^k p = a^k ab = a^{k+1}b$ and $b \in comm(a)$. Hence, $a \in I$ is strongly π-regular, and so (1) holds. □

Remark 2.11 If an element a of a general ring I is strongly π-regular, then b and p in Theorem 2.10 are unique (indeed, as in the proof of Theorem 2.8, we see that b and p are unique).

By Theorem 2.10, the following result is immediate.

Corollary 2.12 *Any strongly π-regular element in a general ring is strongly clean.*

Recall that an element a of a general ring I is *strongly regular* if $a = aba$ and $b \in comm(a)$ for some $b \in I$. I is *strongly regular* if every element in I is strongly regular.

Lemma 2.13 *Let I be a general ring and $a \in I$. Then the following are equivalent:*

(1) *a is strongly regular in I.*

(2) *There exists $b \in comm^2(a)$ such that $a = a^2b$.*

Proof It is similar to the proof of [2, Lemma 1]. □

Theorem 2.14 was proved for a in any ring R in [15].

Theorem 2.14 *For an element a in a general ring I, the following are equivalent:*

(1) *a is strongly π-regular in I.*

(2) *$a \in I$ can be written in the form $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$.*

Proof (1) $\Rightarrow$ (2) Suppose that $a \in I$ is strongly π-regular. It is well known that a is strongly π-regular if and only if a is pseudoinvertible; that is, there exist $c \in I$ and $m \in \mathbb{N}$ such that $ac = ca$, $a^m = a^{m+1}c$, and $c = c^2a$ (see [6, Theorem 4]). Set $s = aca$ and $n = a - aca$. Then $sn = ns = aca(a - aca) = 0$ because $ac = ca$ and ac is idempotent in I. It is easy to check that $s = s^2c$ and so s is strongly regular in I. Write $ca = ac = e = e^2 \in I$. Hence, $(a - aca)^m = (a - ae)^m = a^m - a^m e = a^m - a^m ac = a^m - a^{m+1}c = 0$. Thus, $n \in I$ is nilpotent and so (2) holds.

(2) $\Rightarrow$ (1) Assume that $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$. Since n is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. As s is strongly regular, there exists $x \in I$ such that $s = s^2x$ and $x \in comm^2(s)$ by Lemma 2.13. Then it is easy to see that $a^k = (s+n)^k = s^k$ and $a^{k+1} = (s+n)^k = s^{k+1}$ because $sn = ns = 0$. This gives that $a^k = s^k = s^{k-1}s = s^{k-1}s^2x = s^{k+1}x = a^{k+1}x$. Further, as $as = sa$ and $x \in comm^2(s)$, we have $ax = xa$. Hence, a is strongly π-regular in I. □

The following result is well known for a ring (see [2]).

Corollary 2.15 *If an element a in a general ring I is strongly π-regular, then a^k is strongly regular for some $k \in \mathbb{N}$.*

A new characterization of a quasipolar element in a general ring is given as follows.

Theorem 2.16 *For an element a in a general ring I, the following are equivalent:*

(1) *a is quasipolar in I.*

(2) *$a \in I$ can be written in the form $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$.*

Proof (1) $\Rightarrow$ (2) Assume that $a \in I$ is quasipolar. By Theorem 2.8, there exists $b \in comm^2(a)$ such that $ab^2 = b$ and $a^2b - a \in QN(I)$. Set $s = a^2b$ and $q = a - a^2b$. Further, we have $s \in comm^2(a)$ and $sq = qs = a^2b(a - a^2b) = 0$ because $ab = ba$ and ab is idempotent in I. It is easy to see that $s = s^2b$ and so $s \in I$ is strongly regular.

