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## Positive solutions of first order boundary value problems with nonlinear nonlocal boundary conditions

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**Abstract:** We consider the existence of positive solutions of the nonlinear first order problem with a nonlinear nonlocal boundary condition given by

$$\begin{aligned}x'(t) &= r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1] \\ \lambda x(0) &= x(1) + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1],\end{aligned}$$

where  $r : [0, 1] \rightarrow [0, \infty)$  is continuous, the nonlocal points satisfy  $0 \leq \tau_1 < \tau_2 < \dots < \tau_n \leq 1$ , the nonlinear functions  $f_i$  and  $\Lambda_j$  are continuous mappings from  $[0, 1] \times [0, \infty) \rightarrow [0, \infty)$  for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$  respectively, and  $\lambda > 1$  is a positive parameter. The Leray–Schauder theorem and Leggett–Williams fixed point theorem were used to prove our results.

**Key words:** Positive solutions, Leray–Schauder fixed point theorem, nonlinear boundary conditions

### 1. Introduction

Consider the first order boundary value problem (BVP) with nonlinear nonlocal boundary condition

$$x'(t) = r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1] \tag{1.1}$$

$$\lambda x(0) = x(1) + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1], \tag{1.2}$$

where  $r : [0, 1] \rightarrow [0, \infty)$  is continuous,  $f_i : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ ,  $i = 1, 2, \dots, m$ , and the nonlocal points  $\Lambda_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ ,  $j = 1, 2, \dots, n$  are continuous with  $0 \leq \tau_1 < \tau_2 < \dots < \tau_n \leq 1$ , the nonlinear functions  $\Lambda_j$  satisfy

$$0 \leq \Lambda_j(t, x) \leq x\Psi_j(t, x), \quad t \in [0, 1], \quad x \in [0, \infty) \tag{1.3}$$

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for some positive continuous functions  $\Psi_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ , and the scalar  $\lambda$  satisfies

$$\lambda > \left( 1 + \sum_{j=1}^n \beta_j \right) \exp \left( \int_0^1 r(\eta) d\eta \right), \tag{1.4}$$

where

$$\beta_j = \max_{[0,1] \times [0,c]} \Psi_j(t, x) \tag{1.5}$$

for some real constant  $c$ .

The BVP (1.1)–(1.2) was first studied by Anderson [1], who used the Leggett–William multiple fixed point theorem to obtain three positive solutions of the BVP (1.1)–(1.2). The results obtained by Anderson [1] are motivated by the results obtained for time scales in [2, 5–9, 12, 16–19]. Recently, Padhi et al. [15] used the Leggett–Williams multiple fixed point theorem to establish a sufficient condition for the existence of at least three positive solutions of the BVP (1.1)–(1.2), and improved the results in [1].

Motivated by the work by Anderson [1], Çetin and Topal in [3] used monotone iteration and established an iterative scheme for the existence and approximation of two positive solutions of the nonlinear nonlocal first-order multipoint problem with sign changing nonlinearities

$$\begin{aligned} x'(t) + r(t)x(t) &= \sum_{i=1}^m f_i(t, x(t)), \quad t \in [0, 1] \\ x(0) = x(1) &+ \sum_{j=1}^n g_j(t_j, x(t_j)), \quad \tau_j \in [0, 1], \end{aligned}$$

where  $r : [0, 1] \rightarrow [0, \infty)$  is continuous,  $f_i : [0, 1] \times [0, \infty) \rightarrow (-\infty, \infty)$ ,  $i = 1, 2, \dots, m$ , and the nonlocal points  $g_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ ,  $j = 1, 2, \dots, n$  are continuous, and satisfy  $0 \leq t_1 < t_2 < \dots < t_n \leq 1$ .

