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A two-obstacle problem with variable exponent and measure data

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Abstract: We consider a two-obstacle problem with measure data. For measures that do not charge sets of zero $p(\cdot)$-capacity, we obtain the existence and uniqueness of the solution. On the other hand, for the measure concentrated on a set with zero $p(\cdot)$-capacity, we prove a nonexistence result in the sense that when one looks for solutions via approximation, one cannot find a reasonable solution; see Theorem 2.3 and Remark 2.1 below.

Key words: Variable exponent, obstacle problems, measure data

1. Introduction

Differential equations and variational problems involving variable exponents have attracted more and more attention in recent years. Such problems are interesting from the purely mathematical point of view. Moreover, they have potential applications in various fields such as flow through porous media [1], thermorheological fluids [2], image processing [5, 10], and especially electrorheological fluids (an essential class of non-Newtonian fluids), which have been used not only in fast-acting hydraulic valves and clutches, brakes, and shock absorbers, but also in some new fields such as accurate abrasive polishing, robotics, and space technology[3, 18].

Let Ω be a bounded domain in $\mathbb{R}^N (N > 2)$ and let μ be a bounded Radon measure on Ω. Let $p \in C(\overline{\Omega})$ with $1 < p^- = \min_{x \in \overline{\Omega}} p(x) \le p^+ = \max_{x \in \overline{\Omega}} p(x) < N$. In this paper, we consider the double-obstacle problem involving a variable exponent, which consists of finding a function $u \in K_\psi^\varphi$ such that the following variational inequality holds:

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla (v-u) \ge \int_\Omega \mu (v-u) dx, \forall v \in K_\psi^\varphi, \tag{1.1}$$

where $K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \le v \le \varphi\}, \varphi, \psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \psi \le \varphi$ a.e. in Ω.

We recall that obstacle problems with constant exponents and data of L^1 or measure type have been studied largely; see for example [4, 15, 16] and the references therein. In [19], an obstacle problem with variable exponent and L^1 data was studied. Using smooth approximation, the authors proved the existence and uniqueness of the entropy solution. This result was then extended to obstacle problems with more general type of variable exponents by the authors in [17]. In [8, 11], the regularity and stability results were established for some obstacle problems involving variable exponents.

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More recently, in [20] Rodrigues and Teymurazyanin studied a double-obstacle problem, which included problem (1.1) as a special case. The existence and uniqueness result was obtained when the data involved were regular enough. Motivated by these previous works, in this paper we consider the double-obstacle problem (1.1) involving measure data. Under suitable assumptions, we prove that problem (1.1) admits a unique solution if μ does not charge sets of zero $p(\cdot)$-capacity, and the problem does not admit a "reasonable" solution if μ is concentrated on a set with zero $p(\cdot)$-capacity.

To move on, let us first recall the definitions and properties of the generalized Lebesgue and Sobolev spaces; interested readers may refer to [7, 9, 13] for more details.

For $p \in C(\overline{\Omega})$ with $p^- > 1$, define the variable exponent Lebesgue space as

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R}; u \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < \infty\}$$

with norm $\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\{\rho > 0 : \int_\Omega |\frac{u(x)}{\rho}|^{p(x)} dx \leq 1\}$. We have

$$\min\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\} \leq \int_\Omega |u|^{p(x)} dx \leq \max\{\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}\}.$$

As $p^- > 1$, the space is a reflexive Banach space with dual $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega), v \in L^{p'(\cdot)}(\Omega)$, we have the Hölder-type inequality

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

For positive integer k, the generalized Sobolev space is defined as

$$W^{k,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k\}$$

with norm

$$\|u\|_{W^{k,p(\cdot)}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

In this paper, we will always assume that $p(\cdot)$ satisfies the following log-Hölder continuous condition, i.e. there exists a positive constant C such that

$$|p(x) - p(y)| \leq -\frac{C}{\log|x-y|}, \text{ for every } x, y \in \Omega \text{ with } |x-y| < \frac{1}{2}. \tag{1.2}$$

This condition ensures that smooth functions are dense in the generalized Sobolev spaces. Then $W_0^{k,p(\cdot)}(\Omega)$ can naturally be defined as the completion of $C_c^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p(\cdot)}}$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, the Poincaré-type inequality holds, i.e. $\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$, where the positive constant C depends on p and Ω. Thus $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$.

