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## Simultaneous strong proximality in Banach spaces

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**Abstract:** Several researchers have discussed the problem of strong proximality in Banach spaces. In this paper, we generalize the notion of strong proximality and define simultaneous strong proximality. It is proved that if  $W$  is a simultaneously approximatively compact subset of a Banach space  $X$  then  $W$  is simultaneously strongly proximal and the converse holds if the set of all best simultaneous approximations to every bounded subset  $S$  of  $X$  from  $W$  is compact. We show that simultaneously strongly Chebyshev sets are precisely the sets that are simultaneously strongly proximal and simultaneously Chebyshev. It is also proved that if  $F$  and  $W$  are subspaces of a Banach space  $X$  such that  $F$  is simultaneously strongly proximal,  $W$  is finite dimensional and  $F+W$  is closed then  $F+W$  is simultaneously strongly proximal in  $X$ . How simultaneous strong proximality is transmitted to and from quotient spaces is discussed in this paper.

**Key words:** Strongly proximal, simultaneously strongly proximal, strongly Chebyshev, simultaneously strongly Chebyshev, approximatively compact, simultaneously approximatively compact

### 1. Introduction

Let  $W$  be a nonempty closed subset of a real Banach space  $(X, \|\cdot\|)$  and  $x \in X$ . An element  $w_0 \in W$  is said to be a *best approximation* to  $x$  from  $W$  if

$$\|x - w_0\| = \inf_{w \in W} \|x - w\| \equiv d(x, W).$$

The set of all best approximations to  $x$  from  $W$  is denoted by  $P_W(x)$ . The set  $W$  is called *proximal* if  $P_W(x) \neq \emptyset$  for every  $x \in X$ . If for each  $x \in X$ ,  $P_W(x)$  is a singleton then the set  $W$  is called *Chebyshev*.

A proximal subset  $W$  of a Banach space  $X$  is said to be *strongly proximal* (see [1,5,10]) if for each  $x \in X$  and for any minimizing sequence  $\{y_n\} \subseteq W$  for  $x$ , i.e.  $\lim_{n \rightarrow \infty} \|x - y_n\| = d(x, W)$ , there is a subsequence  $\{y_{n_k}\}$  and a sequence  $\{z_k\} \subseteq P_W(x)$  such that  $\|y_{n_k} - z_k\| \rightarrow 0$ .

A subset  $W$  of a Banach space  $X$  is said to be *approximatively compact* if for any  $x \in X$ , every minimizing sequence  $\{y_n\} \subseteq W$  for  $x$ , i.e.  $\|x - y_n\| \rightarrow d(x, W)$  has a convergent subsequence in  $W$ .

A subset  $W$  of a Banach space  $X$  is said to be *strongly Chebyshev* (see [1]) if for any  $x \in X$ , every minimizing sequence  $\{y_n\} \subseteq W$  for  $x$  is convergent in  $W$ .

Sometimes, it may happen that an element to be approximated is not known exactly but is known to lie in a bounded set  $S$ . In that case, it is reasonable to approximate simultaneously all  $s \in S$  by a single element

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of  $W$  by solving

$$\inf_{w \in W} \sup_{s \in S} \|s - w\| \equiv d(S, W).$$

An element  $w_0 \in W$  is said to be a *best simultaneous approximation* (see [6]) to  $S$  from  $W$  if

$$\sup_{s \in S} \|s - w_0\| = d(S, W).$$

The set of all best simultaneous approximations to  $S$  from  $W$  is denoted by  $L_W(S)$ . The set  $W$  is called *simultaneously proximal* if for each bounded subset  $S$  of  $X$ ,  $L_W(S) \neq \phi$ . If for each bounded subset  $S$  of  $X$ ,  $L_W(S)$  is a singleton then the set  $W$  is called *simultaneously Chebyshev*. For any  $\delta > 0$ , the set  $\{y \in W : \sup_{s \in S} \|s - y\| < \sup_{s \in S} \|s - w\| + \delta \text{ for all } w \in W\}$  is denoted by  $L_W(S, \delta)$ .

Motivated by the notions of best simultaneous approximation and strong proximality, we define the notion of simultaneous strong proximality.

