Turkish Journal of Mathematics

Volume 41 | Number 3 Article 15

1-1-2017

Evolution equations with a parameter and application to transport-convection differential equations

EMILE FRANC DOUNGMO GOUFO

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 636 – 654
© TÜBİTAK
doi:10.3906/mat-1603-107

Evolution equations with a parameter and application to transport-convection differential equations

Emile Franc DOUNGMO GOUFO*
Department of Mathematical Sciences, University of South Africa, Florida, South Africa

Received: 24.03.2016 • **Accepted/Published Online:** 27.06.2016 • **Final Version:** 22.05.2017

Abstract: We deeply investigate the well-posedness of models taking the form ${}_0^A D_t^\beta u(t) = Au(t), \quad u(0) = f, \quad 0 < \beta \leq 1, \quad t > 0$ where ${}_0^A D_t^\beta$ is a derivative with the fractional parameter β and A is a closed densely defined operator in a Banach space. We show that, unlike other systems, solutions of our models are not governed by Mittag–Leffler functions and their variants. We extend and adapt Peano's idea to our models and establish conditions for existence and uniqueness of solutions. In particular, relations between the two-parameter solution operator, its resolvent, and its generator are provided; the issue of subordination and prolongation principles are addressed; and a way to approximate the generalized solution is presented. Finally, application to transport-convection differential equations is performed in the space of distributions with finite higher moments to show how their well-posedness can be addressed.

Key words: Derivative with a new parameter, Cauchy problem, solution operators with two parameters, revamped time, β-exponentially boundedness, well-posedness

1. Motivation and definition

Today, it is widely known that the Newtonian concept of a derivative can no longer satisfy all the complexity of the natural occurrences. A couple of complex phenomena and features happening in some areas of sciences or engineering are still (partially) unexplained by the traditional existing methods and remain open problems. Usually in the mathematical modeling of a natural phenomenon that changes, the evolution is described by a family of time-parameter operators that map an initial given state of the system to all subsequent states that take the system during the evolution. A way of looking at that time evolution as a transition from one state to another has been widely predominant among applied scientists. Hence, this is how the theory of semigroups was developed [15, 24], providing mathematicians with very interesting tools to investigate and analyze resulting mathematical models. However, most of the phenomena that scientists try to analyze and describe mathematically are complex and very hard to handle. Some of them like depolymerization, rock fractures, and fragmentation processes are difficult to analyze [12, 29] and often involve the evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble. Then, though linear, they display nonlinear features such as phase transition (called "shattering") causing the appearance of a "dust" of "zero-size" particles with nonzero mass.

Another example is the groundwater flowing within a leaky aquifer. Recall that an aquifer is an underground layer of water-bearing permeable rock or unconsolidated materials (gravel, sand, or silt) from

*Correspondence: dgoufef@unisa.ac.za
2010 *AMS Mathematics Subject Classification:* 35F10, 26A33, 35D05.

which groundwater can be extracted using a water well. Then, how do we explain accurately the observed movement of water within the leaky aquifer? As an attempt to answer this question, Hantush [16, 17] proposed an equation with the same name and his model has since been used by many hydrogeologists around the world. However, it is necessary to note that the model does not take into account all the nonusual details surrounding the movement of water through a leaky geological formation. Indeed, due to the deformation of some aquifers, the Hantush equation is not able to account for the effect of the changes in the mathematical formulation [2]. Hence, all those nonusual features are beyond the usual models' resolutions and need other techniques and methods of modeling with more parameters involved.

Furthermore, time's evolution and changes occurring in some systems do not happen in the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of a second, causing a major change that may affect the whole system's state forever. Indeed, it has turned out recently that many phenomena in different fields, including sciences, engineering, and technology, can be described very successfully by models using fractional order differential equations [6, 7, 10, 11, 13, 14, 18, 21, 22, 27]. Hence, differential equations with fractional derivatives have become a useful tool for describing nonlinear phenomena that are involved in many branches of chemistry, engineering, biology, ecology, and numerous domains of applied sciences. Many mathematical models, including those in acoustic dissipation, mathematical epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian, porous media, fractional filters motion, and nonlocal phenomena, have proved to provide a better description of the phenomenon under investigation than models with the conventional integer-order derivative [7, 22, 26].

One of the attempts to enhance mathematical models was to introduce the concept of derivatives with fractional order. There exist in the literature a number of definitions of fractional derivatives, including Riemann–Liouville and Caputo derivatives, respectively defined as:

$$ {}_0D_x^{\alpha}\left(f(x)\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dx}\right)^n \int_0^x \left(x-t\right)^{n-\alpha-1} f\left(t\right)dt, \tag{1} $$

$n-1<\alpha\leq n$ and

$$ {}_0^CD_x^{\alpha}\left(f(x)\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_0^x \left(x-t\right)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f\left(t\right)dt, \tag{2} $$

$n-1<\alpha\leq n$. A new fractional derivative with no singular kernel was recently proposed by Caputo et al. in [9]. However, the Caputo fractional derivative [8], for instance, is the one mostly used for modeling real-world problems in the field [6, 7, 13, 14]. However, this derivative exhibits some limitations like not obeying the traditional chain rule, the chain rule representing one of the key elements of the match asymptotic method [4, 5, 19, 28]. Recall that the match asymptotic method has never been used to solve any kind of fractional differential equations because of the nature and properties of fractional derivatives. Hence, the conformable fractional derivative was proposed [1, 20]. This fractional derivative is theoretically easier to handle and obeys the chain rule, but it also exhibits a huge failure that is expressed by the fact that the fractional derivative of any differentiable function at the point zero is zero. This does not make any sense from a physical point of view and then a modified new version, the β–derivative, was proposed in order to skirt the noticed weakness. The main aim of this new derivative was, first of all, to investigate the well-known match asymptotic method [4, 5, 19, 28] in the scope of differential equations with fractional parameter and later to describe the boundary layers problems within the same scope.

