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Topological entropies of a class of constrained systems

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Abstract: In this paper, we consider a class of constrained systems named double upper bounds (p,q)-constrained systems ((p,q)-DUB systems in brief), which are one-dimensional subshifts of finite type. We determinate the topological entropies (Shannon capacities) $C(p,q)$ of all (p,q)-DUB systems and consequently order all (p,q)-DUB systems according to the size of topological entropies. In particular, $C(p,\infty)=C(p+1,p+1)$ are the only equalities possible among the topological entropies of (p,q)-DUB systems.

Key words: Constrained systems, topological entropy, Shannon capacity, subshifts of finite type, order

1. Introduction

Subshifts of finite type are an important branch in topologically dynamical systems. As a special class of subshifts of finite type, some constrained systems are widely studied, especially run-length-limited (d,k)-constrained systems. Given two nonnegative integers d,k with $d<k$, a binary $\{0,1\}$-sequence is called (d,k)-constrained if it has at least d zeros and at most k zeros between any two successive ones. A run-length-limited (d,k)-constrained system, or (d,k)-RLL systems in brief, is the set of all (d,k)-constrained binary sequences and the shift on it. (d,k)-RLL systems were first studied by Shannon [9] and are used today in all manners of storage systems [2,7,8]. In particular, the Shannon capacity plays a major role in the research of (d,k)-RLL systems (see, e.g., [1,3–5]). In fact, the Shannon capacity is the topological entropy of shift on a (d,k)-RLL system.

In this article, we are interested in a class of constrained systems named "double upper bounds (p,q)-constrained systems, which are similar to but different from run-length-limited constrained systems. Given two positive integers p,q, we say that a bilateral or unilateral $\{0,1\}$-sequence is double upper bounds (p,q)-constrained if it includes neither a run of zeros of length more than p nor a run of ones of length more than q. A double upper bounds (p,q)-constrained system, or (p,q)-DUB system in brief, is the set of all double upper bounds (p,q)-constrained bilateral or unilateral sequences and the shift on it. It is obvious that a (p,q)-DUB system is topologically conjugate to the (q,p)-DUB system. Thus, all through the present paper, we assume $p\le q$. Moreover, we can take p or q to be infinity, which means that a run of zeros or ones of arbitrary length is admitted. Notice that the (∞,∞)-DUB system is the full 2-shift, a bilateral (p,∞)-DUB system is a $(0,p)$-RLL system for every positive integer p, a bilateral $(1,q)$-DUB system is a $(1,q)$-RLL system for every positive integer q, and other (p,q)-DUB systems are not RLL systems.

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Let $S(p,q)$ be a bilateral or unilateral (p,q)-DUB system, where p and q are two positive integers with $p \leq q \leq \infty$. Obviously, it is a subshift of finite type in the full 2-shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ or $(\{0,1\}^{\mathbb{N}}, \sigma)$, where σ is the shift mapping on $\{0,1\}$-sequences space. Denote by $C(p,q)$ the topological entropy or Shannon capacity of $S(p,q)$. Let A_n be the number of n-length codes in $S(p,q)$. Then

$$C(p,q) = \lim_{n\to\infty} \frac{1}{n} \ln A_n.$$

It is easy to see that $C(1,1) = 0$ and $C(\infty,\infty) = \ln 2$. We will determinate $C(p,q)$ for all p and q. Furthermore, we will order all (p,q)-DUB systems according to the size of topological entropies. In particular, $C(p,\infty) = C(p+1,p+1)$ are the only equalities possible among the topological entropies of (p,q)-DUB systems.

2. Topological entropies of (p,q)-DUB systems

For a bilateral or unilateral (p,q)-DUB system $S(p,q)$, let Λ be the set of all q-length codes in $S(p,q)$. One can write $\Lambda = \{\beta_1, \ldots, \beta_m\}$, where each $\beta_i = z_1 \ldots z_q$ is a q-length code in $S(p,q)$. Define an $m \times m$ matrix B by, for any $\beta_i = z_1 \ldots z_q$ and $\beta_j = w_1 \ldots w_q$ in Λ,

$$B_{ij} \triangleq B(\beta_i, \beta_j) = 1$$

if $z_2 \ldots z_q = w_1 \ldots w_{q-1}$ and $z_1 \ldots z_q w_q$ is a $(q+1)$-length code in $S(p,q)$; otherwise, $B_{ij} \triangleq B(\beta_i, \beta_j) = 0$. Moreover, we obtain a subshift of finite type (Σ_B, σ) with transition matrix B, where

$$\Sigma_B = \{(x_i) \in \Lambda^{\mathbb{Z}} \ (or\ \Lambda^{\mathbb{N}}); B(x_i, x_{i+1}) = 1,\ for\ all\ i \in \mathbb{Z}\ (or\ \mathbb{N})\}$$

and σ is the shift on Σ_B. As is known as a classic conclusion in symbolic dynamical systems, (Σ_B, σ) is topologically conjugate to the (p,q)-DUB system $(S(p,q), \sigma)$. Furthermore, if λ is the spectral radius of B, then

$$C(p,q) = \ln \lambda.$$

For instance, let us consider $S(1,2)$. Choose $\Lambda = \{\beta_1, \beta_2, \beta_3\}$, where $\beta_1 = 01$, $\beta_2 = 10$, and $\beta_3 = 11$. Define