(2) $\Rightarrow$ (1) Suppose that $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. Then we have that $sy = ys$ is an idempotent and $yq = qy$. Hence, $a + sy = s + sy + q = (s + sy) * q = q * (s + sy)$ and $(s + sy) * (y^2s + sy) = (y^2s + sy) * (s + sy) = 0$. This implies that $(a + sy) * (y^2s + sy) = (y^2s + sy) * (a + sy) = (s + sy) * q * (y^2s + sy) = (y^2s + sy) * (s + sy) * q = q$. As $q \in Q(I)$, it can be checked that $a + sy \in Q(I)$. Further, $a - asy = s + q - s^2y - qsy = q \in QN(I)$ and $sy \in comm^2(a)$. Thus, $a \in I$ is quasipolar, and so (1) holds. □

The following result is a direct consequence of Theorem 2.16.

Corollary 2.17 *Let R be a ring and let $a \in R$. Then the following are equivalent:*

(1) *a is quasipolar.*

(2) *$a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in R^{qnil}$, and $sq = qs = 0$.*

Proposition 2.18 *A general ring I is strongly regular if and only if I is quasipolar and $QN(I) = 0$.*

Proof Assume that I is strongly regular. Then I is strongly π-regular and so I is quasipolar by Theorem 2.10. Let $a \in QN(I)$. By hypothesis, $a = aba$ and $b \in comm(a)$ for some $b \in I$. Since $ab = ba$, we have $ab \in Q(I)$. This implies that $ab = 0$ and so $a = 0$. Hence, $QN(I) = 0$. Conversely, let $a \in I$. Since $QN(I) = 0$, a is strongly regular by Theorem 2.16. □

The following result follows from Proposition 2.18.

Corollary 2.19 [4, Theorem 2.4] Let R be a ring. Then R is strongly regular if and only if R is quasipolar and $R^{qnil} = 0$.

Remark 2.20 (1) In Proposition 2.18, it was proved that if $a \in QN(I)$ and a is strongly regular, then $a = 0$.

(2) If a is strongly regular, then a^k is strongly regular for any $k \in \mathbb{N}$.

(3) If $a \in QN(I)$ and a^k is strongly regular for some $k \in \mathbb{N}$, then $a \in Nil(I)$.

Proposition 2.21 *A general ring I is strongly π-regular if and only if I is quasipolar and $QN(I) \subseteq Nil(I)$.*

Proof Assume that I is strongly π-regular. Then, by Theorem 2.10, I is quasipolar because $Nil(I) \subseteq QN(I)$. Let $a \in QN(I)$. As I is strongly π-regular, by Theorem 2.14, $a = s + n$ where s is strongly regular, n is nilpotent, and $sn = ns = 0$. Since n is nilpotent, there exists $k \in \mathbb{N}$ such that $n^k = 0$. Hence, we have $a^k = s^k$. As s^k is strongly regular and $a \in QN(I)$, by Remark 2.20, we see that $a \in Nil(I)$. Thus, $QN(I) \subseteq Nil(I)$. Conversely, suppose that I is quasipolar and $QN(I) \subseteq Nil(I)$. In view of Theorem 2.16 and Theorem 2.14, I is strongly π-regular. □

The following result is a direct consequence of Proposition 2.21.

Corollary 2.22 [4, Theorem 2.6] Let R be a ring. Then R is strongly π-regular if and only if R is quasipolar and $R^{qnil} \subseteq Nil(R)$.

An element a of a ring R is called *semiregular* if there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. A ring is a *semiregular ring* if each of its elements is semiregular ([13, Proposition 2.2]).

We give a different proof of [19, Theorem 3.2].