Note that the β-derivative is not considered here as a fractional derivative in the same sense as the Riemann–Liouville or Caputo fractional derivative. It is the conventional derivative with a new (fractional) parameter and, as such, has been proven to have many applications in the applied sciences [4, 5] and mathematical epidemiology [3]. It is defined as:

$$ {}_0^A D_t^\beta u(t) = \begin{cases} \lim\limits_{\varepsilon\to 0} \frac{u\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-u(t)}{\varepsilon} & \text{for all } t\geq 0, \quad 0<\beta\leq 1 \\ u(t) & \text{for all } t\geq 0,\ \beta=0, \end{cases} \tag{3} $$

where u is a function such that $u:[0,\ \infty)\to\mathbb{R}$ and Γ the gamma-function

$$ \Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}dt. $$

If the above limit exists then u is said to be β-differentiable.

Note that for $\beta=1$, it obviously becomes the conventional first-order derivative so that ${}_0^A D_t^\beta u(t) = \frac{d}{dt}u(t)$. Moreover, unlike other derivatives with fractional parameters, the β-derivative of a function can be locally defined at a certain point, the same way like the first-order derivative. For a general order, let us say $m\beta$, the $m\beta$-derivative of u is defined as:

$$ {}_0^A D_t^{m\beta} u(t) = {}_0^A D_t^\beta \left({}_0^A D_t^{(m-1)\beta} u(t)\right) \quad \text{for all } t\geq 0,\ m\in\mathbb{N},\ 0<\beta\leq 1. \tag{4} $$

Note that the $m\beta$-derivative of a given function provides information about the previous $(m-1)\beta$-derivatives of the same function. For instance, we have:

$$ \begin{aligned} {}_0^A D_t^{2\beta} u(t) &= {}_0^A D_t^\beta \left({}_0^A D_t^\beta u(t)\right) \\ &= \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[(1-\beta)\left(t+\frac{1}{\Gamma(\beta)}\right)^{-\beta}u' + \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}u''\right]. \end{aligned} \tag{5} $$

This gives the β-derivative a unique property of memory that is not provided by any other derivative. It is also easy to verify that for $\beta=1$, we recover the second derivative of u. For more properties and details on this new derivative, the readers can consult the references [4, 5].

The goal of this article is to deeply investigate systems using the β-derivative and taking the form

$$ \begin{aligned} &{}_0^A D_t^\beta u(x,t) = [\mathbb{A}u(\cdot,t)](x), \quad 0<\beta\leq 1,\ \ x,\ t>0 \\ &u(x,0) = \tilde{f}(x)\ ,\ \ x>0, \end{aligned} \tag{6} $$

where $\mathbb{A}$ is a certain differential and (or) integral expression that can be evaluated at any point $x>0$ for functions u belonging to a certain subset of the domain of $\mathbb{A}$.

2. Two-parameter matrix solution operators

To proceed we can define a Banach space H endowed with the norm $\|\cdot\|_H$, express the model (6) in the form

$$ \begin{aligned} &{}_0^A D_t^\beta u(t) = Au(t), \quad 0<\beta\leq 1,\ \ t>0 \\ &u(0) = f, \end{aligned} \tag{7} $$

and define the domain

$$D(A) := \{v \in H : Av \in H\} \tag{8}$$

on which the realization operator A of the expression $\mathbb{A}$ is defined. To study (7), we can exploit the differential system

$$\begin{aligned} &{}_0^A D_t^\beta u(t) = \mu u(t), \quad 0 < \beta \le 1, \ \ t > 0, \ \mu \in \mathbb{C} \\ &u(0) = \ f_0. \end{aligned} \tag{9}$$

It is easy to check that, instead of the Mittag–Leffler function or one of its variants, the following expression, new in the literature, uniquely solves the model (9):

$$\mathcal{E}_\beta(t) = f_0 \exp\left[\mu\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]. \tag{10}$$

We note that for $\beta = 1$ the following well-known classical result holds:

$$u(t) = f_0 e^{\mu t}.$$

Remark 2.1 *If we set a certain $T_\beta = T = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$, then the expression $v(T) = f_0 e^{\mu T}$ uniquely solves*

$$\begin{aligned} &\partial_T u(T) = \mu u(T), \ \ t > 0, \ \mu \in \mathbb{C} \\ &u(0) = \ f_0, \end{aligned} \tag{11}$$

where ∂_T means a partial derivative (normal derivative) with respect to T. Hence, expression (10) uniquely solves (9) always implying that there exists a function at least in $C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ solving (11)

This remark will be very important in our analysis, with special attention to the expression of T. Next we consider the system of linear differential equations using the β-derivative with constant coefficients:

$$\begin{aligned} &{}_0^A D_t^\beta u_1 = \mu_{11} u_1 + \mu_{12} u_2 + \cdots + \mu_{1n} u_n, \\ &\vdots \\ &{}_0^A D_t^\beta u_n = \mu_{n1} u_1 + \mu_{n2} u_2 + \cdots + \mu_{nn} u_n, \end{aligned} \tag{12}$$

where $0 < \beta \le 1, \ \ t > 0, \ \mu \in \mathbb{C}$. The linearity of the operator ${}_0^A D_t^\beta$ allows us to write system (12) in the matrix form

$${}_0^A D_t^\beta U(t) = MU(t) \tag{13}$$

with U being an n-vector whose components are the unknown functions u_i and M being the $n \times n$ matrix $(\mu_{ij})_{1 \le i,j \le n}$. Let $U(0) = U_0$ be the initial condition vector for (13). We extend Peano's idea [25] by stating by analogy to solution (10) that system (13) can be solved explicitly using the formula

$$U(t) = \exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] U_0 \tag{14}$$

where the matrix exponential

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = I + \frac{T_\beta M}{1!} + \frac{T_\beta^2 M^2}{2!} + \cdots \tag{15}$$

with

$$T_\beta = T_\beta(t) = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}. \tag{16}$$

Remark 2.2 *It is easy to see that the function*

$$\begin{aligned} T_\beta : \quad & \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\ & t \longmapsto \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}, \quad 0 < \beta \le 1 \end{aligned}$$

is a topological homeomorphism from $\mathbb{R}_+$ to $\mathbb{R}_+$. Thus, the topological properties of the space $\mathbb{R}$ (endowed with a topology) are preserved when transforming t to $T_\beta(t)$

Now we consider the space $\mathfrak{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices endowed with the matrix-norm. By definition, we have

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = \sum_{k=0}^{\infty} \frac{T_\beta^k M^k}{k!} \tag{17}$$

for all $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \le 1$. It is well known and not difficult to show that the partial sums of the series (17) form a Cauchy sequence, and so the series converges.