Let $K \subset \Omega$ be compact and χ_K be its characteristic function. The $p(\cdot)$-capacity of K with respect to Ω can be defined as follows (see [7, 12]):

$$\text{cap}_{p(\cdot)}(K, \Omega) = \inf\{\int_\Omega |\nabla\varphi|^{p(x)} dx, \varphi \in C_c^\infty(\Omega), \varphi \geq \chi_K\},$$

where we set $\inf \emptyset = \infty$. For any open set $U \subset \Omega$, define

$$\mathrm{cap}_{p(\cdot)}(U,\Omega) = \sup\{\mathrm{cap}_{p(\cdot)}(K,\Omega), K \subset U, K \text{ compact}\}.$$

The definition of $p(\cdot)$-capacity can be extended to any Borel set $E \subset \Omega$ as

$$\mathrm{cap}_{p(\cdot)}(E,\Omega) = \inf\{\mathrm{cap}_{p(\cdot)}(U,\Omega), E \subset U \subset \Omega, U \text{ open}\}.$$

Using truncation and smooth approximation, for every compact set $K \subset \Omega$, one may define its $p(\cdot)$-capacity as

$$\mathrm{cap}_{p(\cdot)}(K,\Omega) = \inf\{\int_\Omega |\nabla\varphi|^{p(x)}dx, \varphi \in C_c^\infty(\Omega), 0 \le \varphi \le 1, \varphi = 1 \text{ in some neighborhood of } K\}.$$

Let μ be a bounded Radon measure concentrated on a set E. Thanks to the Hahn decomposition theorem, μ can be decomposed as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^-, being positive, are the upper and lower variation of μ, respectively, with μ^+ concentrated on E^+, μ^- concentrated on E^-, and $E^+ \cap E^- = \emptyset$.

The following lemmas play an essential role in our analysis in the next.

Lemma 1.1 ([6]) *Let μ be a bounded Radon measure on Ω, which is absolutely continuous with respect to the $p(\cdot)$-capacity. Then μ can be decomposed as $\mu = \mu_1 + \mu_2$ with $\mu_1 \in L^1(\Omega), \mu_2 \in W^{-1,p'(x)}(\Omega)$.*

Lemma 1.2 *[6] Let $\mu = \mu^+ - \mu^-$ be a Radon measure concentrated on a set E of zero $p(\cdot)$-capacity with $1 < p^- \le p^+ \le N$. Then for every $\delta > 0$, there exist two functions $\psi_\delta^+, \psi_\delta^- \in C_c^\infty(\Omega)$ such that*

$$0 \le \psi_\delta^+ \le 1, \quad 0 \le \psi_\delta^- \le 1, \quad \int_\Omega |\nabla\psi_\delta^+|^{p(x)}\,dx \le \delta, \quad \int_\Omega |\nabla\psi_\delta^-|^{p(x)}\,dx \le \delta,$$

$$0 \le \int_\Omega (1-\psi_\delta^+)d\mu^+ \le \delta, \quad 0 \le \int_\Omega (1-\psi_\delta^-)d\mu^- \le \delta, \quad 0 \le \int_\Omega \psi_\delta^- d\mu^+ \le \delta, \quad 0 \le \int_\Omega \psi_\delta^+ d\mu^- \le \delta.$$

2. Existence and nonexistence results

In this section, we prove existence and nonexistence results for problem (1.1) according to the singularity of the data μ. We are mainly concerned with the case $1 < p^- \le p^+ < N$.

Our first result concerns the existence result for problem (1.1).

Theorem 2.1 *If μ is absolutely continuous with respect to the $p(\cdot)$-capacity, problem (1.1) admits a unique solution u.*

Proof If μ is absolutely continuous with respect to the $p(\cdot)$-capacity, thanks to Lemma 2.1 we have

$$\mu = g + \mathrm{div}G \quad \text{with} \quad g \in L^1(\Omega), G \in (L^{p'(\cdot)}(\Omega))^N.$$

Then problem (1.1) is equivalent to the following problem:

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(v-u)dx \ge \int_\Omega g(v-u)dx + \int_\Omega G\cdot\nabla(v-u)dx \quad \forall v \in K_\psi^\varphi, \\ u_n \in K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \le v \le \varphi\}, \end{cases} \tag{2.1}$$

where $\varphi, \psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \psi \le \varphi$ a.e. in Ω.