Theorem 2.23 *Let R be a ring. If R is quasipolar and $R^{qnil} \subseteq J(R)$, then R is semiregular. The converse holds if R is abelian.*

Proof Assume that R is a quasipolar ring and $R^{qnil} \subseteq J(R)$. Then we have $J(R) = R^{qnil}$. In view of Corollary 2.17, $R/J(R)$ is strongly regular. As R is quasipolar, R is strongly clean and so idempotents lift modulo $J(R)$. Then R is semiregular by [13, Theorem 2.9]. Conversely, let $a \in R$. Then there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. Write $a = aba + (a - aba)$, say $s = aba$ and $q = a - aba$. Since $a - aba \in J(R) \subseteq R^{qnil}$ and R is abelian, we see that $s \in comm^2(a)$, $q \in R^{qnil}$, $s = aba = (aba)^2 b = s^2 b$, and $sq = qs = aba(a - aba) = a^2ba - a^2ba = 0$. By Corollary 2.17, a is quasipolar, and so R is quasipolar. Take $x \in R^{qnil}$. By assumption, there exists $y \in R$ with $yxy = y$ and $x - xyx \in J(R)$. Note that $x \cdot 0 = 0$ and $x^2 \cdot 0 - x = -x \in R^{qnil}$. By Theorem 2.8, we get $y = 0$. This gives that $x \in J(R)$. □

3. Extensions of quasipolar general rings

Let S be a ring and I an (S, S)-bimodule, which is a general ring in which $(vw)s = v(ws)$, $(vs)w = v(sw)$, and $(sv)w = s(vw)$ hold for all $v, w \in I$ and $s \in S$. Then the *ideal-extension* (it is also called the Dorroh extension) $I(S; I)$ of S by I is defined to be the additive abelian group $E(S; I) = S \oplus I$ with multiplication $(s, v)(r, w) = (sr, sw + vr + vw)$. In this case, $I \lhd E(S; I)$, and $E(S; I)/I \cong S$. In particular, $E(\mathbb{Z}; I)$ is the standard unitization of the general ring I.

Clean general ideal-extensions were considered in [14, Proposition 7]. Now we deal with quasipolar general ideal-extensions.

Proposition 3.1 *The following are equivalent for a general ring I:*

(1) *I is quasipolar.*

(2) *$(0, a)$ is quasipolar in $E(\mathbb{Z}; I)$ for all $a \in I$.*

(3) *There exists a ring* S *such that* $I =_S I_S$ *and* $(0,a)$ *is quasipolar in* $E(S;I)$ *for all* $a \in I$.

Proof (1) $\Rightarrow$ (2) Let $a \in I$ and $R = E(\mathbb{Z};I)$. By Theorem 2.16, we have $-a = s+q$ where $s \in I$ is strongly regular, $q \in QN(I)$, and $sq = qs = 0$. Write $(0,a) = (0,-s)+(0,-q)$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. This implies that $(0,-s) = (0,-s)^2(0,-y)$ and $(0,-y) \in comm^2\big((0,-s)\big)$, and so, by Lemma 2.13, $(0,-s)$ is strongly regular in R. Assume that $(x,y) \in comm\big((0,q)\big)$. Then we have $x+y \in comm(q)$ and so $(x+y)q \in Q(I)$ because $q \in QN(I)$. This gives $(1,0)+(x,y)(0,-q) = \big(1,-(x+y)q\big) \in U(R)$ (the inverse is $(1,-t)$ where $(x+y)q*t = 0 = t*(x+y)q$). Hence, $(0,-q) \in R^{qnil}$. As $sq = qs = 0$, we see that $(0,-s)(0,-q) = (0,-q)(0,-s) = (0,0)$, and so $(0,a) \in R$ is quasipolar by Corollary 2.17.

(2) $\Rightarrow$ (3) It is clear with $S = \mathbb{Z}$.