Proposition 2.1 *For any $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \le 1$, the map*

$$\begin{aligned} \mathbb{R}_+ \quad & \longrightarrow \quad \mathfrak{M}_n(\mathbb{C}) \\ t \quad & \longmapsto \quad \exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] \end{aligned} \tag{18}$$

is continuous.

Proof The proof follows from the fact that the map $T_\beta \quad \longmapsto \quad \exp[T_\beta M]$ is continuous in T_β and completed by Remark 2.2. □

The following well-known results [15] that apply for exponential functions hold.

Proposition 2.2 *For any* $M \in \mathfrak{M}_n(\mathbb{C})$ *and* $0 < \beta \leq 1$,

$$\exp[(T_\beta + S_\beta)M] = \exp[T_\beta M] \cdot \exp[S_\beta M]$$

$$\exp[0M] = I.$$

Hence, the map $T_\beta \quad \longmapsto \quad \exp[T_\beta M]$ is a homomorphism of the additive semigroup $(\mathbb{R}_+, +)$ into a multiplicative semigroup of matrices $(\mathfrak{M}_n, \cdot)$.

Definition 2.1 *The modified time expressed by* T_β *in (16) is called the revamped time (or GA-revamped time) corresponding to* t *for the model (13).*

Remark 2.3 *Note that* $T_\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is well defined and increasing for* $0 < \beta \leq 1$ *with:*

- $T_\beta(0) = 0$
- $T_1(t) = t$
- $\frac{dT_\beta(t)}{dt} = \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta - 1}$

This means the revamped time always coincides with its corresponding time at the beginning (initial conditions) or when $\beta = 1$ *(conventional first-order derivative).*

Definition 2.2 (Two-parameter matrix solution operators) *Let us fix* $\beta \in (0, 1]$ *and* $t \in \mathbb{R}_+$. *The pair* $(\{S_\beta(t)\}_{t \geq 0,\, 0 < \beta \leq 1}; T_\beta(t))$ *where* $T_\beta(t) = \frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$ *is called the two-parameter matrix solution operator for system (13), where* $\{S_\beta(t)\}_{t \geq 0,\, 0 < \beta \leq 1}$ *is the two-parameter family such that:*

- $S_\beta(t) = G(T_\beta)$ *with* T_β *the revamped time corresponding to* t.
- $\{G(T_\beta)\}_{T_\beta \geq 0}$, *the one-parameter family defined as*

$$G(T_\beta) = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp[T_\beta M] \tag{19}$$

and representing a semigroup (in T_β *) generated by the matrix* $M \in \mathfrak{M}_n(\mathbb{C})$.

3. Strongly continuous two-parameter solution operators

With the previous definition in mind, we come back to model (7):

$$
\begin{aligned}
&{}_0^A D_t^{\beta} u(t) = Au(t), \quad 0 < \beta \leq 1, \;\; t > 0. \\
&u(0) = \, f.
\end{aligned} \tag{20}
$$

If $A : H \to H$ is a bounded linear operator, then we can exploit Definition 2.2 to solve model (20) together with the exponential series represented in (17), which is still convergent with respect to the norm in the space of bounded linear operators $\mathcal{B}(H)$. In this case, the pair $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}, T_\beta(t))$ defined in Definition 2.2 and that solves (20) is simply called the two-parameter solution operator for the system (20). More precisely, we have:

Theorem 3.1 *For system (20), every uniformly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ on a Banach space H induces a solution that is in the form (19):*

$$
u(t) = G(T_\beta) f = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right] f, \qquad f \in D(A),
$$

for some bounded linear operator A.

Proof The proof follows from the previous section and the only point to add is that if $A : H \to H$ is a bounded linear operator, then the series

$$
\sum_{k=0}^{\infty} \frac{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right)^{k} A^k}{k!}
$$

converges in the used norm for every $t > 0$. □

However, the reality is sometime complex and as mentioned in the introduction, the operator A is, in most of the cases, unbounded. Simple examples are differential operators that are not bounded on the whole space H. Then multiple iterates of operator A appearing in series (17) make it impossible to use the series to solve (20). The main reason is that the common domain of those iterates of A could be reduced to the null subspace $\{0\}$. Then, more considerations, in addition to what was developed in the previous section, are necessary.

Definition 3.1 (Strongly continuous two-parameter solution operators) *Let us fix $\beta \in (0, 1]$ and $t \in \mathbb{R}_+$. The pair $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ is said to be a strongly continuous two-parameter solution operator for system (20) if the two-parameter family $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$ is such that:*

- *$S_\beta(t) = G(T_\beta)$ with T_β the revamped time corresponding to t.*
- *$\{G_A(T_\beta)\}_{T_\beta \geq 0}$ is a strongly continuous semigroup (in T_β) generated by the operator A, that is:*

(i) $\quad G_A(0) = I$;

(ii) $G_A(T_\beta + S_\beta) = G_A(T_\beta)G_A(S_\beta)$ *for all* $T_\beta, S_\beta \geq 0$*;*

(iii) $\lim_{T_\beta \to 0^+} G_A(T_\beta)f = f$ *for any* $f \in H$*.*

Remark 3.1 *Note that:*

(a) For $\beta = 1$, $T_\beta(t) = t$ *and the definition here above coincides with the definition of the classical well-known (one-parameter)* C_0*-semigroup.*