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the spectral radius of B is $\lambda = 1.3247\ldots$, and consequently

$$C(1,2) = \ln \lambda = 0.2812\ldots > 0.$$

To determinate the topological entropies of (p,q)-DUB systems, we need to review some conclusions in Perron–Frobenius theory (refer to [6, 10]).

Lemma 2.1 *Let $B \geq 0$ be a square matrix. Then $B^N > 0$ for some positive integer N if and only if B is primitive.*

Lemma 2.2 *Suppose that B is a primitive nonnegative square matrix. Let λ be the spectral radius of B. Then*

$$\lim_{n\to\infty} \frac{B^n}{\lambda^n} = rl,$$

where r and l are the left and right eigenvectors for B normalized so that $lr = 1$.

Denote by a_n the number of n-length codes ending with zero in $S(p,q)$, and denote by b_n the number of n-length codes ending with one in $S(p,q)$. Then, obviously, $A_n = a_n + b_n$.

Proposition 2.3 *The transition matrix B defined as above is primitive. Furthermore, the limit $\lim\limits_{n\to\infty} \frac{a_n}{A_n}$ exists.*

Proof Obviously, B is a square $\{0,1\}$-matrix. To prove that B is primitive, we will show that for $N = q+4$, $B^N > 0$. Given any $\beta_i = (z_1, z_2, \cdots, z_q)$ and $\beta_j = (z_1', z_2', \cdots, z_q')$ in Λ:

(1) If $z_q = 0$ and $z_1' = 0$, then there exists a code

$$C = (z_1, z_2, \cdots, z_{q-1}, 0, 1, 0, 1, 0, z_2', z_3', \cdots, z_q')$$

in $S(p,q)$. Consequently, there exists a $(q+4)$-length code from β_i to β_j in Σ_B and hence $B_{ij}^N = B^N(\beta_i, \beta_j) > 0$.

(2) If $z_q = 0$ and $z_1' = 1$, then there exists a code

$$C = (z_1, z_2, \cdots, z_{q-1}, 0, 1, 1, 0, 1, z_2', z_3', \cdots, z_q')$$

in $S(p,q)$. Consequently, there exists a $(q+4)$-length code from β_i to β_j in Σ_B and hence $B_{ij}^N = B^N(\beta_i, \beta_j) > 0$.

(3) If $z_q = 1$ and $z_1' = 0$, then there exists a code

$$C = (z_1, z_2, \cdots, z_{q-1}, 1, 0, 1, 1, 0, z_2', z_3', \cdots, z_q')$$

in $S(p,q)$. Consequently, there exists a $(q+4)$-length code from β_i to β_j in Σ_B and hence $B_{ij}^N = B^N(\beta_i, \beta_j) > 0$.

(4) If $z_q = 1$ and $z_1' = 1$, then there exists a code

$$C = (z_1, z_2, \cdots, z_{q-1}, 1, 0, 1, 0, 1, z_2', z_3', \cdots, z_q')$$

in $S(p,q)$. Consequently, there exists a $(q+4)$-length code from β_i to β_j in Σ_B and hence $B_{ij}^N = B^N(\beta_i, \beta_j) > 0$.

In conclusion, we have $B^N > 0$. Notice that A_n is the sum of all elements of B^n and a_n is the sum of elements in some certain columns of B^n. Then, by Lemma 2.2, the limit $\lim\limits_{n\to\infty} \frac{a_n}{A_n}$ exists. □

According to Proposition 2.3, denote

$$\lim_{n\to\infty} \frac{a_n}{A_n} = x$$

and

$$\lim_{n\to\infty} \frac{b_n}{A_n} = y = 1 - x.$$

In addition, if λ is the the spectral radius of B, then

$$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \lambda.$$

For $0 < p \leq q < \infty$, we have for each $n \in \mathbb{N}$,

$$a_{n+1} = b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1},$$
$$b_{n+1} = a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-q+1}.$$

Then

$$\frac{a_{n+1}}{A_{n-p+1}} = \frac{b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1}}{A_{n-p+1}}$$

and

$$\frac{b_{n+1}}{A_{n-q+1}} = \frac{a_n + a_{n-1} + a_{n-2} + \cdots + a_{n-q+1}}{A_{n-q+1}}.$$