This work has been divided into three sections. Section 1 is the Introduction, which contains the basic notations and the fixed point theorems used in this paper. Statements of two theorems, namely the Leray–Schauder theorem and Leggett–Williams multiple fixed point theorem, are given in this section. In Section 3, we give an example to show that the use of the Leray–Schauder theorem is not sufficient to have a complete study on the existence of positive solutions. This forces us to use the Leggett–Williams multiple fixed point theorem in the example to find the number of positive solutions. Thus, we have applied the Leray–Schauder theorem and Leggett–Williams multiple fixed point theorem in Section 2 to obtain three theorems on the existence of positive solutions of the BVP (1.1)–(1.2). In Section 3, all these three theorems are applied to an example to get a complete study on the existence of positive solutions. The motivation for the study of this example for a particular case came from [4] and [14]. An extensive use the Leggett–Williams fixed point theorem can be found in [13] for the existence of positive periodic solutions of first order functional differential equations.

We shall use the following notations for our use in the sequel: Let  $X$  be a Banach space. For any cone  $K$  on  $X$ , we denote  $K_a = \{x \in K : \|x\| < a\}$ ,  $\bar{K}_a = \{x \in K : \|x\| \leq a\}$  and

$$K(\psi, b, c) = \{x \in K; \psi(x) \geq b \text{ and } \|x\| \leq c\}$$

for any constants  $a > 0, b > 0$ , and  $c > 0$ . With the above notations, we now state the following fixed point theorems.

**Theorem 1.1** ([10], Leray–Schauder) *Let  $K$  be a convex subset of a Banach space  $X$ ,  $0 \in K$ ,  $A : K \rightarrow K$  be a completely continuous operator. Then either (i)  $A$  has at least one fixed point in  $K$ , or (ii) the set  $\{x \in K : x = \mu Ax, 0 < \mu < 1\}$  is unbounded.*

**Theorem 1.2** ([11, Theorem 3.3]) (*Leggett–Williams fixed point theorem*) *Let  $X = (X, \|\cdot\|)$  be a Banach space and  $K \subset X$  be a cone, and  $c_4 > 0$  be a constant. Suppose there exists a concave nonnegative continuous function  $\psi$  on  $K$  with  $\psi(x) \leq \|x\|$  for  $x \in \overline{K}_{c_4}$  and let  $A : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$  be a continuous compact map. Assume that there are numbers  $c_1, c_2$ , and  $c_3$  with  $0 < c_1 < c_2 < c_3 \leq c_4$  such that*

- (i)  $\{x \in K(\psi, c_2, c_3); \psi(x) > c_2\} \neq \emptyset$  and  $\psi(Ax) > c_2$  for all  $x \in K(\psi, c_2, c_3)$ ;
- (ii)  $\|Ax\| < c_1$  for all  $x \in \overline{K}_{c_1}$ ;
- (iii)  $\psi(Ax) > c_2$  for all  $x \in K(\psi, c_2, c_4)$  with  $\|Ax\| > c_3$ .

*Then  $A$  has at least three fixed points  $x_1, x_2$ , and  $x_3$  in  $\overline{K}_{c_4}$ . Furthermore, we have  $x_1 \in \overline{K}_{c_1}$ ,  $x_2 \in \{x \in K(\psi, c_2, c_4) : \psi(x) > c_2\}$ , and  $x_3 \in \overline{K}_{c_4} \setminus \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$ .*

**Remark 1** *According to the localization of the fixed points in Theorem 1.2, one of them is possibly a zero (namely  $x_1 \in \overline{K}_{c_1}$ ). Thus, the operator  $A$ , stated in Theorem 1.2, has at least two positive fixed points and a zero fixed point as can be easily observed. Accordingly, the BVP (1.1)–(1.2) has two positive  $T$ -periodic solutions and a possible trivial solution (if the conditions of Theorem 1.2 are satisfied).*

**2. Main results**

Clearly, the BVP (1.1)–(1.2) is equivalent to the integral equation

$$x(t) = \sum_{i=1}^m \int_0^1 G(t, s) f_i(s, x(s)) ds + \frac{\exp(\int_0^t r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp(\int_0^1 r(\eta) d\eta)}, \tag{2.1}$$

where  $G(t, s)$  is Green’s kernel, given by

$$G(t, s) = \frac{\exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \times \begin{cases} \lambda & ; \text{ if } 0 \leq s \leq t \leq 1 \\ \exp(\int_0^1 r(\eta) d\eta) & ; \text{ if } 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