(b) If $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (20) generated by* A*, then*

$$Af = \lim_{t\to 0} \frac{S_\beta(t)f - f}{t} = \lim_{T_\beta \to 0} \frac{G_A(T_\beta)f - f}{T_\beta}, \tag{21}$$

where the domain of A*,* $D(A)$*, is chosen to be defined as the set of all* $f \in H$ *for which this limit exists. The latter equality is due to the above Definition 3.1 and the fact that* $T_\beta(t) \to 0$ *as* $t \to 0$*.*

(c) –If $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (20) generated by* A*, then for* $f \in D(A)$ *the function* $t \longrightarrow S_\beta(t)f = G_A(T_\beta)f$ *is a classical solution of the fractional Cauchy problem (20).*
–For $f \in H \setminus D(A)$*, however, the function* $u(t) = S_\beta(t)f$ *is continuous but, in general, not differentiable, nor* $D(A)$*-valued, and therefore not a classical solution.*

(d) The strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$ *thanks to properties (ii) and (iii) here above and the Banach–Steinhaus theorem, which shows that any* C_0*-semigroup like* $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$*.*

(e) If $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (20) generated by* A*, then for* $f \in D(A)$ *the function* $T_\beta \longrightarrow G_A(T_\beta)f$ *a classical solution of*

$$\begin{aligned} \partial_t u(t) &= Au(t), \quad t > 0. \\ u(0) &= f. \end{aligned} \tag{22}$$

More precisely, we have the following statement:

Proposition 3.2 *Let* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *be a strongly continuous two-parameter solution operator for the system (20) generated by* $(A, D(A))$*. Then* $t \to S_\beta(t)f = G_A(T_\beta)f, \quad f \in D(A)$*, is the only solution of (20) taking values in* $D(A)$*.*

Proof To prove it we set $u(t) = v(T_\beta) \in D(A)$ for all $t > 0$, where $T_\beta = T_\beta(t)$ is the revamped time corresponding to t, $v \in C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ and ${}_0^A D_t^\beta u(t) = Au(t), \quad t > 0$. Then, by Definition (3.1), $v(T_\beta)$ satisfies $\partial_t u(t) = Au(t), \ t > 0$. Let us define the function

$$\begin{array}{rccc} z: & (0, T_\beta) & \longrightarrow & H \\ & S_\beta & \longmapsto & G_A(T_\beta - S_\beta)v(S_\beta) \end{array}$$

and make use of the well known property of semigroups [15]:

$$\partial_{T_\beta} G_A(T_\beta)v(T_\beta) = AG_A(T_\beta)v(T_\beta) = G_A(T_\beta)Av(T_\beta),$$

to state that z is differentiable and

$$0 = \partial_{S_\beta} z(S_\beta) = G_A(T_\beta - S_\beta)(\partial_{S_\beta} v(S_\beta) - (Av)(S_\beta)). \tag{23}$$

Thus, z is constant on $(0, T_\beta)$, meaning that for any $\varepsilon, \eta \in (0, T_\beta)$ we have

$$G_A(T_\beta - \varepsilon)v(\varepsilon) = G_A(T_\beta - \eta)v(\eta)$$

which tends to

$$G_A(T_\beta)v(0) = v(T_\beta),$$

as ε tends to 0 and η tends to T_β. This proves that v is defined by the semigroup $\{G_A(T_\beta)\}_{T_\beta \geq 0} = \{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$. Hence, by Definition (3.1), u is also defined by the strongly continuous two-parameter solution operator $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$, which concludes the proof. □

It is now clear that for $f \in D(A)$,

$$D_t^\beta S_\beta(t)f = \frac{d}{dT_\beta} G_A(T_\beta)f.$$

Hence, making use of the well-known properties of strongly continuous semigroups, we have the following corollary:

Corollary 3.3 *Let $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ be a strongly continuous two-parameter solution operator for the system (5) generated by $(A, D(A))$. Then, for $f \in D(A)$, $S_\beta(t)f \in D(A)$ and*

$$D_t^\beta S_\beta(t)f = AS_\beta(t)f = S_\beta(t)Af \tag{24}$$

for all $t \geq 0$.

Definition 3.2 ***(Two-parameter solution operators β-exponentially bounded)***

- *The strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ for the system (20) is said to be β-exponentially bounded if there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S_\beta(t)\|_H \leq M \exp\left[\omega\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]. \tag{25}$$

- *If system (20) admits a strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ satisfying (25), then we say that the operator $A \in \mathcal{G}^\beta(M, \omega)$.*

- *$(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ is said to be contractive if*

$$\|S_\beta(t)\|_H \leq 1, \tag{26}$$

and we say $A \in \mathcal{G}^\beta(1, 0)$.

- *As in [27], we say that problem (20) is well-posed if it admits a strongly continuous two-parameter solution operator.*

Let us set

$$\mathcal{G}^{\beta}(\omega) := \bigcup\{\mathcal{G}^{\beta}(M,\omega), M \geq 1\},$$

$$\mathcal{G}^{\beta} := \bigcup\{\mathcal{G}^{\beta}(\omega), \omega \geq 0\}$$

and denote by

$$\mathcal{B}(H) := \mathcal{B}(H;H)$$

the space of all bounded linear operators from H to H.

Remark 3.2 *Condition (25) holds if and only if the one-parameter family $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ given in Definition (3.1) satisfies*

$$\|G_A(T_\beta)\|_H \leq M e^{\omega T_\beta}. \tag{27}$$

Corollary 3.4 *Problem (20) is well-posed if $A \in \mathcal{B}(H)$.*

Proof This is a direct consequence of Theorem 3.1 and Proposition 3.2. □

Next let us recall the following definition:

Definition 3.3 *The set $\rho(A)$ is called the resolvent set of operator A and is defined as:*

$$\rho(A) = \{\lambda \in \mathbb{R}; \quad \lambda I - A : D(A) \to X \text{ is invertible and } \quad (\lambda I - A)^{-1} \in \mathcal{B}(H)\}. \tag{28}$$

Then, for $\lambda \in \rho(A)$, the inverse $R(\lambda, A) := (\lambda I - A)^{-1}$ is, by the closed graph theorem, a bounded operator on H and is termed as the resolvent of A at point λ.