As $n \to \infty$,

$$\lambda^p x = \lambda^{p-1} y + \lambda^{p-2} y + \lambda^{p-3} y + \cdots + \lambda y + y$$

and

$$\lambda^q y = \lambda^{q-1} x + \lambda^{q-2} x + \lambda^{q-3} x + \cdots + \lambda x + x.$$

Consequently,

$$x = \frac{\lambda^{p-1} + \lambda^{p-2} + \lambda^{p-3} + \cdots + \lambda + 1}{\lambda^p + \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1}$$

and

$$x = \frac{\lambda^q}{\lambda^q + \lambda^{q-1} + \lambda^{q-2} + \cdots + \lambda + 1}.$$

Thus,

$$\frac{\lambda^{p-1} + \lambda^{p-2} + \lambda^{p-3} + \cdots + \lambda + 1}{\lambda^p + \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1} = \frac{\lambda^q}{\lambda^q + \lambda^{q-1} + \lambda^{q-2} + \cdots + \lambda + 1},$$

and hence

$$\frac{\lambda^p - 1}{\lambda^{p+1} - 1} + \frac{\lambda^q - 1}{\lambda^{q+1} - 1} = 1. \tag{2.1}$$

Equation (2.1) is said to be the characteristic equation of $S(p,q)$ for $0 < p \leq q < \infty$.

Similarly, for $0 < p < \infty$ and $q = \infty$, we have for each $n \in \mathbb{N}$,

$$a_{n+1} = b_n + b_{n-1} + b_{n-2} + \cdots + b_{n-p+1},$$
$$b_{n+1} = A_n,$$

and then

$$\lambda^p x = \lambda^{p-1} y + \lambda^{p-2} y + \lambda^{p-3} y + \cdots + \lambda y + y$$

and

$$\lambda y = 1.$$

Consequently,

$$\frac{\lambda^{p-1}+\lambda^{p-2}+\lambda^{p-3}+\cdots+\lambda+1}{\lambda^{p}+\lambda^{p-1}+\lambda^{p-2}+\cdots+\lambda+1}=\frac{\lambda-1}{\lambda}.$$

Thus,

$$\frac{\lambda^p-1}{\lambda^{p+1}-1}+\frac{1}{\lambda}=1. \tag{2.2}$$

Equation (2.2) is said to be the characteristic equation of $S(p,\infty)$ for $0<p<\infty$.

For $(p,q)\neq(1,1),(\infty,\infty)$, it is not difficult to see that $S(p,q)$ is a subsystem of $S(\infty,\infty)$ and $S(1,2)$ is a subsystem of $S(p,q)$. Then

$$0<C(1,2)\leq C(p,q)=\ln\lambda\leq C(\infty,\infty)=\ln 2.$$

Thus, $\lambda\in(1,2]$. We will prove that λ is the unique root of the characteristic equation in $(1,2)$.

Theorem 2.4 *For $(p,q)\neq(1,1),(\infty,\infty)$, there exists one and only one root λ of the characteristic equation (2.1) or (2.2) in the open interval $(1,2)$. Furthermore, $C(p,q)=\ln\lambda$.*

Proof Let $f(\lambda)=\dfrac{\lambda^p-1}{\lambda^{p+1}-1}$. Since

$$\begin{aligned}f'(\lambda)&=\frac{-\lambda^{2p}+(p+1)\lambda^p-p\lambda^{p-1}}{(\lambda^{p+1}-1)^2}\\&=\frac{\lambda^{p-1}((p+1)\lambda-\lambda^{p+1}-p)}{(\lambda^{p+1}-1)^2}<0,\end{aligned}$$

one can see that $f(\lambda)$ is a strictly decreasing function, and so is the function $F(\lambda)=\dfrac{\lambda^p-1}{\lambda^{p+1}-1}+\dfrac{\lambda^q-1}{\lambda^{q+1}-1}$. Notice

$$F(\lambda)=\frac{\lambda^p-1}{\lambda^{p+1}-1}=\frac{1+\lambda+\lambda^2+\cdots+\lambda^{p-1}}{1+\lambda+\lambda^2+\cdots+\lambda^p}+\frac{1+\lambda+\lambda^2+\cdots+\lambda^{q-1}}{1+\lambda+\lambda^2+\cdots+\lambda^q},$$

and then

$$F(1)=\frac{p}{p+1}+\frac{q}{q+1}=\frac{1}{1+\frac{1}{p}}+\frac{1}{1+\frac{1}{q}}\geq\frac{1}{2}+\frac{1}{2}=1,$$

