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Numerical method for solving linear stochastic Itô–Volterra integral equations driven by fractional Brownian motion using hat functions

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Abstract: In this paper, we present a numerical method to approximate the solution of linear stochastic Itô–Volterra integral equations driven by fractional Brownian motion with Hurst parameter $H \in (0,1)$ based on a stochastic operational matrix of integration for generalized hat basis functions. We obtain a linear system of algebraic equations with a lower triangular coefficients matrix from the linear stochastic integral equation, and by solving it we get an approximation solution with accuracy of order $O(h^2)$. This numerical method shows that results are more accurate than the block pulse functions method where the rate of convergence is $O(h)$. Finally, we investigate error analysis and with some examples indicate the efficiency of the method.

Key words: Brownian and fractional Brownian motion process, linear stochastic integral equation, hat functions

1. Introduction

Recently there has been an increasing demand for numerical methods to solve stochastic differential and stochastic integral equations. Stochastic Itô–Volterra integral equations appear in models of various problems in science and engineering events and so on. For many of them there is no exact solution, so numerical computation and analysis will become important. As an example, in [10], Heydari et al. used hat functions for solving stochastic Itô–Volterra integral equations, and others have tried to solve them either numerically or theoretically [5,6,11,12,13,17,19].

For stochastic differential and integral equations caused by fractional Brownian motion, there exist several ways to solve them: path-wise and related techniques, Dirichlet forms, Euler approximations, Malliavin calculus, and the Skorokhod integral, but almost all methods have very poor numerical convergence [3,9,14,16,18]. It is important to find approximation solutions for them, because they cannot be solved analytically in most cases and have many applications in models of physics problems, telecommunication networks, and finance [4]. Ezzati et al. used block pulse functions for solving stochastic differential equations with Hurst parameter $H \in (\frac{1}{2}, 1)$ [8].

In this paper we consider the following linear stochastic Itô–Volterra integral equation, which has been caused by a fractional Brownian motion:

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$$X(t) = f(t) + \int_0^t K_1(s,t)X(s)ds + \int_0^t K_2(s,t)X(s)dB^{(H)}(s), \; t \in [0,T], \tag{1}$$

where $X(t)$, $f(t)$, $K_1(s,t)$, and $K_2(s,t)$, for $t, s \in [0,T]$, are stochastic processes defined on the same probability space (Ω, F, P); $X(t)$ is an unknown function; and $B^{(H)}(t)$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. We try to solve it numerically by using hat functions, which are more accurate and efficient than block pulse basis functions where the rate of convergence is $O(h)$ [8].

In order to compute the approximation solution of this equation, we first define some properties of hat functions, and then we get the operational matrix of stochastic integration driven by fractional Brownian motion and get a linear system of algebraic equations with a lower triangular coefficients matrix. Finally the convergence and error analysis of the suggested method are given, along with some examples that show the efficiency of this method.

2. Fractional Brownian motion and its properties

2.1. Fractional Brownian motion

A standard fractional Brownian motion $(B^{(H)}(t))_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a continuous Gaussian process with zero mean and a covariance function:

$$Cov(B^{(H)}(s), B^{(H)}(t)) = \frac{1}{2}(s^{2H} + t^{2H} - \mid t-s \mid^{2H}).$$

Fractional Brownian motions have the following properties:

(a) $B^{(H)}(0) = 0$ and $E(B^{(H)}(t)) = 0$ for all $t \geq 0$.

(b) $B^{(H)}$ has homogeneous increments.

(c) $E(B^{(H)}(t)^2) = t^{2H}, t \geq 0$.

(d) $B^{(H)}$ has continuous trajectories.

If $H = 1/2$, we get to standard Brownian motion [4].