Proposition 3.5 *If the strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ for system (20) is β-exponentially bounded in terms of Definition 3.2, then $S_\beta(t)$ is related to its resolvent by the formula*

$$R(\lambda, A)f = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \exp\left[-\lambda\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] S_\beta(t) f dt, \tag{29}$$

for $f \in H$ and $Re\lambda > \omega$.

Proof The proof follows from Definition 3.1 where $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ is a strongly continuous semigroup with operator A as an infinitesimal generator and satisfying (27). Then, from the semigroup theory, we have that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda T_\beta} G_A(T_\beta) dT_\beta.$$

Substituting the revamped time T_β and using Remark 2.3 leads to the formula. □

We can therefore propose the following diagram for system (20) presenting the relations between the two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$, its generator, and its resolvent.

$$\begin{tikzcd}
& (\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t)) \arrow[dl, "R(\lambda, A)f = \int_0^\infty \left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} \exp\left[-\lambda\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{\beta}\right)\right] S_\beta(t) f dt"'] \arrow[dr, "Af = \lim_{t\to 0}\frac{S_\beta(t)f-f}{t}"] & \\
(R(\lambda, A))_{\lambda\in\rho(A)} & & (A, D(A)) \arrow[ll, "R(\lambda, A) = (\lambda I - A)^{-1}"']
\end{tikzcd}$$

4. Exponential approximation and application

For dynamical systems (20) with unbounded operators A, analysis can be done by using the following exponential approximation:

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{\beta}\right)A\right]f = \lim_{p\to\infty}\left[I-\frac{1}{p}\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{\beta}\right)A\right]^{-p} f. \tag{30}$$

If the above limit exists, then it defines a strongly continuous two-parameter solution operator as given in Definition (3.1). Conditions of the existence of the limit (30) are given by making use of the Hille–Yosida theorem (see [15, Chap. 2, Section 3]) in the theory of semigroups and completed by Remark 3.2. Then we have the following theorem that applies to model (20) with the fractional parameter β:

Theorem 4.1 *$A \in \mathcal{G}^\beta(M, \omega)$ if and only if: (a) A is closed and densely defined, (b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda-\omega)^n}, \tag{31}$$

where $\rho(A)$ is the resolvent set of the operator A as defined above.

Proposition 4.2 *Let $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ be the a strongly continuous two-parameter solution operator for system (20) generated by A. Then*

$$S_\beta(t)f = \lim_{p\to\infty}\left[I-\frac{1}{p}\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{\beta}\right)A\right]^{-p} f, \quad \textit{for } f \in H,$$

and the limit is uniform in t on any bounded interval.

Proof Considering the revamped time corresponding to t, $T_\beta = T_\beta(t)$, we have by definition $S_\beta(t)f = G_A(T_\beta)f$. Since the one-parameter family $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ is a C_0-semigroup generated by A, we make use of [15, Corollary III 5.5] to write

$$G_A(T_\beta)f = \lim_{p\to\infty}\left(I - \frac{T_\beta}{p}A\right)^{-p} f, \quad \text{for } f \in H$$

and the proposition is proved. □

As an application, we can approximate the solution for system (20) by considering the alternate model given by

$$\frac{u_p\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right] - u_p\left[(k-1)\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right]}{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)} = Au_p\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)\right] \tag{32}$$

$$u_p(0) = f$$

for $0 < \beta \leq 1$, $t > 0$. The explicit solution of problem (32) is given by

$$u_p(t) = \left[I - \left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{p\beta}\right)A\right]^{-p} f$$

, which represents an approximation of the solution for model (20). Making use of Proposition 4.2, we see that $\lim_{p\to\infty} u_p(t) = S_\beta(t)f$. Hence, the difference system (32) is very important in solving the model (20) since their solutions converge to the solution of (20), and from Proposition 3.2, this solution $S_\beta(t)f$ is unique if f is taken from $D(A)$.

5. Subordination and prolongation principles for evolution equations with β-derivatives

In this section, we address the issue of the subordination principle for evolution equations with fractional parameters. This principle has been proved only for models with Caputo fractional derivatives [6, 27] and the opposite principle has been proved not to be true. Hence, we go farther by also addressing the opposite principle, named here the prolongation principle. Recall that these principles study the existence of two-parameter solution operators for problems (5) with different values of derivative orders. We note that if we have a strongly continuous semigroup $\{G_A(T)\}_{T\geq 0}$ generated by operator A, we can always identify the Cauchy problem for which it is a solution. This yields the following lemma:

Lemma 5.1 *Consider model (5) and T_β the GA-revamped time corresponding to t. If there is a strongly continuous semigroup (in T_β), say $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ generated by the operator A, then the family $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$, such that $S_\beta(t) = G(T_\beta)$, is a strongly continuous two-parameter solution operator for the system (5).*

Theorem 5.2 *Consider the models (5) with two different orders β and δ such that $0 < \delta < \beta \leq 1$. Let $\omega \geq 0$; then $A \in \mathcal{G}^\beta(\omega)$ if and only if $A \in \mathcal{G}^\delta(\omega)$.*

Proof Suppose $A \in \mathcal{G}^{\beta}(\omega)$; then (5) admits a strongly continuous two-parameter solution operator $(\{S_{\beta}(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_{\beta}(t))$ satisfying (25). Hence, by definition we have $\{S_{\beta}(t)\}_{t\geq 0,\,0<\beta\leq 1} = \{G_A(T_{\beta})\}_{T_{\beta}\geq 0}$ where T_{β} is GA-revamped time $\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta}-\Gamma(\beta)^{-\beta}}{\beta}$, corresponding to t, and $\{G_A(T_{\beta})\}_{T_{\beta}\geq 0}$ is a strongly continuous semigroup (in T_{β}) generated by the operator A. Moreover, by Remark 3.2, we have $G_A(T_{\beta})$ satisfying (27). For $0<\delta<\beta\leq 1$, let us define $T_{\delta}=T_{\delta}(t)=\frac{\left(t+\frac{1}{\Gamma(\delta)}\right)^{\delta}-\Gamma(\delta)^{-\delta}}{\delta}$, the GA-revamped time (of order δ) corresponding to t, and then $\{G_A(T_{\delta})\}_{T_{\delta}\geq 0}$ is also a strongly continuous semigroup (in T_{δ}) generated by the operator A since $\{G_A(T_{\beta})\}_{T_{\beta}\geq 0}$ is. Moreover, by (27) we have

$$\|G_A(T_{\delta})\|_{\mathcal{X}} \leq M e^{\omega T_{\delta}}, \tag{33}$$

and Lemma 5.1 concludes the first part of the proof, showing the subordination principle for the model (5). Conversely, to prove the prolongation principle, we suppose $A \in \mathcal{G}^{\delta}(\omega)$ and the rest of the proof follows the same steps as above. □