**Theorem 2.1** *Let*

$$f_{i0} = \limsup_{x \rightarrow 0+, 0 \leq t \leq 1} \frac{f_i(t, x)}{x} = 0, \quad i = 1, 2, \dots, m \tag{2.3}$$

*hold. Then the BVP (1.1)–(1.2) has at least one positive solution.*

**Proof** Let  $X = C[0, 1]$ ; then  $X$  is a Banach space endowed with the sup. norm. From (2.3), there exist constants  $\epsilon$  and  $B > 0$  such that

$$f_i(t, x) < \epsilon x \text{ for } 0 < x \leq B, 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m$$

holds, where  $\epsilon > 0$  is chosen such that it satisfies

$$0 < \epsilon < \frac{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right)}{\lambda m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) ds}. \tag{2.4}$$

On  $X$ , we define a convex set  $K$  by

$$K = \{x(t); x(t) \in X, x(t) \geq 0, x(t) \text{ is nondecreasing}, x(t) \leq B, 0 \leq t \leq 1\}, \tag{2.5}$$

and an operator  $A : K \rightarrow X$  by

$$(Ax)(t) = \sum_{i=1}^m \int_0^1 G(t,s) f_i(s, x(s)) ds + \frac{\exp(\int_0^t r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))}, \tag{2.6}$$

where  $G(t, s)$  is Green's kernel given in (2.2). The operator  $A$  in (2.6) can be rewritten as

$$\begin{aligned} Ax(t) &= \sum_{i=1}^m \int_0^t \frac{\lambda \exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \\ &\quad + \sum_{i=1}^m \int_t^1 \frac{\exp(\int_0^1 r(\eta) d\eta) \exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \\ &\quad + \frac{\exp(\int_0^t r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{\lambda - \exp(\int_0^1 r(\eta) d\eta)}. \end{aligned} \tag{2.7}$$

Let  $x \in K$ . Then the positivity of  $f_i, i = 1, 2, \dots, m$  and  $\Lambda_j, j = 1, 2, \dots, n$  shows that  $(Ax)(t) \geq 0$  for all  $t \in [0, 1]$ . Now we show that the fixed points of the operator  $A$  are the solutions of the BVP (1.1)–(1.2). In fact, if  $x = Ax$ , then from (2.7), we have

$$\begin{aligned} \lambda x(0) - x(1) &= \sum_{i=1}^m \int_0^1 \frac{\lambda \exp(\int_s^1 r(s) ds)}{(\lambda - \exp(\int_0^1 r(s) ds))} f_i(s, x(s)) ds \\ &\quad + \lambda \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\ &\quad - \sum_{i=1}^m \int_0^1 \frac{\lambda \exp(\int_s^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \\ &\quad - \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \\ &= \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)), \quad \tau_j \in [0, 1]. \end{aligned}$$

Hence the boundary condition (1.2) is satisfied. Again differentiating (2.7) with respect to  $t$ , with  $Ax = x$ , we obtain

$$\begin{aligned}
 x'(t) &= \sum_{i=1}^m \int_0^t \frac{\lambda r(t) \exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds + \sum_{i=1}^m \frac{\lambda}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(t, x(t)) \\
 &+ \sum_{i=1}^m \int_t^1 \frac{r(t) \exp(\int_0^1 r(\eta) d\eta) \exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \\
 &- \sum_{i=1}^m \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(t, x(t)) + \frac{r(t) \exp(\int_0^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \\
 &= r(t) \left[ \sum_{i=1}^m \int_0^t \lambda \frac{\exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \right] \\
 &+ r(t) \left[ \sum_{i=1}^m \int_t^1 \frac{\exp(\int_0^1 r(\eta) d\eta) \exp(\int_s^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} f_i(s, x(s)) ds \right] \\
 &+ r(t) \left[ \frac{\exp(\int_0^t r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \right] + \sum_{i=1}^m f_i(t, x(t)) \\
 &= r(t)x(t) + \sum_{i=1}^m f_i(t, x(t)), \text{ (using (2.1)),}
 \end{aligned}$$

which shows that  $x(t)$  satisfies (1.1). Moreover,  $(Ax)'(t) = r(t)x(t) + \sum_{i=1}^m f_i(t, x(t))$ ,  $t \in [0, 1]$  shows that  $Ax$  is nondecreasing,  $t \in [0, 1]$ .