A simultaneously proximal subset  $W$  of a Banach space  $X$  is said to be *simultaneously strongly proximal* if for each bounded subset  $S$  of  $X$  and for any minimizing sequence  $\{y_n\} \subseteq W$  for  $S$ , i.e.  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - y_n\| = d(S, W)$ , there is a subsequence  $\{y_{n_k}\}$  and a sequence  $\{z_k\} \subseteq L_W(S)$  such that  $\|y_{n_k} - z_k\| \rightarrow 0$ .

A subset  $W$  of a Banach space  $X$  is said to be *simultaneously approximatively compact* (see also [7]) if for any bounded subset  $S$  of  $X$ , every minimizing sequence  $\{y_n\} \subseteq W$  for  $S$  has a convergent subsequence in  $W$ .

A subset  $W$  of a Banach space  $X$  is said to be *simultaneously strongly Chebyshev* if for any bounded subset  $S$  of  $X$ , every minimizing sequence  $\{y_n\} \subseteq W$  for  $S$  is convergent in  $W$ .

In this paper, we prove that if  $W$  is simultaneously approximatively compact then  $W$  is simultaneously strongly proximal and the converse holds if  $L_W(S)$  is compact for every bounded subset  $S$  of  $X$ . We shall also show that simultaneously strongly Chebyshev sets are precisely the sets that are simultaneously strongly proximal and simultaneously Chebyshev.

The following question was raised by Cheney in [2]:

If  $F$  and  $W$  are proximal subspaces of a Banach space  $X$ , and  $F + W$  is closed, does it follow that  $F + W$  is proximal in  $X$ ?

Feder [4] gave a negative answer to this problem and proved that if  $W$  is reflexive and  $F$  is proximal such that  $W \cap F$  is finite dimensional then  $F + W$  is proximal. Lin [9] improved this result and proved that  $W$  is reflexive if and only if for every Banach space  $X$  with  $W \subseteq X$ ,  $F$  is proximal in  $X$  and  $W + F$  is closed implies  $W + F$  is proximal in  $X$ . Sun and Luo [13] proved that if  $F$  is strongly proximal and  $W$  is a finite dimensional subspace of  $X$  such that  $F + W$  is closed then  $F + W$  is a strongly proximal subspace of  $X$ . Rawashdeh [11] generalized the problem of sum of proximal subspaces to sum of simultaneously proximal subspaces and proved that if  $F$  and  $W$  are subspaces of a Banach space  $X$  such that  $F$  is simultaneously proximal and  $W$  is of finite dimension then  $F + W$  is simultaneously proximal. Thus, the question is:

Can we prove an analogous result for the sum of simultaneously strongly proximal subspaces?

In this paper, we take up this problem and prove that if  $F$  and  $W$  are subspaces of a Banach space  $X$  such that  $F$  is simultaneously strongly proximal,  $W$  is finite dimensional, and  $F + W$  is closed then  $F + W$  is simultaneously strongly proximal in  $X$ . We also prove that if  $F$  is a simultaneously proximal subspace

and  $W$  a subspace of  $X$  such that  $W + F$  is closed then  $(W + F)/F$  is simultaneously strongly proximal in  $X/F$  if  $W + F$  is simultaneously strongly proximal in  $X$ .

The results proved in this paper generalize several results of [1,3,8,10,11,13].

## 2. Simultaneous strong proximality and simultaneous approximative compactness

It was shown in [11] that if  $W$  is a reflexive subspace of a Banach space  $X$  then  $W$  is simultaneously proximal in  $X$ . Thus, it is natural to ask whether we can prove a similar result for simultaneous strong proximality? In this section, we prove that if  $W$  is a finite dimensional subspace of a Banach space  $X$  then  $W$  is simultaneously strongly proximal. We also give some relationships between simultaneous strong proximality, simultaneous approximative compactness, and simultaneous strong Chebyshevity.

**Proposition 2.1** *Let  $W$  be a finite dimensional subspace of a Banach space  $X$ ; then  $W$  is simultaneously approximatively compact.*