The following corollary appears as an immediate consequence.

Corollary 5.3 *Consider any $\beta \in (0,1)$. Then there are constants $\omega \geq 0$ and $M \geq 1$ such that the operator A in model (5) is the infinitesimal generator of a C_0-semigroup $G(t)$ satisfying $\|G(t)\| \leq Me^{\omega t}$, $t\geq 0$ if and only if $A \in \mathcal{G}^{\beta}(M,\omega)$ with the corresponding two-parameter solution operator $(\{S_{\beta}(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_{\beta}(t))$ satisfying (25).*

6. Applications to break-up dynamics in transport-convection processes

6.1. Mathematical settings and model analysis

In this section we address the well-posedness of the model

$${}_0^A D_t^{\beta} p(t,x,n) = -div(\omega(x,n)p(t,x,n)) - a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m), \tag{34}$$

where $t>0$, $0<\beta\leq 1$, $x\in\mathbb{R}^3$, $n=1,2,3,...$ and subject to initial conditions

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n=1,2,3,... \tag{35}$$

by using the concepts defined here above and setting other suitable conditions. Equation (34) modelizes the break-up dynamics of moving groups. In terms of the mass size m and the position x, the state of the system is characterized at any moment t by the particle-mass-position distribution $p=p(t,x,m)$ (p is also called the *density* or *concentration* of particles), with $p:\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{R}_+$, and the velocity $\omega=\omega(x,m)$ of the transport is supposed to be a known quantity depending on m and x. The average fragmentation rate a_n is the average number at which clusters of size n undergo splitting, and $b_{n,m}\geq 0$ is the average number of n-groups produced upon the splitting of m-groups. The space variable x is supposed to vary in the whole of $\mathbb{R}^3=\Omega$. The function $\overset{o}{p}_n$ represents the density of n-groups at the beginning of observation ($t=0$) and it is integrable with respect to x over the full space $\mathbb{R}^3$. The necessary assumptions that will be useful in the analysis are introduced in the following sections.

6.2. Well-posedness for the break-up part of the model

Since a group of size $m \leq n$ cannot split to form a group of size n, we require $b_{n,m} = 0$ for all $m \leq n$ and

$$a_1 = 0, \quad \sum_{m=1}^{n-1} m b_{m,n} = n, \quad (n = 2, 3, ...), \tag{36}$$

meaning that a cluster of size one cannot split and the sum of all individuals obtained by break-up of an n-group is equal to n. Because the total number of individuals in a population is not modified by interactions among groups and the mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$\mathcal{X}_1 := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n|g_n(x)|dx < \infty \right\}. \tag{37}$$

We work in this space because it has many desirable properties, like controlling the norm of its elements, which, in our case, represents the total mass (or total number of individuals) of the system and must be finite. Because the uniqueness of solutions to the systems of type (34)–(35) is proved to be a more difficult problem [12, 23], we restrict our analysis to a smaller class of functions, so we introduce the following class of Banach spaces (of distributions with finite higher moments):

$$\mathcal{X}_r := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r|g_n(x)|dx < \infty \right\}, \tag{38}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x,n) \longrightarrow p(t,x,n) = p_n(t,x)$ is such that $\mathbf{p} = (p_n(t,x))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$ we can rewrite (34)–(35) in more compact form:

$$\begin{aligned} {}_0^A D_t^{\beta} \mathbf{p} &:= \mathbf{D}\mathbf{p} + \mathbf{F}\mathbf{p}, \\ \mathbf{p}_{|_{t=0}} &= \mathring{\mathbf{p}}, \end{aligned} \tag{39}$$

where $t > 0$, $0 < \beta \leq 1$, $x \in \mathbb{R}^3$, $n = 1, 2, 3, ...$ Here $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathbf{D}$ the transport expression defined as

$$(p(t,x,n))_{n\in\mathbb{N}} \longrightarrow (-\mathrm{div}(\omega(x,n)p(t,x,n)))_{n=1}^{\infty}, \tag{40}$$

$\mathring{\mathbf{p}}$ the initial vector $(\mathring{p}_n(x))_{n\in\mathbb{N}}$ that belongs to $\mathcal{X}_r$, and $\mathbf{F}$ the fragmentation expression defined by

$$(\mathbf{F}\mathbf{p})_{n=1}^{\infty} := \left(-a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m) \right)_{n=1}^{\infty}.$$

In this work, for any subspace $S \subseteq \mathcal{X}_r$, we will denote by S_+ the subset of S defined as $S_+ = \{\mathbf{g} = (g_n)_{n=1}^{\infty} \in S; g_n(x) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{g} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{g}) := \sum_{n=1}^{\infty} n^q g_n$$

of all orders $q \in [0, r]$. In $\mathcal{X}_r$, we define the operators $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{Ag} := (a_n g_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{g} \in \mathcal{X}_r : \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |g_n(x)| dx < \infty\}; \tag{41}$$

$$\mathbf{Bg} := (B_n g_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m g_m \right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{42}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (36). Now let us prove that $\mathbf{B}$ is well defined on $D(\mathbf{A})$ as stated in (42). Making use of condition (36), we have

$$n^r - \sum_{m=1}^{n-1} m^r b_{m,n} \geq n^r - (n-1)^{r-1} \sum_{m=1}^{n-1} m b_{m,n} = n^r - n(n-1)^{r-1} \geq 0.$$

Hence,

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{43}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. For every $\mathbf{g} \in D(\mathbf{A})$, we have then

$$\begin{aligned}
\|\mathbf{Bg}\|_r &= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |g_m(x)| \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{\infty} n^r b_{n,m} \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{m-1} n^r b_{n,m} \right) dx \\
&\leq \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| m^r dx \\
&= \|\mathbf{Ag}\|_r \\
&< \infty,
\end{aligned}$$

where we have used inequality (43). Then $\|\mathbf{Bg}\|_r \leq \|\mathbf{Ag}\|_r$, for all $\mathbf{g} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A} + \mathbf{B}, D(\mathbf{A}))$ is well defined.