2.2. Fractional Itô formula

Let $H \in (0,1)$. Assume that $f(s,x) : R \times R \to R$ belongs to $C^{1,2}(R \times R)$, and assume that the random variables

$$f(t, B^{(H)}(t)), \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds, \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds,$$

all belong to $L^2(\Omega)$. Then:

$$f(t, B^{(H)}(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^{(H)}(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, B^{(H)}(s))dB^H(s) + H\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^{(H)}(s))s^{2H-1}ds. \tag{2}$$

For more details see [4].

3. Hat functions and their properties

[1,2,7,15] The family of the first $(n+1)$ hat functions on [0, T] is defined as follows:

$$\phi_0(t) = \begin{cases} \frac{h-t}{h} & 0 \le t \le h, \\ 0 & \text{otherwise} , \end{cases}$$

$$\phi_i(t) = \begin{cases} \frac{t-(i-1)h}{h} & (i-1)h \le t \le ih, \\ \frac{(i+1)h-t}{h} & ih \le t \le (i+1)h, \\ 0 & otherwise, \end{cases}$$

for which i=1,2,...,n-1 and $h = \frac{T}{n}$. We also have:

$$\phi_n(t) = \begin{cases} \frac{t-(T-h)}{h} & T-h \le t \le T, \\ 0 & otherwise. \end{cases}$$

From the above definitions, we have:

$$\phi_i(jh) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{3}$$

and

$$\phi_i(t)\phi_j(t) = 0, \mid i-j \mid \geq 2. \tag{4}$$

An arbitrary function $f(t) \in L^2[0,T]$ can be expanded by the hat basis functions as:

$$f(t) \simeq \sum_{i=0}^{n} f_i \phi_i(t) = F^T \Phi(t) = \Phi(t)^T F, \tag{5}$$

where

$$F = [f_0, f_1, ..., f_n]^T, \tag{6}$$

and

$$\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_n(t)]^T. \tag{7}$$

The coefficients f_i in (5) are given by:

$$f_i = f(ih), i = 0, 1, ..., n. \tag{8}$$

For an arbitrary function of two variables $k(x,y) \in L^2([0,T] \times [0,T])$, we have the following approximation by the hat basis functions:

$$k(s,t) = \Phi(s)^T \Lambda \Psi(t), \tag{9}$$

in which $\Phi(s)$ and $\Psi(t)$ are $(n+1)$-dimensional generalized hat function vectors and Λ is the $(n+1) \times (n+1)$ generalized hat functions coefficients matrix with entries $a_{ij}, i = 0, ..., n, j = 0, ..., n,$ as follows:

$$a_{ij} = k(ih, jh).$$

From relation (4), we have:

$$\Phi(t)\Phi(t)^T = \begin{pmatrix} \phi_0^2(t) & \phi_0(t)\phi_1(t) & & & & \\ \phi_0(t)\phi_1(t) & \phi_1^2(t) & \phi_1(t)\phi_2(t) & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \phi_{n-1}(t)\phi_n(t) \\ & & & & \phi_{n-1}(t)\phi_n(t) & \phi_n^2(t) \end{pmatrix}.$$

According to (3) and expanding entries of $\Phi(t)\Phi(t)^T$ by the hat functions, we have:

$$\Phi(t)\Phi(t)^T \simeq \begin{pmatrix} \phi_0(t) & 0 & \cdots & 0 \\ 0 & \phi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(t) \end{pmatrix}.$$

Integration of vector $\Phi(t)$ defined in (7) can be expressed as [20]:

$$\int_0^t \Phi(s)ds \simeq \ P\Phi(t), t \in [0,T], \tag{10}$$

where P is an $(n+1)\times(n+1)$ operational matrix for integration and is given by:

$$P = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 0 & 1 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