**Proof** Let  $S$  be a bounded subset of  $X$  and  $\{y_n\} \subseteq W$  a minimizing sequence for  $S$ , i.e.

$$\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - y_n\| = d(S, W).$$

Then the sequence  $\{a_n\}$  given by  $a_n = \sup_{s \in S} \|s - y_n\|$  is a convergent sequence of real numbers and hence is bounded. Therefore, there exist  $M_1 > 0$  such that  $\|a_n\| = \sup_{s \in S} \|s - y_n\| \leq M_1$  for every positive integer  $n$ . The inequality  $\|y_n\| \leq \|y_n - s\| + \|s\|$  for any  $s \in S$  and for all  $n$  gives  $\|y_n\| \leq \sup_{s \in S} \|y_n - s\| + \sup_{s \in S} \|s\| \leq M_1 + M_2 = M$  for all  $n$ . This implies that  $\{y_n\}$  is a bounded sequence in  $W$ . Since  $W$  is finite dimensional, there exists a subsequence  $\{y_{n_k}\} \rightarrow y_0 \in W$ . Thus  $W$  is simultaneously approximatively compact for  $S$ .  $\square$

The above proposition shows that if  $\{y_n\} \subseteq W$  is a minimizing sequence for  $S$  then  $\{y_n\}$  is bounded. Therefore, if  $W$  is a closed subset of a finite dimensional normed linear space  $X$  then  $W$  is simultaneously approximatively compact.

Concerning simultaneous approximative compactness and simultaneous strong proximality, we have

**Theorem 2.2** *Let  $W$  be a closed subset of a Banach space  $(X, \|\cdot\|)$ . Then  $W$  is simultaneously approximatively compact if and only if  $W$  is simultaneously strongly proximal and  $L_W(S)$  is compact for every bounded subset  $S$  of  $X$ .*

**Proof** Suppose  $W$  is simultaneously approximatively compact and  $S$  a bounded subset of  $X$ . Let  $\{y_n\} \subseteq W$  be a minimizing sequence for  $S$ , i.e.

$$\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - y_n\| = d(S, W). \tag{2.1}$$

Since  $W$  is simultaneously approximatively compact,  $\{y_n\}$  has a subsequence  $\{y_{n_k}\} \rightarrow y_0 \in W$ . From (2.1), we have  $\sup_{s \in S} \|s - y_0\| = d(S, W)$ , i.e.  $y_0 \in L_W(S)$  and so  $W$  is simultaneously proximal for  $S$ . Then for the constant sequence  $\{y_0\} \subseteq L_W(S)$ , we have  $\|y_{n_k} - y_0\| \rightarrow 0$ . Hence  $W$  is simultaneously strongly proximal for  $S$ .

Now suppose that  $\{z_n\}$  is any sequence in  $L_W(S)$ , i.e.  $\sup_{s \in S} \|s - z_n\| = d(S, W)$  for every  $n \in \mathbb{N}$ . Then  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - z_n\| = d(S, W)$  and so  $\{z_n\}$  is a minimizing sequence for  $S$ . Since  $W$  is simultaneously approximatively compact,  $\{z_n\}$  has a convergent subsequence  $\{z_{n_k}\}$ . Hence  $L_W(S)$  is compact.

Conversely, suppose that  $W$  is simultaneously strongly proximal and  $L_W(S)$  is compact for every bounded subset  $S$  of  $X$ . Let  $\{y_n\} \subseteq W$  be a minimizing sequence for  $S$ . Since  $W$  is simultaneously strongly proximal, there exist a subsequence  $\{y_{n_k}\}$  of  $\{y_n\}$  and sequence  $\{z_k\} \subseteq L_W(S)$  such that  $\|y_{n_k} - z_k\| \rightarrow 0$ . Since  $L_W(S)$  is compact,  $\{z_k\}$  has a subsequence  $\{z_{k_l}\} \rightarrow z_0 \in W$ . This gives  $\{y_{n_k}\} \rightarrow z_0$  and so  $W$  is simultaneously approximatively compact.  $\square$

Using Proposition 2.1, we obtain

**Corollary 2.3** *Let  $W$  be a finite dimensional subspace of a Banach space  $(X, \|\cdot\|)$ . Then  $W$  is simultaneously strongly proximal and  $L_W(S)$  is compact for every bounded subset  $S$  of  $X$ .*

Since a closed subset of a finite dimensional normed linear space is simultaneously approximatively compact, we have

**Corollary 2.4** *Let  $W$  be a closed subset of a finite dimensional normed linear space  $(X, \|\cdot\|)$ . Then  $W$  is simultaneously strongly proximal and  $L_W(S)$  is compact for every bounded subset  $S$  of  $X$ .*

A closed subset  $W$  of a Banach space  $X$  is called *simultaneously quasi-Chebyshev* (see [8]) if  $L_W(S)$  is nonempty and compact for every bounded subset  $S$  of  $X$ . Therefore, we have

**Corollary 2.5** *Let  $W$  be a closed subset of a Banach space  $(X, \|\cdot\|)$ ; then  $W$  is simultaneously approximatively compact if and only if  $W$  is simultaneously strongly proximal and simultaneously quasi-Chebyshev.*

The following theorem gives relationships between simultaneous approximative compactness, simultaneous strong Chebyshevity, and simultaneous strong proximality.