6.3. Well-posedness for the transport part of the model

Our primary objective in this section is to analyze the solvability of the Cauchy problem for the transport equation

$$ {}_0^A D_t^{\beta} p(t, x, n) = -div(\omega(x, n)\, p(t, x, n)), \tag{44}$$

$$p(0, x, n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ...$$

in the space $\mathcal{X}_r$, where $t>0,\ 0<\beta\leq 1,\ x\in\mathbb{R}^3,\ \ n=1,2,3,...$
To do so we need the following:

Let us fix $n\in\mathbb{N}$. We consider the function $\omega_n:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ defined by $\omega_n(x)=\omega(x,n)$ and $\tilde{\mathcal{D}}_n$ the expression appearing on the right-hand side of the equation (44). Then

$$\begin{aligned}\tilde{\mathcal{D}}_n[p(t,x,n)] &:= -div\,(\omega(x,n)\,p(t,x,n))\\ &= (\nabla\cdot\omega(x,n))p(t,x,n)+\omega(x,n)\cdot(\nabla p(t,x,n)).\end{aligned}\tag{45}$$

We assume that ω_n is divergence-free and globally Lipschitz continuous. Then
$div\,\omega_n(x):=\nabla\cdot\omega(x,n)=0$ and (45) becomes

$$\tilde{\mathcal{D}}_n[p(t,x,n)]:=\omega(x,n)\cdot(\nabla p(t,x,n)).\tag{46}$$

We note that the operators on the right-hand side of (39) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus, we need to work with parameter-dependent operators that can be "glued"together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter [12, 23]. Let us consider the space $\mathcal{X}:=L_p(S,X)$ where $1\leq p<\infty$, (S,m) is a measure space, and X is a Banach space. Let us suppose that we are given a family of operators $\{(A_s,D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A},D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae:

$$\mathcal{D}(\mathbb{A}):=\{g\in\mathcal{X};g(s)\in D(A_s)\text{ for almost every }s\in S,\ \ \mathbb{A}g\in\mathcal{X}\},\tag{47}$$

and, for $g\in\mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}g)(s):=A_sg(s),\tag{48}$$

for every $s\in S$.

We set

$$X_x:=L_1(\mathbb{R}^3,dx):=\{\psi:\|\psi\|=\int\limits_{\mathbb{R}^3}|\psi(x)|dx<\infty\}$$

and define in X_x the operators $(\mathcal{D}_n,D(\mathcal{D}_n))$ as

$$\begin{aligned}\mathcal{D}_np_n=\tilde{\mathcal{D}}_np_n,&\quad\text{with }\tilde{\mathcal{D}}_np_n\ \text{ represented by (46)}\\ D(\mathcal{D}_n):=\{p_n\in X_x,\ \ &\mathcal{D}_np_n\in X_x\},\quad n\in\mathbb{N}.\end{aligned}\tag{49}$$

Then, in $\mathcal{X}_r$, we can define for the operator $\mathbf{D}$ (40) the domain

$$D(\mathbf{D})=\{\mathbf{p}=(p_n)_{n\in\mathbb{N}}\in\mathcal{X}_r,\,p_n\in D(\mathcal{D}_n)\,\text{for almost every}\,n\in\mathbb{N},\,\mathbf{Dp}\in\mathcal{X}_r\}.\tag{50}$$

Theorem 6.1 *Let us fix any $\beta\in(0,1]$. If for each $n\in\mathbb{N}$ the function $\omega_n:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ is globally Lipschitz continuous and divergence-free, then the operator $(D(\boldsymbol{D}),\boldsymbol{D})$ is the generator of a contractive strongly continuous two-parameter solution operator for the system (44).*

Proof To prove it we apply the subordination principle of Theorem 5.2, by considering the model (44) with $\beta = 1$ to have the compact form

$$\partial_t \mathbf{P} = \mathbf{DP}, \tag{51}$$

subject to the initial condition

$$\mathbf{p}|_{t=0} = \overset{\circ}{\mathbf{p}}, \tag{52}$$

where $\mathbf{D}$ is the transport expression defined in (40). Making use of [23, Theorem 2] or [12, Theorem 3.4.2], it is proved that if the conditions of Theorem 6.1 are satisfied then there exists a strongly continuous stochastic (positive and contractive) semigroup generated by $(D(\mathbf{D}), \mathbf{D})$. Hence, $\mathbf{D} \in \mathcal{G}^1(1,0)$ and exploiting the subordination principle of Theorem 5.2, we have $\mathbf{D} \in \mathcal{G}^\beta(1,0)$, which proves the theorem. □

6.4. Existence results for the full model

Attention is now shifted to the transport problem with the loss part of the break-up process. We assume that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1 \alpha_n \leq a_n(x) \leq \theta_2 \alpha_n, \tag{53}$$

with $\alpha_n \in \mathbb{R}_+$ and independent of the state variable x. Then a_n is bounded for each $n \in \mathbb{N}$ and the loss operator $(A_n, D(A_n))$ can be defined in X_x as $A_n(x) = a_n(x)$ with $D(A_n) = X_x = L_1(\mathbb{R}^3)$. The corresponding abstract Cauchy problem for the full model (34)–(35) reads as:

$$\begin{aligned} {}_0^A D_t^\beta \mathbf{p} &= \mathbf{Dp} + \mathbf{Fp} \\ \mathbf{p}_{|_{t=0}} &= \overset{\circ}{\mathbf{p}}. \end{aligned} \tag{54}$$

The following theorem holds.