4. Stochastic operational matrix

Theorem 1 [10] *The Itô integral of $\Phi(t)$, which is given by (7), yields:*

$$\int_0^t \Phi(s)dB(s) \simeq \ P_s\Phi(t), \tag{11}$$

where the matrix P_s is $(n+1)\times(n+1)$ and called the operational matrix of stochastic integration for the generalized hat functions, and it is given by:

$$P_s = \begin{pmatrix} 0 & \alpha_0 & \alpha_0 & & \cdots & \alpha_0 & \alpha_0 \\ 0 & B(h)+\beta_1 & \beta_1+\alpha_1 & & \cdots & \beta_1+\alpha_1 & \beta_1+\alpha_1 \\ 0 & 0 & B(2h)+\beta_2 & \beta_2+\alpha_2 & \cdots & \beta_2+\alpha_2 & \beta_2+\alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B((n-1)h)+\beta_{n-1} & \beta_{n-1}+\alpha_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & B(T)+\beta_n \end{pmatrix},$$

and

$$\begin{cases} \alpha_i = \frac{1}{h} \int\limits_{ih}^{(i+1)h} B(s)ds & \mathrm{i} = 0, 1, 2, ..., \mathrm{n}-1, \\ \beta_i = -\frac{1}{h} \int\limits_{(i-1)h}^{ih} B(s)ds & \mathrm{i} = 1, 2, ..., \mathrm{n}. \end{cases}$$

Theorem 2 *Let $\Phi(t)$ be the vector defined in (7). The integral of $\Phi(t)$ according to fractional Brownian motion can be expressed as:*

$$\int\limits_0^t \Phi(s) dB^{(H)}(s) \simeq \; P_{sH} \Phi(t), \tag{12}$$

where $(n+1) \times (n+1)$ matrix P_{sH} is called the operational matrix of stochastic integration driven by fractional Brownian motion for the generalized hat functions and is given by:

$$P_{sH} = \begin{pmatrix} 0 & \alpha_0 & \alpha_0 & \alpha_0 & \cdots & \alpha_0 & \alpha_0 \\ 0 & B^{(H)}(h) + \beta_1 & \beta_1 + \alpha_1 & \beta_1 + \alpha_1 & \cdots & \beta_1 + \alpha_1 & \beta_1 + \alpha_1 \\ 0 & 0 & B^{(H)}(2h) + \beta_2 & \beta_2 + \alpha_2 & \cdots & \beta_2 + \alpha_2 & \beta_2 + \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B^{(H)}((n-1)h) + \beta_{n-1} & \beta_{n-1} + \alpha_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & B^{(H)}(T) + \beta_n \end{pmatrix},$$

and

$$\begin{cases} \alpha_i = \frac{1}{h} \int\limits_{ih}^{(i+1)h} B^{(H)}(s)ds & \mathrm{i} = 0, 1, 2, ..., \mathrm{n}-1, \\ \beta_i = -\frac{1}{h} \int\limits_{(i-1)h}^{ih} B^{(H)}(s)ds & \mathrm{i} = 1, 2, ..., \mathrm{n}. \end{cases}$$

Proof In order to compute $\int\limits_0^t \phi(s) dB^{(H)}(s)$, choose $X_t = B^{(H)}(t)$ and $f(t,x) = \phi_i(t) \times x$. Then according to relation (2), we have:

$$Y_t = f(t, B^{(H)}(t)) = \phi_i(t) \times B^{(H)}(t).$$

So:

$$d(\phi_i(t) \times B^{(H)}(t)) = B^{(H)}(t) \times \phi_i'(t)dt + \phi_i(t) dB^{(H)}(t).$$

By integrating from 0 to t, we have:

$$\phi_i(t) B^{(H)}(t) - \phi_i(0) B^{(H)}(0) = \int\limits_0^t B^{(H)}(y) \phi_i'(y) dy + \int\limits_0^t \phi_i(y) dB^{(H)}(y).$$

Therefore:

$$\int\limits_0^t \phi_i(y) dB^{(H)}(y) = \phi_i(t) B^{(H)}(t) - \int\limits_0^t B^{(H)}(y) \phi_i'(y) dy. \tag{13}$$