The uniqueness of the solution to problem (2.1) follows from rather standard arguments. Indeed, assume that u_1, u_2 are two solutions of (2.1). Taking u_1 as a test function in the formulation of solution u_2, and taking u_2 as a test function in the formulation of solution u_1, we can easily deduce that $u_1 = u_2$.

The existence of the solution u for problem (2.1) can be obtained as the limit of the solution sequence $\{u_n\}$ for the following approximate problem:

$$\begin{cases} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(v-u_n)dx \geq \int_\Omega g_n(v-u_n)dx + \int_\Omega G\cdot\nabla(v-u_n)dx, \quad \forall v \in K_\psi^\varphi, \\ u_n \in K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \leq v \leq \varphi\}, \end{cases} \tag{2.2}$$

where g_n is a sequence of smooth functions that converges to g in $L^1(\Omega)$ with $\|g_n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}$. Since the analysis is rather similar to those in [6, 19], we omit the details. $\square$

In the above theorem, we have found that when μ is a 'smooth' measure, we can find a solution for problem (1.1) via approximations. However, for a singular measure μ we cannot expect to find a reasonable solution for problem (1.1) in such a way.

Let $\mu(=\mu^+ - \mu^-)$ be a bounded Radon measure concentrated on a set $E(\subset\Omega)$ with zero $p(\cdot)$-capacity. Let $f_n = f_n^\oplus - f_n^\ominus$ be a sequence of smooth data with $f_n^\oplus, f_n^\ominus$ being positive and converging to μ^+, μ^- respectively in the narrow topology of measures, i.e.

$$\lim_{n\to\infty}\int_\Omega f_n^\oplus \phi dx = \int_\Omega \phi d\mu^+, \quad \lim_{n\to\infty}\int_\Omega f_n^\ominus \phi dx = \int_\Omega \phi d\mu^- \qquad \text{for any } \phi \in C(\overline{\Omega}). \tag{2.3}$$

Let $G \in (L^{p'(\cdot)}(\Omega))^N, g \in L^1(\Omega)$, and g_n be a sequence of smooth functions that converges to g in $L^1(\Omega)$ with $\|g_n\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega)}$. Consider the following obstacle problem:

$$\begin{cases} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(v-u_n)dx \geq \int_\Omega g_n(v-u_n)dx + \int_\Omega G\cdot\nabla(v-u_n)dx \\ +\int_\Omega f_n(v-u_n)dx, \forall v \in K_\psi^\varphi, \\ u_n \in K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \leq v \leq \varphi\}, \end{cases} \tag{2.4}$$

where $\varphi, \psi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \psi \leq \varphi$ a.e. in Ω. As n tends to infinity, we have the following convergence result for the solution sequence $\{u_n\}$ (the existence and uniqueness of the solution u_n to (2.2) for each n follows from standard results for monotone, coercive operators; see [14, 20]).

Theorem 2.2 *Let u_n be the solution to problem (2.2). Then when n tends to infinity, u_n converges in $W_0^{1,p(\cdot)}(\Omega)$ to a function u, which is the unique solution of the following double obstacle problem:*

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla(v-u) \geq \int_\Omega g(v-u)dx + \int_\Omega G\cdot\nabla(v-u), \forall v \in K_\psi^\varphi, \\ u \in K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \leq v \leq \varphi\}. \end{cases} \tag{2.5}$$

As a special case, we have the following nonexistence result for problem (1.1) immediately.

Theorem 2.3 *Let $\mu(=\mu^+ - \mu^-)$ be a bounded Radon measure concentrated on a set with zero $p(\cdot)$-capacity, and f_n be a sequence of smooth functions converging to μ in the sense of (2.3). Then as n tends to infinity,* *the solution u_n for the following obstacle problem*

$$\begin{cases} \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(v-u_n)dx \geq \int_\Omega f_n(v-u_n)dx, \forall v \in K_\psi^\varphi, \\ u_n \in K_\psi^\varphi = \{v \in W_0^{1,p(\cdot)}(\Omega), \psi \leq v \leq \varphi\}. \end{cases} \tag{2.6}$$

converges to 0 *in* $W_0^{1,p(\cdot)}(\Omega)$*.*

Remark 2.1 *As we see, if μ is a bounded Radon measure concentrated on a set with zero $p(\cdot)$-capacity, we cannot find a solution to problem (1.1) by approximation. The 'singular' measure disappears when we pass to the limit in the approximate problem. Moreover, 0 is not a solution to the original problem (1.1). Thus Theorem 2.3 implies a nonexistence result for problem (1.1) in the sense that when one looks for solutions by approximation, one cannot find a reasonable solution.*

