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CC -normal topological spaces

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Abstract: A topological space X is called CC -normal if there exist a normal space Y and a bijective function $f : X \rightarrow Y$ such that the restriction $f|_A : A \rightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$. We will investigate this property and produce some examples to illustrate the relation between CC -normality and other weaker kinds of normality.

Key words: Normal, L -normal, C -normal, CC -normal, mildly normal, almost normal, epinormal, submetrizable, π -normal

1. Introduction

A. V. Arhangel'skiĭ introduced in 2012, when he was visiting the Department of Mathematics at King Abdulaziz University, a new weaker version of normality, called C -normality [2]. A topological space X is called C -normal if there exist a normal space Y and a bijective function $f : X \rightarrow Y$ such that the restriction $f|_C : C \rightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. We use the idea of this definition to introduce another new weaker version of normality that will be called CC -normality. The purpose of this paper is to investigate this property. We prove that normality implies CC -normality but the converse is not true in general. We present some examples to show relationships between CC -normality and other weaker versions of normality such as C -normality, L -normality, almost normality, mild normality, epinormality, and π -normality. Throughout this paper, we denote an ordered pair by $\langle x, y \rangle$, the set of positive integers by \mathbb{N} , and the set of real numbers by \mathbb{R} . A T_4 space is a T_1 normal space and a Tychonoff space is a T_1 completely regular space. We do not assume T_2 in the definition of compactness and countable compactness. We do not assume regularity in the definition of Lindelöfness. For a subset A of a space X , $\text{int}A$ and \bar{A} denote the interior and the closure of A , respectively. An ordinal γ is the set of all ordinals α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 , the first uncountable ordinal is ω_1 , and the successor cardinal of ω_1 is ω_2 .

2. CC -normality

Definition 2.1 A topological space X is called CC -normal if there exist a normal space Y and a bijective function $f : X \rightarrow Y$ such that the restriction $f|_A : A \rightarrow f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.

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We do not assume T_2 in the above definition. Recall that a topological space X is called L -normal if there exist a normal space Y and a bijective function $f : X \rightarrow Y$ such that the restriction $f|_A : A \rightarrow f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$ [9]. Any normal space is CC -normal, just by taking $X = Y$ and f to be the identity function. We will give an example of a CC -normal space that is neither normal nor locally compact, but first we give a theorem that will be used in the example.

Theorem 2.2 *If X is an L -normal space such that each countably compact subspace is contained in a Lindelöf subspace, then X is CC -normal.*

Proof Let X be any L -normal space such that if A is any countably compact subspace of X ; then there exists a Lindelöf subspace B such that $A \subseteq B$. Let Y be a normal space and $f : X \rightarrow Y$ be a bijective function such that $f|_C : C \rightarrow f(C)$ is a homeomorphism for each Lindelöf subspace C of X . Now let A be any countably compact subspace of X . Pick a Lindelöf subspace B of X such that $A \subseteq B$. Then $f|_B : B \rightarrow f(B)$ is a homeomorphism, and hence $f|_A : A \rightarrow f(A)$ is a homeomorphism as $(f|_B)|_A = f|_A$. \square

By similar arguments, we obtain the following corollary.

Corollary 2.3 (a) *If X is C -normal and any Lindelöf subspace of X is contained in a compact subspace of X , then X is L -normal [9].*

(b) *If X is C -normal and any countably compact subspace of X is contained in a compact subspace of X , then X is CC -normal.*

(c) *If X is CC -normal and any Lindelöf subspace of X is contained in a countably compact subspace of X , then X is L -normal.*

Example 2.4 *We modify the Dieudonné plank [15] to define a new topological space. Let*

$$X = ((\omega_2 + 1) \times (\omega_0 + 1)) \setminus \{\langle \omega_2, \omega_0 \rangle\}.$$