By expanding $\int\limits_0^t \phi_i(y)dB^{(H)}(y)$ in terms of hat functions, we will have:

$$\int\limits_0^t \phi_i(y)dB^{(H)}(y) \simeq \sum_{j=0}^{n} a_{ij}\phi_j(t) = \sum_{j=0}^{n}\left(\int\limits_0^{jh} \phi_i(y)dB^{(H)}(y)\right)\phi_j(t).$$

Using (13), we have:

$$a_{ij} = \int\limits_0^{jh} \phi_i(y)dB^{(H)}(y) = \phi_i(jh)B^{(H)}(jh) - \int\limits_0^{jh} B^{(H)}(y)\phi_i'(y)dy.$$

By using the definition and properties of hat functions mentioned in Section 3, a_{ij} have the following form:

$$a_{0j} = \begin{cases} 0 & j = 0, \\ \frac{1}{h}\int\limits_0^h B^{(H)}(y)dy & j \geq 1, \end{cases}$$

$$a_{ij} = \begin{cases} 0 & j \leq i-1, \\ B^{(H)}(ih) - \frac{1}{h}\int\limits_{(i-1)h}^{ih} B^{(H)}(y)dy & j = i, \\ -\frac{1}{h}\left(\int\limits_{(i-1)h}^{ih} B^{(H)}(y)dy - \int\limits_{ih}^{(i+1)h} B^{(H)}(y)dy\right) & j \geq i+1 \text{ and } i \neq n, \end{cases}$$

where $i = 1, ..., n$ and $j = 0, 1, ..., n$.

Therefore, by substituting $\alpha_i = \frac{1}{h}\int\limits_{ih}^{(i+1)h} B^{(H)}(s)ds$ and $\beta_i = -\frac{1}{h}\int\limits_{(i-1)h}^{ih} B^{(H)}(s)ds$, the matrix P_{sH} and so the Itô integral driven by fractional Brownian motion of $\Phi(x)$ will be obtained. □

In this paper we will work with matrix P_{sH} and its entries.

5. Numerical method using stochastic operational matrix

In this section, we apply the operational matrices of integration and stochastic integration caused by fractional Brownian motion with Hurst parameter $H \in (0,1)$. By using hat basis functions and their properties we try to solve the following equation:

$$X(t) = f(t) + \int\limits_0^t K_1(s,t)X(s)ds + \int\limits_0^t K_2(s,t)X(s)dB^{(H)}(s), \; t \in [0,T]. \tag{14}$$

We will approximate $X(t)$, $f(t)$, $K_1(s,t)$, and $K_2(s,t)$ as follows:

$$X(t) \simeq X^T\Phi(t) = \Phi(t)^T X, \tag{15}$$

$$f(t) \simeq F^T\Phi(t) = \Phi(t)^T F, \tag{16}$$

$$K_1(s,t) \simeq \Phi(s)^T K_1 \Phi(t) = \Phi(t)^T K_1^T \Phi(s), \tag{17}$$

$$K_2(s,t) \simeq \Phi(s)^T K_2 \Phi(t) = \Phi(t)^T K_2^T \Phi(s), \tag{18}$$

where X and F are the generalized hat coefficients vectors, and K_1 and K_2 are generalized hat coefficient matrices.

By substituting the above relations in (14), we have:

$$\begin{aligned} X^T\Phi(t) \simeq F^T\Phi(t) + X^T \left(\int_0^t \Phi(s)\Phi(s)^T ds \right) K_1\Phi(t) + \\ X^T \left(\int_0^t \Phi(s)\Phi(s)^T dB^{(H)}(s) \right) K_2\Phi(t). \end{aligned} \tag{19}$$