Theorem 6.2 *Assume that (53) is satisfied for each $n \in \mathbb{N}$.*
There is an extension $(\mathcal{K}, D(\mathcal{K}))$ of $(\boldsymbol{D} + \boldsymbol{F}, D(\boldsymbol{D}) \cap D(\boldsymbol{A}))$ that generates, on $\mathcal{X}_r$, a strongly continuous two-parameter solution operator for the system (34)–(35), which is contractive.

Proof The proof follows the same steps as the proof of Theorem 6.1 where we apply the subordination principle on reference [23, Theorem 5] or [12, Theorem 3.5.2]. □

7. Concluding remarks

We have presented a concise analysis of new linear evolution equations containing the β-derivative, a new derivative recently developed in order to extend the traditional match asymptotic method to the scope of the fractional differential equation and describe the boundary layers problems within the framework of fractional calculus. In the process, we have extended Peano's idea and used concepts like revamped time, two-parameter solution operators, subordination, and prolongation principles to address the problem of well-posedness for the model and provide a method to approximate the generalized unique solution to the model. As an application, the well-posedness of an integrodifferential equation modeling convection and break-up processes has been analyzed. It is certain that this work will inspire more than one author with the introduction of a new derivative and thus emerges as a breakthrough that might help in solving the open problems mentioned here above or lead to more complex analysis of evolutions equations often describing phenomena more and more intricate.

Acknowledgment

The author would like to thank Dr Abdon Atangana for his valuable comments, support, and guidance for the accomplishment of this work.

References

[1] Abu Hammad M, Khalil R. Conformable fractional heat differential equation. Int J Pure Appl Math 2014; 94: 215-221.

[2] Atangana A. Drawdown in prolate spheroidal-spherical coordinates obtained via Green's function and perturbation methods. Commun Nonlinear Sci 2014; 19: 1259-1269.

[3] Atangana A. A novel model for the Lassa hemorrhagic fever: deathly disease for pregnant women. Neural Comput Appl 2015; 26: 1895-1903.

[4] Atangana A. Derivative with a New Parameter: Theory, Methods and Applications. New York, NY, USA: Academic Press, 2015.

[5] Atangana A, Doungmo Goufo EF. Extension of match asymptotic method to fractional boundary layers problems. Math Probl Eng 2014; 2014: 107535.

[6] Bazhlekova EG. Subordination principle for fractional evolution equations. Fract Calc Appl Anal 2000; 3: 213-230.

[7] Brockmann D, Hufnagel L. Front propagation in reaction-superdiffusion dynamics: taming Lévy flights with fluctuations. Phys Review Lett 2007; 98: 178301.

[8] Caputo M. Linear models of dissipation whose Q is almost frequency independent II. Geophys J R Ast Soc 1967; 13: 529539.

[9] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Progr Fract Differ Appl 2015; 1: 1-13.

[10] Demirci E, Unal A, Özalp N. A fractional order SEIR model with density dependent death rate. Hacet J Math Stat 2011; 40: 287-295.

[11] Diethelm K. The Analysis of Fractional Differential Equations. Berlin, Germany: Springer, 2010.

[12] Doungmo Goufo EF. Non-local and non-autonomous fragmentation-coagulation processes in moving media. PhD, North-West University, Potchefstroom, South Africa, 2014.

[13] Doungmo Goufo EF. A mathematical analysis of fractional fragmentation dynamics with growth. J Funct Space 2014; Article ID 201520: 1-7.

[14] Doungmo Goufo EF. A biomathematical view on the fractional dynamics of cellulose degradation. Fract Calc Appl Anal 2015; 18: 554-564.

[15] Engel KJ, Nagel R. One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics (Book 194). New York, NY, USA: Springer-Verlag, 2000.

[16] Hantush MS. Analysis of data from pumping tests in leaky aquifers. T Am Geo Union 1956; 37: 702-714.

[17] Hantush MS, Jacob CE. Non-steady radial flow in an infinite leaky aquifer. T Am Geo Union 1955; 36: 95-100.

[18] Hilfer R. Application of Fractional Calculus in Physics. Singapore: World Scientific, 1999.

[19] Kestin J, Persen LN. The transfer of heat across a turbulent boundary layer at very high Prandtl numbers. Int J Heat Mass Transfer 1962; 5: 355-371.

[20] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math 2014; 264: 65-70.

[21] Khan Y, Sayevand K, Fardi M, Ghasemi M. A novel computing multi-parametric homotopy approach for system of linear and nonlinear Fredholm integral equations. Appl Math Comput 2014; 249: 229-236.

[22] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam, the Netherlands: Elsevier, 2006.

[23] Oukouomi Noutchie SC, Doungmo Goufo EF. Global solvability of a continuous model for nonlocal fragmentation dynamics in a moving medium. Math Probl Eng 2013; 2013: 320750.

[24] Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations. Berlin, Germany: Springer-Verlag, 1983.

[25] Peano G. Intégration par séries des équations différentielles linéaires. Math Ann 1888; 32: 450-456 (in French).

[26] Pooseh S, Rodrigues HS, Torres DFM. Fractional derivatives in dengue epidemics. In: Simos TE, Psihoyios G, Tsitouras C, Anastassi Z, editors. AIP Conference Proceedings, ICNAAM. Melville, NY, USA: American Institute of Physics, 2011, pp. 739-742.

[27] Prüss J. Evolutionary Integral Equations and Applications, Basel, Switzerland: Birkhäuser, 1993.

[28] Schlichting H. Boundary-Layer Theory. 7th ed. New York, NY, USA: McGraw-Hill, 1979.

[29] Ziff RM, McGrady ED. Shattering transition in fragmentation. Phys Rev Lett 1987: 58: 892-895.