Write $X = A \cup B \cup N$, where $A = \{\langle \omega_2, n \rangle : n < \omega_0\}$, $B = \{\langle \alpha, \omega_0 \rangle : \alpha < \omega_2\}$, and $N = \{\langle \alpha, n \rangle : \alpha < \omega_2$ and $n < \omega_0\}$. The topology \mathcal{T} on X is generated by the following neighborhood system: for each $\langle \alpha, n \rangle \in N$, let $\mathcal{B}(\langle \alpha, n \rangle) = \{\{\langle \alpha, n \rangle\}\}$. For each $\langle \omega_2, n \rangle \in A$, let $\mathcal{B}(\langle \omega_2, n \rangle) = \{V_\alpha(n) = \langle \alpha, \omega_2 \rangle \times \{n\} : \alpha < \omega_2\}$. For each $\langle \alpha, \omega_0 \rangle \in B$, let $\mathcal{B}(\langle \alpha, \omega_0 \rangle) = \{V_n(\alpha) = \{\alpha\} \times (n, \omega_0] : n < \omega_0\}$. Then X is a Tychonoff nonnormal space that is not locally compact as any basic open neighborhood of any element in A is not Lindelöf, and hence not compact. Now a subset $C \subseteq X$ is countably compact if and only if C satisfies all of the following conditions: (1) $C \cap A$ and $C \cap B$ are both finite; (2) the set $\{\langle \alpha, n \rangle \in C \cap N : \langle \omega_2, n \rangle \in C \cap A\}$ is finite; and (3) the set $\{\langle \alpha, n \rangle \in C \cap N : \langle \alpha, \omega_0 \rangle \notin C \cap B, \langle \omega_2, n \rangle \notin C \cap A\}$ is finite. This means that any countably compact subspace is countable and hence Lindelöf. Since the modified Dieudonné plank is L -normal, see [9], by Theorem 2.2, it is CC -normal.

Theorem 2.5 *If X is a T_1 space such that the only countably compact subspaces of X are the finite subsets, then X is CC -normal.*

Proof Let X be a T_1 space such that the only countably compact subspaces of X are the finite subsets of X . By T_1 , we conclude that any countably compact subspace of X is discrete. Thus, let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y works. \square

C -normality is a generalization of CC -normality because any compact space is countably compact; hence, any CC -normal space is C -normal. Obviously, any countably compact CC -normal space must be normal. Thus, $\omega_1 \times I^{\omega_1}$, where $I = [0, 1]$ is the closed unit interval with its usual metric topology and I^{ω_1} is an uncountable product of I , is not CC -normal because it is a countably compact nonnormal space [15], but $\omega_1 \times I^{\omega_1}$ is C -normal being locally compact [2]. The space $\omega_1 \times (\omega_1 + 1)$ is an example of an L -normal space, see [9], which is not CC -normal because it is a countably compact nonnormal space. Here is an example of a CC -normal space that is not L -normal.

Example 2.6 Consider $(\mathbb{R}, \mathcal{CC})$, where \mathcal{CC} is the countable complement topology [15]. The space $(\mathbb{R}, \mathcal{CC})$ is T_1 as any singleton is closed. Let C be any countably infinite subset of \mathbb{R} . For each $c \in C$, define $V_c = (\mathbb{R} \setminus C) \cup \{c\}$. Then the family $\{V_c : c \in C\}$ is a countable open cover for C that has no finite subcover, and hence C is not countably compact. Thus, the only countably compact subspaces are the finite sets. Therefore, by Theorem 1.5, $(\mathbb{R}, \mathcal{CC})$ is CC -normal. It is not L -normal because it is a Lindelöf nonnormal space.

In Example 2.10, we give a Tychonoff separable first countable locally compact CC -normal space that is not L -normal.

Theorem 2.7 CC -normality is a topological property.

Proof Let X be a CC -normal space and $X \cong Z$. Let Y be a normal space and $f : X \rightarrow Y$ be a bijective function such that $f|_C : C \rightarrow f(C)$ is a homeomorphism for each countably compact subspace C of X . Let $g : Z \rightarrow X$ be a homeomorphism. Then $f \circ g : Z \rightarrow Y$ satisfies all requirements. \square

Theorem 2.8 CC -normality is an additive property.