Remark 2.2 *For obstacle problems with general Leray–Lions type operators involving variable exponent (see for example [21]), with very minor modifications, we can prove the similar result.*

Proof of Theorem 2.2. Now we are in a position to prove Theorem 2.2. Hereafter, we use C to denote some positive constant, which may distinguish with each other even in the same line. We denote by $w(\delta, n)$ any quantity such that

$$\lim_{\delta\to 0^+}\lim_{n\to\infty}|w(\delta,n)| = 0.$$

Furthermore, the convergences may be understood to be taken possibly up to a suitable subsequence extraction, even if we do not explicitly stress it.

Taking a function $v_0 \in K_\psi^\varphi$ as a test function in (2.4), we deduce that

$$\int_\Omega |\nabla u_n|^{p(x)}dx \leq \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v_0 dx + \int_\Omega G\cdot\nabla(u_n - v_0)dx + \int_\Omega (g_n + f_n)(u_n - v_0)dx. \tag{2.7}$$

Using Young's inequality, we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v_0 dx + \int_\Omega G\cdot\nabla(u_n - v_0)dx \\ &\leq C\int_\Omega |G|^{p'(x)}dx + C\int_\Omega |\nabla v_0|^{p(x)}dx + \frac{1}{2}\int_\Omega |\nabla u_n|^{p(x)}dx. \end{aligned}$$

Note that (2.3) implies that $\|f_n\|_{L^1(\Omega)} \leq C$. We then deduce from (2.7) that

$$\int_\Omega |\nabla u_n|^{p(x)}dx \leq C\int_\Omega (|G|^{p'(x)} + |\nabla v_0|^{p(x)})dx + 2(\|g\|_{L^1(\Omega)} + C)\max\{\|\varphi\|_{L^\infty(\Omega)}, \|\psi\|_{L^\infty(\Omega)}\} \leq C. \tag{2.8}$$

Noting that $\psi \leq u_n \leq \varphi$, we know that $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, and there exists a function $u \in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ with $\psi \leq u \leq \varphi$ such that

$$\begin{aligned} &u_n \longrightarrow u \quad \text{strongly in } L^1(\Omega), \text{ weakly}^* \text{ in } L^\infty(\Omega), \\ &u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p(\cdot)}(\Omega), \nabla u_n \longrightarrow \nabla u \quad \text{weakly in } L^{p(\cdot)}(\Omega). \end{aligned}$$

Next, let us prove that u_n converges to u strongly in $W_0^{1,p(\cdot)}(\Omega)$.

Let $\Psi_\delta = \psi_\delta^+ + \psi_\delta^-$. Taking $u_n\Psi_\delta + u(1-\Psi_\delta)$ (of course belongs to K_ψ^φ) as a test function in (2.4), we deduce that

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u)(1-\Psi_\delta)dx - \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\Psi_\delta(u_n-u)dx \\
&\leq \int_\Omega g(u_n-u)(1-\Psi_\delta)dx + \int_\Omega G\cdot\nabla(u_n-u)(1-\Psi_\delta)dx - \int_\Omega G\cdot\nabla\Psi_\delta(u_n-u)dx \\
&+ \int_\Omega f_n^{\oplus}(u_n-u)(1-\Psi_\delta)dx - \int_\Omega f_n^{\ominus}(u_n-u)(1-\Psi_\delta)dx. \qquad (2.9)
\end{aligned}$$

Denote the seven terms in the above inequality by A_1 to A_7 sequentially. Thanks to Lemma 2.2, both ψ_δ^+ and ψ_δ^- converge to 0 weakly* in $L^\infty(\Omega)$, strongly in $W_0^{1,p(\cdot)}(\Omega)$, almost everywhere in Ω, as δ vanishes. Since $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$, we are ready to obtain that

$$A_2, A_5 = w(\delta, n). \qquad (2.10)$$

From the weakly* convergence of u_n to u, we have $A_3 = w(\delta,n)$. On the other hand, from the weak convergence of ∇u_n to ∇u in $L^{p(\cdot)}(\Omega)$, we know that $A_4 = w(n)$. By Lemma 2.2 we have

$$|A_6| \leq C\int_\Omega f_n^{\oplus}(1-\Psi_\delta)dx = C\int_\Omega f_n^{\oplus}(1-\psi_\delta^+)dx - C\int_\Omega f_n^{\oplus}\psi_\delta^- dx = w(\delta,n).$$