Next, for  $0 < x \leq B$ , we have

$$\begin{aligned}
 (Ax)(t) \leq \|Ax\| &= Ax(1) = \sum_{i=1}^m \int_0^1 G(1, s) f_i(s, x(s)) ds + \frac{\exp(\int_0^1 r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\
 &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) f_i(s, x(s)) ds + x(t) \sum_{j=1}^n \beta_j \right] \tag{2.8} \\
 &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \epsilon \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) \|x\| ds + \|x\| \sum_{j=1}^n \beta_j \right] \\
 &\leq \frac{B \exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \epsilon m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) ds + \sum_{j=1}^n \beta_j \right] \\
 &\leq B. \tag{2.9}
 \end{aligned}$$

This proves that  $A(K) \subset K$ . One may verify that  $A$  is completely continuous.

In order to use Theorem 1.1, we consider  $x \in K$  with  $x(t) = \mu(Ax)(t)$ ,  $0 < \mu < 1$ . Then, using (2.9), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \leq B,$$

which implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 1.1, the operator  $A$  has a fixed point in  $X$ , which is a positive solution of the BVP (1.1)–(1.2). This completes the proof of the theorem.  $\square$

**Theorem 2.2** *Let*

$$f_{i\infty} = \limsup_{x \rightarrow \infty, 0 \leq t \leq 1} \frac{f_i(t, x)}{x} = 0, \quad i = 1, 2, \dots, m \tag{2.10}$$

*hold. Then the BVP (1.1)–(1.2) has at least one positive solution.*

**Proof** Let  $X = C[0, 1]$ ; then  $X$  is a Banach space endowed with the sup. norm. From (2.10), there exist positive constants  $\epsilon$  and  $N$  such that

$$f_i(t, x) < \epsilon x \text{ for } x \geq N, \quad 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m,$$

where  $\epsilon$  is chosen such that (2.4) is satisfied. Let

$$\gamma = \max_{0 \leq t \leq 1, 0 \leq x \leq N, 1 \leq i \leq m} f_i(t, x).$$

Then

$$f_i(t, x) < \epsilon x + \gamma \text{ for } x \geq 0, \quad 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m.$$

For the above choice of  $\epsilon$  and  $\gamma$ , we consider a constant  $B$  by

$$B \geq \frac{\lambda \gamma m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) ds}{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right) - \lambda \epsilon m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) ds}. \tag{2.11}$$

Now we define a convex set  $K$  on  $X$  by (2.5) and an operator  $A : K \rightarrow X$  by (2.6), where  $G(t, s)$  is Green’s kernel given in (2.2). One may verify that  $A$  is completely continuous. Proceeding as in Theorem 2.1, we can prove that a fixed point of the operator  $A$  in the cone  $K$  is equivalent to the existence of a positive solution of the BVP (1.1)–(1.2),  $(Ax)(t) \geq 0$  and  $Ax$  is nondecreasing for  $0 \leq t \leq 1$ . Now we show that  $Ax \leq B$  for  $0 \leq t \leq 1$ , where  $B$  is defined in (2.11). For  $0 < x \leq B$ , from (2.8), we have

$$\begin{aligned} (Ax)(t) &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) (\epsilon \|x\| + \gamma) ds + \|x\| \sum_{j=1}^n \beta_j \right] \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda m (\epsilon B + \gamma) \int_0^1 \exp(-\int_0^s r(\eta) d\eta) ds + B \sum_{j=1}^n \beta_j \right] \\ &\leq B. \end{aligned} \tag{2.12}$$

This proves that  $A(K) \subset K$ .