Proof Let X_α be a CC -normal space for each $\alpha \in \Lambda$. We show that their sum $\bigoplus_{\alpha \in \Lambda} X_\alpha$ is CC -normal. For each $\alpha \in \Lambda$, pick a normal space Y_α and a bijective function $f_\alpha : X_\alpha \rightarrow Y_\alpha$ such that $f_{\alpha|_{C_\alpha}} : C_\alpha \rightarrow f_\alpha(C_\alpha)$ is a homeomorphism for each countably compact subspace C_α of X_α . Since Y_α is normal for each $\alpha \in \Lambda$, the sum $\bigoplus_{\alpha \in \Lambda} Y_\alpha$ is normal ([4], 2.2.7). Consider the function sum ([4], 2.2.E), $\bigoplus_{\alpha \in \Lambda} f_\alpha : \bigoplus_{\alpha \in \Lambda} X_\alpha \rightarrow \bigoplus_{\alpha \in \Lambda} Y_\alpha$ defined by $\bigoplus_{\alpha \in \Lambda} f_\alpha(x) = f_\beta(x)$ if $x \in X_\beta, \beta \in \Lambda$. Now a subspace $C \subseteq \bigoplus_{\alpha \in \Lambda} X_\alpha$ is countably compact if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : C \cap X_\alpha \neq \emptyset\}$ is finite and $C \cap X_\alpha$ is countably compact in X_α for each $\alpha \in \Lambda_0$. If $C \subseteq \bigoplus_{\alpha \in \Lambda} X_\alpha$ is countably compact, then $(\bigoplus_{\alpha \in \Lambda} f_\alpha)|_C$ is a homeomorphism because $f_{\alpha|_{C \cap X_\alpha}}$ is a homeomorphism for each $\alpha \in \Lambda_0$. \square

CC -normality is not a multiplicative property. For example, the normal spaces ω_1 and $\omega_1 + 1$ are both CC -normal, but $\omega_1 \times (\omega_1 + 1)$ is not CC -normal. CC -normality is not hereditary. For example, the space $\omega_1 \times (\omega_1 + 1)$ is not CC -normal while it is a subspace of its Stone–Čech compactification $(\omega_1 + 1)^2$.

It is clear that a function $f : X \rightarrow Y$ bearing the CC -normality of X need not be continuous. For example, consider the modified Dieudonné plank X , see Example 1.4. Let $Y = X$ with the topology generated by the following neighborhood system: points of B and N have the same local base as in X and each point of A is isolated. Then the identity function from X onto Y is not continuous, but it bears the CC -normality of X . A function $f : X \rightarrow Y$ bearing the CC -normality of X will be continuous if X is Fréchet. Recall that a space X is called *Fréchet* if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of points of A such that $a_n \rightarrow x$, see [4].

Theorem 2.9 *If X is CC -normal and Fréchet and $f : X \rightarrow Y$ bears the CC -normality of X , then f is continuous.*

Proof Assume that X is CC -normal and Fréchet. Let $f : X \rightarrow Y$ bear of the CC -normality of X . Let $A \subseteq X$ and pick $y \in f(\overline{A})$. Pick the unique $x \in X$ such that $f(x) = y$. Thus, $x \in \overline{A}$. Since X is Fréchet, there exists a sequence $(a_n) \subseteq A$ such that $a_n \rightarrow x$. The subspace $B = \{x, a_n : n \in \mathbb{N}\}$ of X is countably compact being compact, and thus $f|_B : B \rightarrow f(B)$ is a homeomorphism. Now let $W \subseteq Y$ be any open neighborhood of y . Then $W \cap f(B)$ is open in the subspace $f(B)$ containing y . Since $f(\{a_n : n \in \mathbb{N}\}) \subseteq f(B) \cap f(A)$ and $W \cap f(B) \neq \emptyset$, $W \cap f(A) \neq \emptyset$, hence $y \in \overline{f(A)}$, and thus $f(\overline{A}) \subseteq \overline{f(A)}$. Therefore, f is continuous. \square

We conclude from the above proof that if X is C -normal and Fréchet and $f : X \rightarrow Y$ bears the C -normality of X , then f is continuous. Since any first countable space is Fréchet, the statements are true if X is first countable. In fact, for C -normality the statement is true if X is a k -space, see [9]. For a function that bears L -normality, the following is true: “If X is L -normal and of countable tightness and $f : X \rightarrow Y$ bears the L -normality of X , then f is continuous.”, see [9].