If we assume K_1^i, K_2^i, R^i, and R_{sH}^i be the ith rows of matrices K_1, K_2, P, and P_{sH} and $D_{K_1^i}$ to be a diagonal matrix with K_1^i as its diagonal entries and $D_{K_2^i}$ a diagonal matrix with K_2^i as its diagonal entries, we can simplify the above relation as follows:

$$\left(\int_0^t \Phi(s)\Phi(s)^T ds \right) K_1\Phi(t) \simeq \begin{pmatrix} R^1\Phi(t)K_1^1\Phi(t) \\ R^2\Phi(t)K_1^2\Phi(t) \\ \vdots \\ R^{n+1}\Phi(t)K_1^{n+1}\Phi(t) \end{pmatrix} \simeq \begin{pmatrix} R^1 D_{K_1^1} \\ R^2 D_{K_1^2} \\ \vdots \\ R^{n+1} D_{K_1^{n+1}} \end{pmatrix} \Phi(t) =$$

$$B_1\Phi(t),$$

where

$$B_1 = \frac{h}{2} \begin{pmatrix} 0 & k_{01}^1 & k_{02}^1 & \cdots & k_{0n}^1 \\ 0 & k_{11}^1 & 2k_{12}^1 & \cdots & 2k_{1n}^1 \\ 0 & 0 & k_{22}^1 & \cdots & 2k_{2n}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_{nn}^1 \end{pmatrix}.$$

We also have:

$$\left(\int_0^t \Phi(s)\Phi(s)^T dB^{(H)}(s) \right) K_2\Phi(t) \simeq \begin{pmatrix} R_{sH}^1\Phi(t)K_2^1\Phi(t) \\ R_{sH}^2\Phi(t)K_2^2\Phi(t) \\ \vdots \\ R_{sH}^{n+1}\Phi(t)K_2^{n+1}\Phi(t) \end{pmatrix} \simeq \begin{pmatrix} R_{sH}^1 D_{K_2^1} \\ R_{sH}^2 D_{K_2^2} \\ \vdots \\ R_{sH}^{n+1} D_{K_2^{n+1}} \end{pmatrix} \Phi(t) =$$

$$B_2\Phi(t),$$

where

$$B_2 = \begin{pmatrix} 0 & \alpha_0 k_{01}^2 & \alpha_0 k_{02}^2 & \cdots & \alpha_0 k_{0n}^2 \\ 0 & (B^{(H)}(h)+\beta_1)k_{11}^2 & (\beta_1+\alpha_1)k_{12}^2 & \cdots & (\beta_1+\alpha_1)k_{1n}^2 \\ 0 & 0 & (B^{(H)}(2h)+\beta_2)k_{22}^2 & \cdots & (\beta_2+\alpha_2)k_{2n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (B^{(H)}(T)+\beta_n)k_{nn}^2 \end{pmatrix}.$$

Thus, equation (19) will be:

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^T B_1 \Phi(t) + X^T B_2 \Phi(t). \tag{20}$$

Therefore, we have:

$$X^T(I - B_1 - B_2) \simeq F^T. \tag{21}$$

By putting $M = (I - B_1 - B_2)^T$ and replacing $\simeq$ by $=$, we obtain the following linear lower triangular system of the algebraic equation:

$$MX = F. \tag{22}$$

By solving this system, we can get the approximation solution of equation (14).

6. Error analysis

In this section we get error analysis of the proposed method. First we will provide a theorem to prove that $\|B^{(H)}(t)\|$ is bounded on $[0,T]$, in which $\|.\|$ is sup-norm and is defined as:

$$\|f(t)\| = \sup_{t\in[0,T]} |f(t)|.$$

Theorem 3 *For every $x > 0$*

$$P(M(t) \geq x) = 2P(B^{(H)}(t) \geq x) = 2(1 - \phi(\frac{x}{\sqrt{t^{2H}}})), \tag{23}$$

in which $M(t) = max_{0\leq s\leq t} B^{(H)}(s)$ and $\phi(x)$ is the cumulative standard normal distribution function.