Next suppose that  $x \in K$  with  $x(t) = \mu(Ax)(t)$ ,  $0 < \mu < 1$ . Then, using (2.12), we have

$$x(t) = \mu(Ax)(t) < (Ax)(t) \leq B,$$

which, in turn, implies that the set

$$\{x \in K : x = \mu Ax, 0 < \mu < 1\}$$

is bounded. Hence, by Theorem 1.1, the operator  $A$  has a fixed point in  $X$ , which is a positive solution of the BVP (1.1)–(1.2). This completes the proof of the theorem.  $\square$

**Remark 2** Theorems 2.1 and 2.2 require at least one of the conditions (2.3) or (2.10) to guarantee the existence of a positive solution of the BVP (1.1)–(1.2). In Section 3, we give an example, that is the BVP (3.1)–(3.2), where both Theorems 2.1 and 2.2 are applicable, guaranteeing the existence of a positive solution independently. Therefore, now we are confronted with a question regarding the number of positive solutions admitted by the BVP (3.1)–(3.2). In this case, we shall use the following theorem regarding the number of positive solutions admitted by the BVP (3.1)–(3.2).

**Theorem 2.3** Suppose that the conditions (2.3) and (2.10) are satisfied, and there exists a positive constant  $c_2 > 0$  such that

$$\sum_{i=1}^m \int_0^1 \exp\left(\int_s^1 r(\eta) d\eta\right) f_i(s, x(s)) ds + \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} > c_2[\lambda - \exp(\int_0^1 r(\eta) d\eta)], \quad c_2 \leq x \leq \lambda c_2 \quad (2.13)$$

for  $0 \leq t \leq 1$  holds. Then the BVP (1.1)–(1.2) has at least two positive solutions.

**Proof** Let  $X = C[0, 1]$  be a Banach space endowed with the sup. norm. On the space  $X$ , we define a cone  $K$  by

$$K = \{x \in X; x(t) \geq 0, x(t) \text{ nondecreasing}, t \in [0, 1]\}$$

and an operator  $A : K \rightarrow X$  by (2.6), where  $G(t, s)$  is Green’s kernel given in (2.2). Proceeding as in the lines of Theorem 2.1, we can show that  $A(K) \subset K$  and  $A : K \rightarrow K$  is completely continuous. Further, the existence of a positive solution of the BVP (1.1)–(1.2) is equivalent to the existence of a fixed point of the operator  $A$  in  $K$ .

First we consider (2.10). Then there exist constants  $\epsilon > 0$  and  $N > 0$  such that

$$f_i(t, x) < \epsilon x \text{ for } x \geq N, 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m,$$

where  $\epsilon > 0$  is chosen so that it satisfies the property (2.4). Let

$$\gamma = \max_{0 \leq t \leq 1, 0 \leq x \leq N, 1 \leq i \leq m} f_i(t, x).$$

Then

$$f_i(t, x) < \epsilon x + \gamma \text{ for } x \geq 0, 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m.$$

Choose a constant  $c_4 > 0$  such that

$$c_4 \geq \left\{ \lambda c_2, \frac{\lambda \gamma m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) ds}{\lambda - \left(1 + \sum_{j=1}^n \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right) - \lambda \epsilon m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) ds} \right\}.$$



For  $x \in \overline{K}_{c_4}$ , we have

$$\begin{aligned} (Ax)(t) \leq \|Ax\| = Ax(1) &= \sum_{i=1}^m \int_0^1 G(1, s) f_i(s, x(s)) ds + \frac{\exp(\int_0^1 r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) f_i(s, x(s)) ds + x(t) \sum_{j=1}^n \beta_j \right] \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) (\epsilon \|x\| + \gamma) ds + \|x\| \sum_{j=1}^n \beta_j \right] \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda m (\epsilon c_4 + \gamma) \int_0^1 \exp(-\int_0^s r(\eta) d\eta) ds + c_4 \sum_{j=1}^n \beta_j \right] \\ &\leq c_4, \end{aligned}$$

that is,  $A : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$ .