Any L -normal regular separable space of countable tightness is normal, see [9]. This is not true for CC -normality. Here is an example of a CC -normal Tychonoff separable first countable space that is not normal. For simplicity, we will denote the first infinite ordinal just by ω .

Example 2.10 *We choose a suitable Mrówka space. Recall that two countably infinite sets are said to be almost disjoint [16] if their intersection is finite. Call a subfamily of $[\omega]^\omega = \{A \subset \omega : A \text{ is infinite}\}$ a mad family [16] on ω if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let \mathcal{A} be a pairwise almost disjoint subfamily of $[\omega]^\omega$. The Mrówka space $\Psi(\mathcal{A})$ is defined as follows: the underlying set is $\omega \cup \mathcal{A}$, each point of ω is isolated, and a basic open neighborhood of $W \in \mathcal{A}$ has the form $\{W\} \cup (W \setminus F)$, with $F \in [\omega]^{<\omega} = \{B \subseteq \omega : B \text{ is finite}\}$. It is well known that there exists an almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ such that $|\mathcal{A}| > \omega$ and the Mrówka space $\Psi(\mathcal{A})$ is a Tychonoff, separable, first countable, and locally compact space that is neither countably compact nor normal. \mathcal{A} is a mad family if and only if $\Psi(\mathcal{A})$ is pseudocompact [11]. Since $\Psi(\mathcal{A})$ is locally compact, it is C -normal, see [2]. Now a subspace C of $\Psi(\mathcal{A})$ is countably compact if and only if $C \cap \mathcal{A}$ is finite and the set $(\omega \cap C) \setminus (\bigcup_{A \in (C \cap \mathcal{A})} A)$ is finite. This means that any countably compact subspace is compact. Thus, by Corollary 2.3(b), $\Psi(\mathcal{A})$ is CC -normal.*

3. CC -normality and other properties

Let us recall some definitions.

Definition 3.1 *A subset A of a space X is called a closed domain of X [4] (also called regularly closed, κ -closed) if $A = \overline{\text{int}A}$. A space X is called mildly normal [14] (also called κ -normal [12]) if for any two disjoint closed domains A and B of X there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [6,7]. A space X is called almost normal [13] if for any two disjoint closed subsets A and B of X , one of which is a closed domain, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [8]. A subset A of a space X is called π -closed [7] if A is a finite intersection of closed domains of X . A space X is called π -normal [7] if for any two disjoint closed subsets A and B of X , one of which is π -closed, there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A space X is called quasnormal [17] if for any two disjoint π -closed subsets A and B of X , there exist two disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$, see also [7].*

It is clear from the definitions that

$$\text{normal} \implies \pi\text{-normal} \implies \text{almost normal} \implies \text{mildly normal.}$$

$$\text{normal} \implies \pi\text{-normal} \implies \text{quasinormal} \implies \text{mildly normal.}$$

Example 3.2 Consider $(\mathbb{R}, \mathcal{CF})$, where \mathcal{CF} is the finite complement topology [15]. Since the only closed domains of $(\mathbb{R}, \mathcal{CF})$ are \emptyset and \mathbb{R} , $(\mathbb{R}, \mathcal{CF})$ is π -normal and hence quasinormal, almost normal, and mildly normal, but $(\mathbb{R}, \mathcal{CF})$ is not CC -normal because it is countably compact, being a compact, nonnormal space.

Here is an example of a CC -normal space that is not π -normal.

Example 3.3 The modified Dieudonné plank X of Example 1.4 is CC -normal but neither quasinormal nor π -normal.