**Theorem 2.6** *Let  $W$  be a closed subset of a Banach space  $X$ . Then the following are equivalent:*

- (i)  $W$  is simultaneously strongly Chebyshev.
- (ii)  $W$  is simultaneously strongly proximal and simultaneously Chebyshev.
- (iii)  $W$  is simultaneously approximatively compact and simultaneously Chebyshev.

**Proof** (i)  $\Rightarrow$  (ii). Since  $W$  is simultaneously strongly Chebyshev, it is simultaneously approximatively compact and so by Theorem 2.2,  $W$  is simultaneously strongly proximal. Now suppose  $S$  is any bounded subset of  $X$  and  $w_1, w_2 \in L_W(S)$ ,  $w_1 \neq w_2$ . Then

$$\sup_{s \in S} \|s - w_1\| = d(S, W) = \sup_{s \in S} \|s - w_2\|.$$

Consider the sequence  $\{y_n\}$  in  $W$  such that  $y_{2n} = w_1$  and  $y_{2n+1} = w_2$ . Then  $\{y_n\}$  is a minimizing sequence for  $S$  in  $W$ . Since  $w_1 \neq w_2$ ,  $\{y_n\}$  is not convergent, a contradiction to simultaneous strong Chebyshevity of  $W$ . Thus  $w_1 = w_2$  and hence  $W$  is simultaneously Chebyshev.

(ii)  $\Rightarrow$  (iii) follows from Theorem 2.2.

(iii)  $\Rightarrow$  (i). Let  $\{y_n\} \subseteq W$  be a minimizing sequence for a bounded subset  $S$  of  $X$ , i.e.  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - y_n\| = d(S, W)$ . Since  $W$  is simultaneously approximatively compact,  $\{y_n\}$  has a subsequence  $\{y_{n_k}\} \rightarrow y_0$ . Then  $\sup_{s \in S} \|s - y_0\| = d(S, W)$ , i.e.  $y_0 \in L_W(S)$ . We claim that every subsequence of  $\{y_n\}$  also converges to  $y_0$ . Suppose  $\{y_n\}$  has a subsequence  $\{y_{n_i}\}$  such that  $\{y_{n_i}\} \rightarrow z_0$ ,  $z_0 \neq y_0$ . Then  $\sup_{s \in S} \|s - z_0\| = d(S, W)$ ,

i.e.  $z_0$  is also a best simultaneous approximation to  $S$  from  $W$ . However,  $W$  is simultaneously Chebyshev and so  $y_0 = z_0$ , a contradiction. Therefore, every subsequence of  $\{y_n\}$  converges to  $y_0$  and hence  $\{y_n\} \rightarrow y_0$ .  $\square$

The following example shows that a simultaneously approximatively compact set need not be simultaneously strongly Chebyshev.

**Example 2.7** Let  $X = (\mathbb{R}^2, \|\cdot\|)$ , where  $\|(x, y)\| = \max(|x|, |y|)$ ,  $W = \{(x, 0) : x \in \mathbb{R}\}$  and  $S = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ . Then  $W$  being finite dimensional subspace of  $X$ , is simultaneously approximatively compact. But  $W$  is not simultaneously strongly Chebyshev for  $S$ . Consider the sequence  $\{y_n\}$  such that  $y_{2n} = (2.5, 0)$  and  $y_{2n+1} = (1.5, 0)$ . Then  $\{y_n\}$  is a minimizing sequence for  $S$  which is not convergent.

**Remarks**

1. While an approximatively compact subset of a Banach space is strongly proximal (see [1]), a strongly proximal subset of a Banach space need not be approximatively compact.

Let  $X = l_\infty$ ,  $W = c_0$ . Then  $W$ , being an M-ideal, is strongly proximal in  $X$ . However, for  $x = (1, 1, 1, \dots) \in l_\infty$ , the sequence  $y_n = (1, 1, \dots, 1, 0, 0, \dots) \in W$  is a minimizing sequence for  $x$  but  $\{y_n\}$  has no convergent subsequence (see [1]).