Proof Let T_x denote the first time at which $B^{(H)}(t)$ hits level x, i.e. $T_x = inf\{t > 0 : B^{(H)}(t) = x\}$. Obviously $P(M(t) \geq x) = P(T_x \leq t)$ and we have:

$$P(B^{(H)}(t) \geq x) = P(B^{(H)}(t) \geq x | T_x \leq t)P(T_x \leq t) + P(B^{(H)}(t) \geq x | T_x > t)P(T_x > t).$$

If $T_x \leq t$, the process at the point that belongs to $[0,t]$ will visit x and in accordance with the symmetric property of $B^{(H)}(t)$, the probability of being above and below x at time t for $B^{(H)}(t)$ is equal, so we have:

$$P(B^{(H)}(t) \geq x | T_x \leq t) = \frac{1}{2}.$$

Since $P(B^{(H)}(t) \geq x | T_x > t) = 0$, we have:

$$P(M(t) \geq x) = P(T_x \leq t) = 2P(B^H(t) \geq x) = \frac{2}{\sqrt{2\pi t^{2H}}} \int_x^\infty e^{\frac{-y^2}{2t^{2H}}} dy.$$

If we put $z = \frac{y}{\sqrt{t^{2H}}}$, we will have:

$$P(M(t) \geq x) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t^{2H}}}}^{\infty} e^{\frac{-z^2}{2}} dz = 2(1 - \phi(\frac{x}{\sqrt{t^{2H}}})).$$

Thus, $\|B^H(t)\| < \infty$ almost surely. □

In the following, we use Theorems 4 and 5 from [10] to get the order of convergence for our method, which is obtained in Theorem 6.

Theorem 4 [10] *Suppose* $f(t) \in C^2([0,T])$ *and* $e_n(t) = f(t) - f_n(t), t \in I = [0,T]$, *where* $f_n(t)$ *is the approximation of* $f(t)$ *by the generalized hat functions. Then:*

$$\|e_n(t)\| \leq \frac{T^2}{2n^2} \|f''(t)\|.$$

Thus, we have:

$$\|e_n(t)\| = O\left(\frac{1}{n^2}\right). \tag{24}$$

Theorem 5 [10] *Suppose* $f(s,t) \in C^2([0,T] \times [0,T])$ *and* $e_n(s,t) = f(s,t) - f_n(s,t), (s,t) \in D = [0,T] \times [0,T]$, *where* $f_n(s,t)$ *is the approximation of* $f(s,t)$ *by the generalized hat functions. Then:*

$$\|e_n(s,t)\| \leq \frac{T^2}{2n^2} \left(\| \frac{\partial^2 f(s,t)}{\partial s^2} \| + 2 \| \frac{\partial^2 f(s,t)}{\partial s \partial t} \| + \| \frac{\partial^2 f(s,t)}{\partial t^2} \| \right).$$

Thus, we have:

$$\|e_n(s,t)\| = O\left(\frac{1}{n^2}\right). \tag{25}$$

Theorem 6 *Suppose* $X(t)$ *and* $X_n(t)$ *are the exact and approximation solution of the target equation (14). If*

(a) $\|X(t)\| \leq \rho, t \in I = [0,T]$,

(b) $\|K_1(s,t)\| \leq M_1, (s,t) \in D = I \times I$,

(c) $\|K_2(s,t)\| \leq M_2, (s,t) \in D = I \times I$,

(d) $(d) T(M_1 + \lambda(h)) + (M_2 + \gamma(h))\|B^H(t)\| < 1$,

then we have:

$$\|X(t) - X_n(t)\| \leq \frac{\Gamma(h) + T\rho\lambda(h) + \rho\gamma(h)\|B^H(t)\|}{1 - (T(M_1 + \lambda(h)) + (M_2 + \gamma(h))\|B^H(t)\|)}, \tag{26}$$

where

$$\Gamma(h) = \frac{h^2}{2} \|f''(t)\|,$$

$$\lambda(h) = \frac{h^2}{2}\left(\|\frac{\partial^2 K_1(s,t)}{\partial s^2}\| + 2\|\frac{\partial^2 K_1(s,t)}{\partial s \partial t}\| + \|\frac{\partial^2 K_1(s,t)}{\partial t^2}\|\right),$$

$$\gamma(h) = \frac{h^2}{2}\left(\|\frac{\partial^2 K_2(s,t)}{\partial s^2}\| + 2\|\frac{\partial^2 K_2(s,t)}{\partial s \partial t}\| + \|\frac{\partial^2 K_2(s,t)}{\partial t^2}\|\right).$$