Next we consider (2.3). Then there exist constants  $\epsilon$  and  $c_1 \in (0, c_2)$  such that

$$f_i(t, x) < \epsilon x \text{ for } 0 < x \leq c_1, 0 \leq t \leq 1 \text{ and } 1 \leq i \leq m,$$

where  $\epsilon$  satisfies the property (2.4).

For  $x \in \overline{K}_{c_1}$ , we have

$$\begin{aligned} (Ax)(t) \leq \|Ax\| = Ax(1) &= \sum_{i=1}^m \int_0^1 G(1, s) f_i(s, x(s)) ds + \frac{\exp(\int_0^1 r(\eta) d\eta) \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) f_i(s, x(s)) ds + x(t) \sum_{j=1}^n \beta_j \right] \\ &\leq \frac{\exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \epsilon \sum_{i=1}^m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) \|x\| ds + \|x\| \sum_{j=1}^n \beta_j \right] \\ &\leq \frac{c_1 \exp(\int_0^1 r(\eta) d\eta)}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \lambda \epsilon m \int_0^1 \exp(-\int_0^s r(\eta) d\eta) ds + \sum_{j=1}^n \beta_j \right] \\ &< c_1. \end{aligned}$$

This proves the condition (ii) of Theorem 1.2.

Set  $c_3 = \lambda c_2$ . In order to verify the condition (i) of Theorem 1.2, we set  $\theta(t) = \lambda c_2$  for  $t \in [0, 1]$ . Let  $\psi(t) = \min_{t \in [0, 1]} x(t)$  be a nonnegative concave functional on  $K$ . Since  $\psi(\theta(t)) = \min_{t \in [0, 1]} \theta(t) = \lambda c_2 > c_2; c_2 \leq \psi(x), \|x\| = \lambda c_2$ , then the set  $\{x \in K; c_2 \leq \psi(x), \|x\| \leq \lambda c_2\}$  is nonempty. Let  $x \in (K, \psi, c_2, c_3)$ ; then

$c_2 \leq \psi(x) \leq x \leq \|x\| = x(1) = \lambda c_2 = c_3$ , and hence

$$\begin{aligned} \psi(Ax)(t) &= Ax(0) = \sum_{i=1}^m \int_0^1 G(0, s) f_i(s, x(s)) ds + \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\ &= \frac{1}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \left[ \sum_{i=1}^m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) f_i(s, x(s)) ds + \sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j)) \right] \\ &\geq \frac{1}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \sum_{i=1}^m \int_0^1 \exp(\int_s^1 r(\eta) d\eta) f_i(s, x(s)) ds + \frac{\sum_{j=1}^n \Lambda_j(\tau_j, x(\tau_j))}{(\lambda - \exp(\int_0^1 r(\eta) d\eta))} \\ &> c_2 \text{ (using (2.13))} \end{aligned}$$

holds. Thus, the condition (i) of Theorem 1.2 is satisfied.

Finally, suppose that  $x \in K(\psi, c_2, c_3)$  with  $\|Ax\| > \lambda c_2 = c_3$ . Then

$$\psi(Ax) = (Ax)(0) \geq \frac{1}{\lambda}(Ax)(1) = \frac{\|Ax\|}{\lambda} > \frac{\lambda c_2}{\lambda} = c_2$$

implies that the condition (iii) of Theorem 1.2 is satisfied. Hence the BVP (1.1)–(1.2) has at least three solutions. Consequently, the BVP (1.1)–(1.2) has at least two positive solutions. This completes the proof of the theorem.  $\square$

**Remark 3** Anderson in [1] and Padhi et al. in [15] applied the Leggett–Williams multiple fixed point theorem [11] to obtain sufficient conditions for the existence of three positive solutions of the BVP (1.1)–(1.2). Although the conditions of Theorem 2.3 imply the conditions obtained in [15] and [1], the conditions of Theorem 2.3 are easy to use. We have used Theorem 2.3 to complete the example given in Section 3.