2. A proximal subset of a Banach space need not be strongly proximal. Even a proximal convex subset of a Banach space need not be strongly proximal.

Let  $X = (l_1, \|\cdot\|_H)$ , as constructed by Smith (see [12], Example 5). Then the unit ball  $B(X_H)$  is proximal in  $X$  but not strongly proximal in  $X$  (see [14]).

3. The results proved in this section generalize the corresponding results proved in [1,10] for strong proximality.

**3. Sum and quotient of simultaneously strongly proximal subspaces**

In this section, we discuss simultaneous strong proximality for the sum and quotient subspaces of  $X$  and see how simultaneous strong proximality is transmitted to and from quotient spaces and to sum of subspaces. The results proved in this section are motivated by the corresponding results proved for proximality in [3], simultaneous proximality in [8,11], and strong proximality in [13].

The following two results of [11] will be used in the sequel:

**Lemma 3.1** Let  $F$  and  $W$  be subspaces of a Banach space  $X$  such that  $F$  is simultaneously proximal,  $W$  is finite dimensional, and  $F + W$  is closed; then  $F + W$  is simultaneously proximal in  $X$ .

**Lemma 3.2** Let  $F$  be a simultaneously proximal subspace of a Banach space  $X$  and  $W$  a subspace of  $X$  such that  $W + F$  is closed. If  $W + F$  is simultaneously proximal in  $X$  then  $(W + F)/F$  is simultaneously proximal in  $X/F$ .

Concerning the sum of simultaneously strongly proximal subspaces, we have

**Theorem 3.3** Let  $F$  and  $W$  be subspaces of a Banach space  $X$  such that  $F$  is simultaneously strongly proximal,  $W$  is finite dimensional, and  $F + W$  is closed; then  $F + W$  is simultaneously strongly proximal in  $X$ .

**Proof** Since  $F$  is simultaneously proximal and  $W$  is finite dimensional, it follows from Lemma 3.1 that  $F + W$  is simultaneously proximal. Let  $S$  be a bounded subset of  $X$  and  $\{h_n + w_n\} \subseteq F + W$  is such that  $\sup_{s \in S} \|s - (h_n + w_n)\| \rightarrow d(S, W + F)$ , where  $\{h_n\}$  is a sequence in  $F$  and  $\{w_n\}$  is a sequence in  $W$ . Therefore, the sequence  $\{a_n\}$  given by  $a_n = \sup_{s \in S} \|s - (h_n + w_n)\|$  is a convergent sequence of real numbers and hence bounded. Thus, there exist  $M_1 > 0$  such that  $|a_n| \leq M_1$  for every  $n$ . The inequality  $\|h_n + w_n\| \leq \|h_n + w_n - s\| + \|s\|$  for any  $s \in S$  and for any  $n$  implies that  $\|h_n + w_n\| \leq \sup_{s \in S} \|h_n + w_n - s\| + \sup_{s \in S} \|s\| \leq M_1 + M_2 = M$  for all  $n$  and so  $\{h_n + w_n\}$  is a bounded sequence in  $F + W$ .

Since the projection  $P : F + W \rightarrow W, P(h + w) = w$  is closed and  $P(F) = 0$ , by the closed graph theorem there exists a positive constant  $C$  such that  $\|w_n\| = \|P(h_n + w_n)\| \leq C\|h_n + w_n\| \leq CM$ . This implies that  $\{w_n\}$  is a bounded sequence and so is  $\{h_n\}$ . Since  $W$  is finite dimensional, by passing to a subsequence if necessary, we may assume that  $w_n \rightarrow w_0 \in W$ . We claim  $d(S - w_0, F) = d(S, F + W) = \lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (w_0 + h_n)\|$ .