Proof X is not normal because A and B are closed disjoint subsets that cannot be separated by two disjoint open sets. Note that $\text{int}(A) = \emptyset = \text{int}(B)$. Thus, A and B are not closed domains. We will show that A and B are π -closed sets. Let $E = \{n < \omega_0 : n \text{ is even}\}$ and $O = \{n < \omega_0 : n \text{ is odd}\}$. Let $C = \{\langle \alpha, n \rangle : \alpha < \omega_2, n \in E\} = \omega_2 \times E$ and $D = \{\langle \alpha, n \rangle : \alpha < \omega_2, n \in O\} = \omega_2 \times O$. Then C and D are both open in X , being subsets of N . Thus, \overline{C} and \overline{D} are both closed domains in X , being closures of open sets. Now $\overline{C} = C \cup B \cup \{\langle \omega_2, n \rangle \in A : n \in E\}$ and $\overline{D} = D \cup B \cup \{\langle \omega_2, n \rangle \in A : n \in O\}$, and hence $\overline{C} \cap \overline{D} = B$. Thus, B is π -closed. Now let K and L be subsets of ω_2 such that $K \cap L = \emptyset$, $K \cup L = \omega_2$, and the cofinality of K and L is ω_2 ; for instance, let K be the set of limit ordinals in ω_2 and L be the set of successor ordinals in ω_2 . Let $G = \{\langle \alpha, n \rangle : \alpha \in K, n < \omega_0\} = K \times \omega_0$ and $H = \{\langle \alpha, n \rangle : \alpha \in L, n < \omega_0\} = L \times \omega_0$. Then G and H are both open in X being subsets of N . Thus \overline{G} and \overline{H} are both closed domains in X , being closures of open sets. Now $\overline{G} = G \cup A \cup \{\langle \alpha, \omega_0 \rangle \in B : \alpha \in K\}$ and $\overline{H} = H \cup A \cup \{\langle \alpha, \omega_0 \rangle \in B : \alpha \in L\}$, and hence $\overline{G} \cap \overline{H} = A$. Thus A is π -closed. Therefore, the modified Dieudonné plank X is CC -normal but neither quasinormal nor π -normal. \square

Example 3.4 \mathbb{R} with the particular point topology \mathcal{T}_p , see [15], where the particular point is $p \in \mathbb{R}$, is not CC -normal. Recall that $\mathcal{T}_p = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : p \in U\}$. It is well known that $(\mathbb{R}, \mathcal{T}_p)$ is neither T_1 nor normal and if $A \subseteq \mathbb{R}$, then $\{\{x, p\} : x \in A\}$ is an open cover for A ; thus, a subset A of \mathbb{R} is countably compact if and only if it is finite. To see that $(\mathbb{R}, \mathcal{T}_p)$ is not CC -normal, suppose that $(\mathbb{R}, \mathcal{T}_p)$ is CC -normal. Let Y be a normal space and $f : \mathbb{R} \rightarrow Y$ be a bijection such that the restriction $f|_C : C \rightarrow f(C)$ is a homeomorphism for each countably compact subspace C of $(\mathbb{R}, \mathcal{T}_p)$. For the space Y , we have only two cases:

Case 1: Y is T_1 . Take $C = \{x, p\}$, where $x \neq p$. Then C is a countably compact subspace of $(\mathbb{R}, \mathcal{T}_p)$. By assumption $f|_C : C \rightarrow f(C) = \{f(x), f(p)\}$ is a homeomorphism. Since $f(C)$ is a finite subspace of Y and Y is T_1 , $f(C)$ is a discrete subspace of Y . Thus, we obtain that $f|_C$ is not continuous, which is a contradiction, as $f|_C$ is a homeomorphism.

Case 2: Y is not T_1 . We claim that the topology on Y is coarser than the particular point topology on Y with $f(p)$ as its particular point. To prove this claim, we suppose not. Then there exists a nonempty open set $U \subset Y$ such that $f(p) \notin U$. Pick $y \in U$ and let $x \in \mathbb{R}$ be the unique real number such that $f(x) = y$. Consider $\{x, p\}$. Note that $x \neq p$ because $f(x) = y \in U$, $f(p) \notin U$, and f is one-to-one. Consider $f|_{\{x, p\}} : \{x, p\} \rightarrow \{y, f(p)\}$. Now $\{y\}$ is open in the subspace $\{y, f(p)\}$ of Y because $\{y\} = U \cap \{y, f(p)\}$,

but $f^{-1}(\{y\}) = \{x\}$ and $\{x\}$ is not open in the subspace $\{x, p\}$ of (\mathbb{R}, τ_p) , which means $f|_{\{x, p\}}$ is not continuous. This is a contradiction, and our claim is proved. However, any topology coarser than the particular point topology has no disjoint nonempty open sets and therefore cannot be normal, so we get a contradiction as Y is assumed to be normal. Therefore, (\mathbb{R}, τ_p) is not CC -normal.