(3) $\Rightarrow$ (1) Let $a \in I$ and $R = E(S;I)$. By (3), $(0,-a)+(e,p) = (e,p-a)$ where $(e,p)^2 = (e,p) \in comm^2\big((0,-a)\big)$, $(e,p-a) \in U(R)$, and $(0,-a)(e,p) = \big(0,-a(e+p)\big) \in R^{qnil}$. Since $(e,p)^2 = (e,p)$, we have $e^2 = e$ and $p = ep+pe+p^2$. This gives that $e = 1_S$ because $(e,p-a) \in U(R)$, so $-p$ is an idempotent in I. As $(-1,a-p) \in U(R)$, there exists $q \in I$ such that $q*(a-p) = 0 = (a-p)*q$. This implies that $a+(-p) \in Q(I)$. If $ax = xa$, then we have $(0,x) \in comm\big((0,-a)\big)$ and so $xp = px$ because $(1,p) \in comm^2\big((0,-a)\big)$. Hence, $-p \in comm^2(a)$. Now we show that $a+ap \in QN(I)$. Let $x(a+ap) = (a+ap)x$. As $\big(0,-a(1_S+p)\big) \in R^{qnil}$, it follows that $x(a+ap) \in Q(I)$, so $a \in I$ is quasipolar. The proof is completed. □

Theorem 3.2 *Let* I *be a quasipolar general ring and* $A \lhd I$. *Then* A *is quasipolar.*

Proof Let $R = E(\mathbb{Z};I)$ and $a \in A$. By Theorem 2.16, we have $-a = s+q$ where $s \in I$ is strongly regular, $s \in comm^2(a)$, $q \in QN(I)$, and $sq = qs = 0$. Write $(0,a) = (0,-s)+(0,-q)$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. This implies that $(0,-s) = (0,-s)^2(0,-y)$ and $(0,-y) \in comm^2\big((0,-s)\big)$, and so, by Lemma 2.13, $(0,-s)$ is strongly regular in R. Assume that $(m,n) \in comm\big((0,q)\big)$. Then we have $x+y \in comm(q)$ and so $(m+n)q \in Q(I)$ because $q \in QN(I)$. This gives $(1,0)+(m,n)(0,-q) = \big(1,-(m+n)q\big) \in U(R)$ (the inverse is $(1,-t)$ where $(m+n)q*t = 0 = t*(m+n)q$). Hence, $(0,-q) \in R^{qnil}$. As $sq = qs = 0$, we see that $(0,-s)(0,-q) = (0,-q)(0,-s) = (0,0)$. Let $(u,v)(0,a) = (0,a)(u,v)$. Then $(u+v) \in comm(a)$ and so $(u+v) \in comm(s)$ since $s \in comm^2(a)$. This proves $(u,v)(0,-s) = (0,-s)(u,v)$. That is, $(0,-s) \in comm^2\big((0,a)\big)$, so $(0,a) \in R$ is quasipolar by Corollary 2.17. As $A \cong (0,A) \lhd R$, A is quasipolar by Proposition 3.1 and Remark 2.3. □

This result shows that any ideal of a quasipolar general ring is a quasipolar general ring. However, the converse need not be true in general, as the following example shows.

Given a ring R, the set $I = \{(a,b) \mid a,b \in R\}$ becomes a general ring (without identity) with addition defined componentwise and multiplication defined by $(a,b)(c,d) = (ac,ad)$. Then $I \cong \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} = J$ where J is a right ideal of $M_2(R)$.

Example 3.3 Consider the local ring $R = \mathbb{Z}_{(2)} = \{\frac{m}{n} \in \mathbb{Q} \mid 2 \nmid n\}$ and $(a,b) \in I$. If $a \in J(R)$, then it is easy to verify that $(a,b) \in J(I)$ and so (a,b) is quasipolar in I. If $a \notin J(R)$, then $a \in 1+J(R)$, so $(a,b)+(1,a^{-1}b) =$

$(a+1, b+a^{-1}b)$ where $(1, a^{-1}b)^2 = (1, a^{-1}b) \in comm^2\big((a,b)\big)$ and $(a+1, b+a^{-1}b) \in J(I) \subseteq Q(I)$. Further, since $(a,b)-(a,b)(1,a^{-1}b) = (0,0) \in QN(I)$, (a,b) is quasipolar in I. Hence, I is a quasipolar general ring. On the other hand, $M_2(R)$ is not a quasipolar ring because $M_2(R)$ is not a strongly clean ring (see [16]).