### 3. A complete example

Consider the first order BVP

$$x'(t) = x(t) + \frac{x^n(t)}{1 + x^m(t)}, \quad t \in [0, 1] \tag{3.1}$$

$$\lambda x(0) = x(1) + \Lambda(\tau, x(\tau)), \quad \tau \in [0, 1], \tag{3.2}$$

where  $\lambda > \frac{3e}{2}$ ,

$$\Lambda(t, x) = \begin{cases} \frac{x}{4} \left(1 + e^{-\frac{1}{x(t)-1}}\right), & x > 1 \\ \frac{x}{2}, & x \leq 1, \end{cases}$$

and  $m$  and  $n$  are nonnegative integers satisfying the property

$$1 > \frac{(\lambda - e)}{(e - 1)} \lambda^{n-1} \frac{m}{m - n + 1} \left(\frac{m - n + 1}{n - 1}\right)^{\frac{n-1}{m}} \text{ for } 1 < n < m. \tag{3.3}$$

Here  $r(t) \equiv 1$  and  $f(t, x) = \frac{x^n}{1+x^m}$ ,  $t \in [0, 1]$ . Clearly  $\Lambda(t, x) \leq \frac{x}{2}$  with  $\beta = \frac{1}{2}$ . Consequently,  $\lambda > (1+\beta)e = \frac{3e}{2}$ .

First we consider the case  $0 \leq n \leq 1$ . Then  $\lim_{x \rightarrow \infty} \frac{f(t,x)}{x} = 0$  implies that Theorem 2.2 can be applied to the BVP (3.1)–(3.2). On the other hand,  $\liminf_{x \rightarrow 0^+} \frac{f(t,x)}{x} \neq 0$  implies that Theorem 2.1 cannot be applied to the BVP (3.1)–(3.2). Hence, by Theorem 2.2, the BVP (3.1)–(3.2) has a positive solution.

Next we consider the case when  $n \geq m + 1$ . Then

$$\lim_{x \rightarrow \infty} \frac{f(t,x)}{x} = \lim_{x \rightarrow \infty} \frac{x^{n-1}}{1+x^m} = \lim_{x \rightarrow \infty} \frac{x^{n-m-1}}{\frac{1}{x^m} + 1} \neq 0$$

implies that Theorem 2.2 cannot be applied to this example. On the other hand,

$$\lim_{x \rightarrow 0} \frac{f(t,x)}{x} = \lim_{x \rightarrow 0} \frac{x^{n-1}}{1+x^m} = 0 \tag{3.4}$$

implies, by Theorem 2.1, that the BVP (3.1)–(3.2) has a positive solution. Note that (3.4) holds for any  $n > 1$ . Thus, for any  $n > 1$ , the BVP (3.1)–(3.2) has a positive solution.

Finally, we consider the case  $1 < n < m$ . Since both the conditions (2.3) or (2.10) are satisfied, we shall use Theorem 1.2 to find the number of positive solutions of the BVP (3.1)–(3.2). For this, we need to find a constant  $c_2 > 0$  such that

$$\int_0^1 e^{1-s} \frac{x^n(s)}{1+x^m(s)} ds > c_2(\lambda - e) \text{ for } c_2 \leq \|x\| \leq \lambda c_2 \tag{3.5}$$

holds. For  $x \in K$  and  $c_2 \leq \|x\| \leq \lambda c_2$ , we have

$$\int_0^1 e^{1-s} \frac{x^n(s)}{1+x^m(s)} ds > \frac{c_2^n}{1+\lambda^m c_2^m} (e - 1).$$

This, in turn, implies that (3.5) holds if

$$1 > \frac{(\lambda - e)}{(e - 1)} \frac{(1 + \lambda^m c_2^m)}{c_2^{n-1}} \tag{3.6}$$

holds. Set  $c_2 = \frac{1}{\lambda} \left( \frac{n-1}{m-n+1} \right)^{1/m}$ , which is the minimizer of  $\frac{(\lambda - e)}{(e - 1)} \frac{(1 + \lambda^m c_2^m)}{c_2^{n-1}}$ . Then the inequality (3.6) follows from (3.3). Thus for the case  $1 < n < m$ , the BVP (3.1)–(3.2) has at least two positive solutions.

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