Since  $|\sup_{s \in S} \|s - (h_n + w_n)\| - \sup_{s \in S} \|s - (w_0 + h_n)\|| \leq \sup_{s \in S} \| \|s - (h_n + w_n)\| - \|s - (w_0 + h_n)\| \| \leq \|w_n - w_0\|$ , we have  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (h_n + w_n)\| = \lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (h_n + w_0)\|$ . Therefore,  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (h_n + w_n)\| = d(S, F + W) = \lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (h_n + w_0)\| \geq d(S - w_0, F)$ . We also have  $d(S, F + W) = \inf_{h+w \in F+W} \sup_{s \in S} \|s - (h + w)\| \leq d(S - w_0, F)$ . Hence the claim holds. Therefore,  $\{h_n\} \subseteq F$  is a minimizing sequence for  $S - w_0$ . Since  $F$  is simultaneously strongly proximal, there exist a subsequence  $\{h_{n_k}\}$  and a sequence  $\{z_k\} \subseteq L_F(S - w_0)$  such that  $\|h_{n_k} - z_k\| \rightarrow 0$ . As  $z_k \in L_F(S - w_0)$ ,  $\sup_{s \in S} \|(s - w_0) - z_k\| = d(S - w_0, F) = d(S, F + W)$ , i.e.  $w_0 + z_k \in L_{F+W}(S)$ . Now  $\{w_0 + z_k\} \subseteq L_{F+W}(S)$  and  $\|h_{n_k} + w_{n_k} - (w_0 + z_k)\| \rightarrow 0$ . Thus  $F + W$  is a simultaneously strongly proximal subspace of  $X$ . □

If we take  $L_F(S)$  to be compact for every bounded subset  $S$  of  $X$  then from the proof of the above theorem, we obtain

**Corollary 3.4** *Let  $F$  and  $W$  be subspaces of a Banach space  $X$  such that  $F$  is simultaneously strongly proximal,  $W$  is finite dimensional, and  $F + W$  is closed; then  $F + W$  is simultaneously approximatively compact.*

We require the following lemma in the proof of the next theorem, which shows that if  $W + F$  is simultaneously strongly proximal then so is its quotient space.

**Lemma 3.5** [8] *Let  $W$  be a proximal subspace of a Banach space  $X$ ; then for any bounded subset  $S$  of  $X$  we have*

$$d(S, W) = \sup_{s \in S} \inf_{w \in W} \|s - w\|.$$

**Theorem 3.6** *Let  $F$  be a simultaneously proximal subspace of a Banach space  $X$  and  $W$  a subspace of  $X$  such that  $W + F$  is closed. If  $W + F$  is simultaneously strongly proximal in  $X$  then  $(W + F)/F$  is simultaneously strongly proximal in  $X/F$ .*

**Proof** Suppose  $W + F$  is simultaneously strongly proximal in  $X$ ; then  $W + F$  is simultaneously proximal and so, using Lemma 3.2,  $(W + F)/F$  is simultaneously proximal in  $X/F$ .

Let  $A$  be any bounded subset of  $X/F$ ; then  $A = S/F$  for some bounded subset  $S$  of  $X$  (see [11]). Let  $\{y_n + F\} \subseteq (W + F)/F$  be any minimizing sequence for  $S/F$ , i.e.  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|(s + F) - (y_n + F)\|$

$F)\| = d(S/F, (W + F)/F)$ . Then  $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$  for any  $\delta_n > 0$  after some stage. Since  $y_n + F \in L_{(W+F)/F}(S/F, \delta_n)$

$$\begin{aligned} \sup_{s \in S} \|(s + F) - (y_n + F)\| &< \sup_{s \in S} \|(s + F) - (g + F)\| + \delta_n \text{ for all } g + F \in (W + F)/F \\ \Rightarrow \sup_{s \in S} \inf_{f \in F} \|(s - y_n) - f\| &< \sup_{s \in S} \inf_{f \in F} \|(s - g) - f\| + \delta_n \text{ for all } g \in (W + F) \end{aligned}$$

Using Lemma 3.5 and the proximality of  $F$ , we can find  $f_n \in F$  such that

$$\sup_{s \in S} \|(s - y_n) - f_n\| < \inf_{f \in F} \sup_{s \in S} \|(s - g) - f\| + \delta_n \text{ for all } g \in (W + F)$$

Letting  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} \sup_{s \in S} \|(s - y_n) - f_n\| \leq d(S, W + F).$$