Since the only closed domains in (\mathbb{R}, τ_p) are \emptyset and \mathbb{R} , (\mathbb{R}, τ_p) is almost normal.

Recall that a topological space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than τ , i.e. $\tau_d \subseteq \tau$, see [5]. A topological space (X, τ) is called *epinormal* if there is a coarser topology τ' on X such that (X, τ') is T_4 [2].

The space $\omega_1 + 1$ is CC -normal being T_2 -compact but it is not submetrizable. Indiscrete spaces with more than one element and $(\mathbb{R}, \mathcal{C})$ are examples of CC -normal spaces that are not epinormal because they are not Hausdorff. It was proved in [2] that any submetrizable space is C -normal and any epinormal space is C -normal. We still do not know if submetrizability implies CC -normality or if epinormality implies CC -normality or not.

We discovered that the Alexandroff duplicate space of a CC -normal space is CC -normal. Recall that the Alexandroff duplicate space $A(X)$ of a space X is defined as follows: let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1 \rangle$ in X' by x' and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U \text{ is open in } X \text{ with } x \in U\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ will generate a unique topology on $A(X)$ such that \mathcal{B} is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff duplicate of X* [1,4].

Theorem 3.5 *If X is CC -normal, then its Alexandroff duplicate $A(X)$ is also CC -normal.*

Proof Let X be any CC -normal space. Pick a normal space Y and a bijective function $f : X \rightarrow Y$ such that $f|_C : C \rightarrow f(C)$ is a homeomorphism for each countably compact subspace $C \subseteq X$. Consider the Alexandroff duplicate spaces $A(X)$ and $A(Y)$ of X and Y respectively. It is well known that the Alexandroff duplicate of a normal space is normal [1], and hence $A(Y)$ is also normal. Define $g : A(X) \rightarrow A(Y)$ by $g(a) = f(a)$ if $a \in X$. If $a \in X'$, let b be the unique element in X such that $b' = a$, and then define $g(a) = (f(b))'$. Then g is a bijective function. Now a subspace $C \subseteq A(X)$ is countably compact if and only if $C \cap X$ is countably compact in X and for each open set U in X with $C \cap X \subseteq U$, we have that $(C \cap X') \setminus U'$ is finite. Let $C \subseteq A(X)$ be any countably compact subspace. We show that $g|_C : C \rightarrow g(C)$ is a homeomorphism. Let $a \in C$ be arbitrary. If $a \in C \cap X'$, let $b \in X$ be the unique element such that $b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$ we have that $\{a\}$ is open in C and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in C \cap X$, let W be any open set in Y such that $g(a) = f(a) \in W$. Consider $H = (W \cup (W' \setminus \{(f(a))'\})) \cap g(C)$, which is a basic open neighborhood of $f(a)$ in $g(C)$. Since $f|_{C \cap X} : C \cap X \rightarrow f(C \cap X)$ is a homeomorphism, there exists an open set U in X with $a \in U$ and $f|_{C \cap X}(U \cap C) \subseteq W \cap f(C \cap X)$. Now $(U \cup (U' \setminus \{a'\})) \cap C = G$ is open in C such that $a \in G$ and $g|_C(G) \subseteq H$. Therefore, $g|_C$ is continuous. Now we show that $g|_C$ is open. Let $K \cup (K' \setminus \{k'\})$, where $k \in K$ and K is open in X , be any basic open set in $A(X)$; then $(K \cap C) \cup ((K' \cap C) \setminus \{k'\})$ is a basic open set in C . Since $X \cap C$ is countably compact in X , $g|_C(K \cap (X \cap C)) = f|_{X \cap C}(K \cap (X \cap C))$ is open in $Y \cap f(C \cap X)$ as $f|_{X \cap C}$ is a homeomorphism. Thus $g|_C(K \cap C)$ is open in $Y \cap f(X \cap C)$. Also, $g((K' \cap C) \setminus \{k'\})$ is open in $Y' \cap g(C)$ being a set of isolated points. Thus, $g|_C$ is an open function. Therefore, $g|_C$ is a homeomorphism. \square

The following problems are still open:

1. Is CC -normality hereditary with respect to closed subspaces?
2. If X is a Dowker space, is $X \times I$ then CC -normal?

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