Similarly, we have $A_7 = w(\delta,n)$. Lastly, from the weak convergence of ∇u_n to ∇u in $L^{p(\cdot)}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(u_n-u)(1-\Psi_\delta)dx = w(\delta,n).$$

Then we deduce from the convergences of A_2 to A_7 that

$$0 \leq \int_\Omega |\nabla u - \nabla u_n|^{p(x)}(1-\Psi_\delta)dx \leq A_1 - \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla(u_n-u)(1-\Psi_\delta)dx = w(\delta,n). \qquad (2.11)$$

To obtain the strong convergence of u_n to u in $W_0^{1,p(\cdot)}(\Omega)$, we need only to show that

$$\int_\Omega |\nabla u - \nabla u_n|^{p(x)}\Psi_\delta dx = w(\delta,n). \qquad (2.12)$$

Since $\int_\Omega |\nabla u|^{p(\cdot)}\Psi_\delta dx \longrightarrow 0$ as $\delta\to 0$, we need only to prove

$$\int_\Omega |\nabla u_n|^{p(x)}\Psi_\delta dx \longrightarrow 0 \quad \text{as } n\to\infty, \delta\to 0. \qquad (2.13)$$

Taking $\varphi\psi_\delta^+ + (1-\psi_\delta^+)u_n$ ($\in K_\psi^\varphi$) as a test function in (2.4), we deduce that

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-\varphi)\psi_\delta^+ dx + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\delta^+(u_n-\varphi)dx \\
&\leq \int_\Omega g(u_n-\varphi)\psi_\delta^+ dx + \int_\Omega G\cdot\nabla(u_n-\varphi)\psi_\delta^+ dx + \int_\Omega G\cdot\nabla\psi_\delta^+(u_n-\varphi)dx \\
&+ \int_\Omega f_n^{\oplus}(u_n-\varphi)\psi_\delta^+ dx - \int_\Omega f_n^{\ominus}(u_n-\varphi)\psi_\delta^+ dx. \qquad (2.14)
\end{aligned}$$

Denote the seven terms in (2.14) by B_1 to B_7 sequentially. Similar to the analysis of (2.9), thanks to the convergence of ψ_δ^+ and the estimate for u_n, we have

$$B_2, B_3, B_4, B_5 = w(\delta, n).$$

Note that B_6 is nonpositive and

$$|B_7| = |\int_\Omega f_n^\ominus (u_n - \varphi)\psi_\delta^+ dx| \leq C\int_\Omega f_n^\ominus \psi_\delta^+ dx = w(\delta, n),$$

Note that $|\nabla u_n|^{p(x)-2}\nabla u_n$ converges to some function χ weakly in $L^{p'(\cdot)}(\Omega)$. We have

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi\psi_\delta^+ dx = \int_\Omega \chi\nabla\varphi\psi_\delta^+ dx + w(n) = w(\delta, n).$$

Then we deduce from (2.14) that

$$\int_\Omega |\nabla u_n|^{p(x)}\psi_\delta^+ dx = B_1 + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\varphi\psi_\delta^+ dx = w(\delta, n). \tag{2.15}$$

Similarly, taking $\psi\psi_\delta^- + (1-\psi_\delta^-)u_n (\in K_\psi^\varphi)$ as a test function in (2.4), we can prove that

$$\int_\Omega |\nabla u_n|^{p(x)}\psi_\delta^- dx = w(\delta, n),$$

which, combined with (2.15), implies (2.13). Then we conclude from (2.11), (2.12) that u_n converges to u strongly in $W_0^{1,p(\cdot)}(\Omega)$.

For any $v \in K_\psi^\varphi$, taking $u_n\Psi_\delta + v(1-\Psi_\delta)$ in (2.4), thanks to the convergence of u_n and Ψ_δ, it is easy to obtain that u is a solution to problem (2.5) by passing to the limit on n and δ. The proof is completed.

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