Moreover,  $d(S, W + F) \leq \lim_{n \rightarrow \infty} \sup_{s \in S} \|(s - y_n) - f_n\|$  and so  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|(s - y_n) - f_n\| = d(S, W + F)$ , i.e.  $\{y_n + f_n\}$  is a minimizing sequence for  $S$  in  $W + F$ . Since  $W + F$  is simultaneously strongly proximal for  $S$ , there exist a subsequence  $\{y_{n_k} + f_{n_k}\}$  and  $\{z_k\} \subseteq L_{W+F}(S)$  such that  $\|(y_{n_k} + f_{n_k}) - z_k\| \rightarrow 0$ . Now  $\|(y_{n_k} + F) - (z_k + F)\| = \inf_{f \in F} \|y_{n_k} - z_k + f\| \leq \|y_{n_k} - z_k + f_{n_k}\| \rightarrow 0$ . Hence  $(W + F)/F$  is simultaneously strongly proximal for  $S/F$ .  $\square$

Using Theorem 3.3, we have

**Corollary 3.7** *Let  $F$  and  $W$  be subspaces of a Banach space  $X$  such that  $F$  is simultaneously strongly proximal,  $W$  is finite dimensional, and  $F+W$  is closed; then  $(F+W)/F$  is simultaneously strongly proximal in  $X/F$ .*

Concerning strong proximality in quotient spaces, we have

**Theorem 3.8** *Let  $W$  and  $F$  be subspaces of a Banach space  $X$  and  $F \subseteq W$  is proximal in  $X$ . If  $W$  is simultaneously strongly proximal in  $X$  then  $W/F$  is simultaneously strongly proximal in  $X/F$ .*

**Proof** Since  $W$  is simultaneously proximal,  $W/F$  is simultaneously proximal (see [8]). Let  $S/F$  be any bounded subset of  $X/F$  and  $\{y_n + F\} \subseteq W/F$  be any minimizing sequence for  $S/F$ , i.e.

$$\lim_{n \rightarrow \infty} \sup_{s+F \in S/F} \|(s + F) - (y_n + F)\| = d(S/F, W/F).$$

Thus  $y_n + F \in L_{W/F}(S/F, \delta_n)$  for any  $\delta_n > 0$  after some stage. Then  $\sup_{s+F \in S/F} \|(s + F) - (y_n + F)\| < \sup_{s+F \in S/F} \|(s + F) - (w + F)\| + \delta_n$  for all  $w + F \in W/F$ , i.e.  $\sup_{s \in S} \inf_{f \in F} \|(s - y_n) - f\| < \sup_{s \in S} \inf_{f \in F} \|(s - w) - f\| + \delta_n$  for all  $w \in W$ . Using Lemma 3.5 and proximality of  $F$ , we can find  $f_n \in F$  such that  $\sup_{s \in S} \|(s - y_n) - f_n\| < \inf_{f \in F} \sup_{s \in S} \|(s - w) - f\| + \delta_n < \sup_{s \in S} \|(s - w)\| + \delta_n$  for all  $w \in W$ . Therefore, letting  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (y_n + f_n)\| \leq d(S, W)$ . We also have  $d(S, W) \leq \lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (y_n + f_n)\|$ . Therefore,  $\lim_{n \rightarrow \infty} \sup_{s \in S} \|s - (y_n + f_n)\| = d(S, W)$ . This implies that  $\{y_n + f_n\} \subseteq W$  is a minimizing sequence for  $S$ . Since  $W$  is simultaneously strongly proximal



for  $S$  there exist a subsequence  $\{y_{n_k} + f_{n_k}\}$  and  $\{z_k\} \subseteq L_W(S)$  such that  $\|(y_{n_k} + f_{n_k}) - z_k\| \rightarrow 0$ . Now  $\|(y_{n_k} + F) - (z_k + F)\| = \inf_{f \in F} \|(y_{n_k} - z_k) - f\| \leq \|y_{n_k} - z_k + f_{n_k}\| \rightarrow 0$ . Hence  $W/F$  is simultaneously strongly proximal for  $S/F$ .  $\square$

It was proved in [8] that for subspaces  $W$  and  $F$  of  $X$  such that  $F \subseteq W$  is simultaneously Chebyshev then  $W$  is simultaneously Chebyshev if and only if  $W/F$  is simultaneously Chebyshev. Therefore, we have

**Corollary 3.9** *Let  $W$  and  $F$  be subspaces of a Banach space  $X$  and  $F \subseteq W$  is simultaneously Chebyshev in  $X$ . If  $W$  is simultaneously strongly Chebyshev then  $W/F$  is simultaneously strongly Chebyshev.*

The authors do not know whether the converse parts of Theorems 3.6 and 3.8 hold.

**Remarks** Taking  $S$  to be a singleton set, we obtain several results of strong proximality proved in [1,10,13].

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