Lemma 3.4 *Let $e^2 = e \in I$. Then $QN(eIe) = eIe \cap QN(I)$.*

Proof Let $a \in QN(eIe)$ and $ab = ba$ for some $b \in I$. Then $a \cdot ebe = abe = bae = ba$ and $ebe \cdot a = eba = eab = ab$, so $ebe \in comm(a)$. Since $a \in QN(eIe)$, we have $ab * x = 0 = x * ab$ for some $x \in eIe$. Hence, $a \in eIe \cap QN(I)$. This gives that $QN(eIe) \subseteq eIe \cap QN(I)$. Conversely, let $a \in eIe \cap QN(I)$ and $aere = erea$ for some $ere \in eIe$. This implies that $ae = ea = a$. Since $a \in QN(I)$, $are + y - arey = 0 = are + y - yare$ for some $y \in I$. Then $are + eye - areye = 0 = are + eye - eyare$ and so $are \in Q(eIe)$. Therefore, $eIe \cap QN(I) \subseteq QN(eIe)$. We complete the proof. □

Theorem 3.5 *Let I be a quasipolar general ring with $e^2 = e \in I$. Then eIe is quasipolar.*

Proof Let $a \in eIe$. Then there exists $p^2 = p \in comm^2(a)$ such that $a + p = q \in Q(I)$ and $a - ap \in QN(I)$. Since $ae = ea$, we have $ep = pe$. This implies that $a + epe = eqe$ where $epe^2 = epe$ and $eqe \in Q(I) \cap eIe = Q(eIe)$. It is easy to see that $epe \in comm^2(a)$ because $p^2 = p \in comm^2(a)$. As $a - ap \in QN(I)$, we have $a - ap = a - aep = a - aepe = a - ape = e(a-ap)e \in QN(I) \cap eIe = QN(eIe)$ by Lemma 3.4. Hence, eIe is quasipolar. □

Corollary 3.6 [19, Proposition 3.6] Let R be a ring with $e^2 = e \in R$. If R is quasipolar, then so is eRe.

4. Pseudopolar elements

An element a of R is *pseudo-Drazin invertible* if there exist $b \in R$ and $k \in \mathbb{N}$ satisfying $ab^2 = b$, $b \in comm^2(a)$, and $(a - a^2b)^k \in J(R)$. Such a b, if it exists, is unique; it is called a *pseudo-Drazin inverse* of a. Wang and Chen [18] showed that an element $a \in R$ is pseudo-Drazin invertible if and only if a is pseudopolar; that is, there exist $p \in R$ and $k \in \mathbb{N}$ such that $p^2 = p \in comm^2(a)$, $a + p \in U(R)$, and $a^k p \in J(R)$.

A characterization of pseudopolar elements can be given as follows.

Theorem 4.1 *Let R be a ring and let $a \in R$. Then the following are equivalent:*

(1) *a is pseudopolar.*

(2) *$a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in J^{\#}(R)$, and $sq = qs = 0$.*

Proof (1) ⇒ (2) Assume that $a \in R$ is pseudopolar. Then there exist $b \in comm^2(a)$ and $k \in \mathbb{N}$ such that $ab^2 = b$ and $(a - a^2b)^k \in J(R)$. Set $s = a^2b$ and $q = a - a^2b$. This gives $s \in comm^2(a)$, $q \in J^{\#}(R)$ and $sq = qs = a^2b(a - a^2b) = 0$. It is easy to see that $s = s^2b$ and so $s \in R$ is strongly regular.