and the equality holds if and only if $p=q=1$. In addition,

$$\begin{aligned}F(2)&=\frac{2^p-1}{2^{p+1}-1}+\frac{2^q-1}{2^{q+1}-1}\\&=\frac{2^{p+q+2}-3\cdot2^p-3\cdot2^q+2}{(2^{p+1}-1)(2^{q+1}-1)}\\&<\frac{2^{p+q+2}-3\cdot2^p-3\cdot2^q+2}{(2^{p+1}-1)(2^{q+1}-1)}+\frac{2^p+2^q-1}{(2^{p+1}-1)(2^{q+1}-1)}\\&=\frac{2^{p+q+2}-2^{q+1}-2^{p+1}+1}{(2^{p+1}-1)(2^{q+1}-1)}\\&=1.\end{aligned}$$

Therefore, the characteristic equation (2.1) has a unique root in the open interval $(1,2)$. Similarly, the characteristic equation (2.2) has a unique root in the open interval $(1,2)$. It follows from the discussions before this theorem that the unique root λ is the spectral radius of B corresponding to $S(p,q)$, and hence $C(p,q) = \ln\lambda$. □

Now we will order all (p,q)-DUB systems according to the size of topological entropies. First, let us consider the equalities possible among the topological entropies of (p,q)-DUB systems.

Proposition 2.5 *For every positive integer p,*

$$C(p,\infty) = C(p+1,p+1).$$

Proof For $S(p,\infty)$, the characteristic equation (2.2) can be written as follows:

$$\lambda^{p+2} - 2\lambda^{p+1} + 1 = 0.$$

For $S(p+1,p+1)$, the characteristic equation is

$$\frac{\lambda^{p+1}-1}{\lambda^{p+2}-1} = \frac{1}{2},$$

that is also

$$\lambda^{p+2} - 2\lambda^{p+1} + 1 = 0.$$

Therefore, we have

$$C(p,\infty) = C(p+1,p+1).$$

□

Next, we will prove some strict inequalities.

Proposition 2.6 *For any p, q, and q' with $q < q' \leq \infty$, we have*

$$C(p,q) < C(p,q').$$

Proof Let $\lambda_0, \lambda_1 \in (1,2)$ with $C(p,q) = \ln\lambda_0$ and $C(p,q') = \ln\lambda_1$. Let $g_q(\lambda) = \frac{\lambda^q-1}{\lambda^{q+1}-1}$ for positive integer q and $g_\infty(\lambda) = \frac{1}{\lambda}$. For any $\lambda_0 \in (1,2)$, one can see

$$\begin{aligned} g_{q+1}(\lambda_0) - g_q(\lambda_0) &= \frac{\lambda_0^{q+1}-1}{\lambda_0^{q+2}-1} - \frac{\lambda_0^{q}-1}{\lambda_0^{q+1}-1} \\ &= \frac{\lambda_0^{q+2}+\lambda_0^{q}-2\lambda_0^{q+1}}{(\lambda_0^{q+2}-1)(\lambda_0^{q+1}-1)} \\ &= \frac{\lambda_0^{q}(\lambda_0-1)^2}{(\lambda_0^{q+2}-1)(\lambda_0^{q+1}-1)} \\ &> 0, \end{aligned}$$

and

$$g_\infty(\lambda_0) - g_q(\lambda_0) = \frac{\lambda_0 - 1}{\lambda_0(\lambda_0^{q+1}-1)} > 0.$$

Consequently, if $\lambda_0 \in (1,2)$ satisfies equation (2.1), i.e.

$$\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{\lambda_0^q - 1}{\lambda_0^{q+1} - 1} = 1,$$

then for q' with $q < q' < \infty$,

$$\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{\lambda_0^{q'} - 1}{\lambda_0^{q'+1} - 1} > 1$$

and

$$\frac{\lambda_0^p - 1}{\lambda_0^{p+1} - 1} + \frac{1}{\lambda_0} > 1.$$

Since the functions $\frac{\lambda^p - 1}{\lambda^{p+1}-1} + \frac{\lambda^q - 1}{\lambda^{q+1}-1}$ and $\frac{\lambda^p - 1}{\lambda^{p+1}-1} + \frac{1}{\lambda}$ are strictly decreasing on $(1,2)$, we have $\lambda_0 < \lambda_1$. In conclusion, $C(p,q) < C(p,q')$. □

Following from the two above propositions, we obtain the complete size relationship of the topological entropies of all (p,q)-DUB systems.

Theorem 2.7

$$\begin{aligned} 0 = C(1,1) < C(1,2) < \ldots < C(1,\infty) = C(2,2) < C(2,3) < \ldots < C(2,\infty) \\ = C(3,3) < C(3,4) < \ldots\ldots < C(\infty,\infty) = \ln 2. \end{aligned}$$

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