(2) ⇒ (1) Suppose that $a = s + q$ where s is strongly regular, $s \in comm^2(a)$, $q \in J^{\#}(R)$, and $sq = qs = 0$. Since s is strongly regular, there exists $y \in comm^2(s)$ such that $s = s^2y$ by Lemma 2.13. Then we have that $1 - p = sy = ys$ is an idempotent, $p \in comm^2(a)$, and $yq = qy$. As $q \in J^{\#}(R)$, we see that

$q^n \in J(R)$ and so $1+q \in U(R)$ for some $n \in \mathbb{N}$. Hence, $(a+p)(y^2s+p) = 1+q \in U(R)$ and so $a+p \in U(R)$. Moreover, $a^np = (s^n+q^n)(1-sy) = q^n \in J(R)$ because $s^n = s^{n+1}y$, so (1) holds. □

Note that if R is pseudopolar, then R is quasipolar by Theorem 4.1 and Corollary 2.17. Further, if $-a$ is pseudopolar, then so is a by Theorem 4.1.

Combining Theorem 2.10 with Theorem 4.1, we obtain the following result.

Corollary 4.2 [18, Theorem 2.1] Let R be a ring. Then R is strongly π-regular if and only if R is pseudopolar and $J(R)$ is nil.

We give a different proof of the [18, Theorem 2.4].

Theorem 4.3 *Let R be a ring. If R is pseudopolar and $J^{\#}(R) = J(R)$, then R is semiregular. The converse holds if R is abelian.*

Proof Assume that R is pseudopolar and $J^{\#}(R) = J(R)$. According to Theorem 4.1, $R/J(R)$ is strongly regular. Hence, R is semiregular by [13, Theorem 2.9]. Conversely, let $a \in R$. Then there exists $b \in R$ with $bab = b$ and $a - aba \in J(R)$. Write $a = aba + (a - aba)$, say $s = aba$ and $q = a - aba$. Since $a - aba \in J(R) \subseteq J^{\#}(R)$ and R is abelian, we see that $s \in comm^2(a)$, $q \in J^{\#}(R)$, $s = aba = (aba)^2b = s^2b$, and $sq = qs = aba(a - aba) = a^2ba - a^2ba = 0$. By Theorem 4.1, a is pseudopolar. In view of Theorem 2.23, we see that $J^{\#}(R) = J(R)$. □

Recall that an element $a \in R$ is *strongly π-rad clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e \in U(R)$ and $a^ne \in J(R)$ for some $n \in \mathbb{N}$. A ring R is *strongly π-rad clean* if every element in R is strongly π-rad clean (see [5]). We now give the relations among quasipolarity, strong π-rad cleanness, and pseudopolarity.

Theorem 4.4 *Let R be a ring. Then R is pseudopolar if and only if R is strongly π-rad clean and quasipolar.*

Proof The "only if" part is easy to see and so we only have to prove the "if" part. Let $a \in R$. Then there exists $p^2 = p \in comm^2(a)$ such that $a+p \in U(R)$ and $ap \in R^{qnil}$ since R is quasipolar. Further, there exists $q \in comm(a)$ such that $-a-q \in U(R)$ and $a^nq \in J(R)$ for some $n \in \mathbb{N}$ because R is strongly π-rad clean. Since $a^nq \in J(R)$, we have $aq \in R^{qnil}$. By [12, Proposition 2.3], we see that $p = q$. Hence, a is pseudopolar, as desired. □

Corollary 4.5 [18, Corollary 2.12] Let R be a ring with $e^2 = e \in R$. If R is pseudopolar, then so is eRe.

Proof Assume that R is pseudopolar. Then R is strongly π-rad clean and quasipolar by Theorem 4.4. In view of [5, Corollary 4.2.2] and Corollary 3.6, eRe is strongly π-rad clean and quasipolar. Hence, eRe is pseudopolar again by Theorem 4.4. □

Remark 4.6 Let S be a commutative ring and $R = M_2(S)$. By [18, Example 4.3], we have $J^{\#}(R) = R^{qnil}$. Hence, by Theorem 4.1 and Corollary 2.17, R is quasipolar if and only if R is pseudopolar. Further, if S is commutative local, then R is pseudopolar if and only if R is quasipolar if and only if R is strongly clean (by [7, Corollary 2.13]) if and only if R is strongly π-rad clean (by [5, Corollary 4.3.7]).

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