Proof From equation (14), we have:

$$\begin{aligned} X(t) - X_n(t) = f(t) - f_n(t) + \int_0^t (K_1(s,t)X(s) - K_{1n}(s,t)X_n(s))ds + \\ \int_0^t (K_2(s,t)X(s) - K_{2n}(s,t)X_n(s))dB^{(H)}(s). \end{aligned} \tag{27}$$

Thus, we can write:

$$\begin{aligned} \|X(t) - X_n(t)\| \le \|f(t) - f_n(t)\| + t\|K_1(s,t)X(s) - K_{1n}(s,t)X_n(s)\| + \\ B^{(H)}(t)\|K_2(s,t)X(s) - K_{2n}(s,t)X_n(s)\|. \end{aligned} \tag{28}$$

By using Theorems 4 and 5 and assumptions (a) and (b), we will have:

$$\begin{aligned} \|K_1(s,t)X(s) - K_{1n}(s,t)X_n(s)\| &\le \|K_1(s,t)\|\|X(t) - X_n(t)\| + \\ &\|K_1(s,t) - K_{1n}(s,t)\|\left(\|X(t) - X_n(t)\| + \|X(t)\|\right) \\ &\le (M_1 + \lambda(h))\|X(t) - X_n(t)\| + \rho\lambda(h), \end{aligned} \tag{29}$$

and also we have:

$$\begin{aligned} \|K_2(s,t)X(s) - K_{2n}(s,t)X_n(s)\| &\le \|K_2(s,t)\|\|X(t) - X_n(t)\| + \\ &\|K_2(s,t) - K_{2n}(s,t)\|\left(\|X(t) - X_n(t)\| + \|X(t)\|\right) \\ &\le (M_2 + \gamma(h))\|X(t) - X_n(t)\| + \rho\gamma(h). \end{aligned} \tag{30}$$

Therefore, we conclude:

$$\begin{aligned} \|X(t) - X_n(t)\| \le \Gamma(h) + t((M_1 + \lambda(h))\|X(t) - X_n(t)\| + \rho\lambda(h)) + \\ B^{(H)}(t)((M_2 + \gamma(h))\|X(t) - X_n(t)\| + \rho\gamma(h)). \end{aligned} \tag{31}$$

Thus, we have:

$$\|X(t) - X_n(t)\| \le \frac{\Gamma(h) + T\rho\lambda(h) + \rho\gamma(h)\|B^H(t)\|}{1 - (T(M_1 + \lambda(h)) + (M_2 + \gamma(h))\|B^H(t)\|)}. \tag{32}$$

From the above relation and by using Theorem 3, since $\|B^H(t)\| < \infty$ almost surely, we conclude that $\|X(t) - X_n(t)\| = O(\frac{1}{n^2})$. □

7. Some numerical examples

To demonstrate the method, we consider the following examples, the exact solutions of which exist. Note that n is the number of hat basis functions and m is the number of iterations.

7.1. Example 1

Consider the following stochastic Itô–Volterra integral equation, which is caused by fractional Brownian motion and has an exact solution:

$$X(t) = -\frac{1}{8} - \int_0^t \frac{1}{4} s \times X(s)ds - \int_0^t \frac{1}{40} X(s)dB^{(H)}(s), t \in (0,T], T < 1.$$

The exact solution of the above equation is:

$$X(t) = \frac{-1}{8} exp(\frac{-1}{40} B^H(t) - \frac{t^2}{8} - \frac{1}{3200} t^{2H}).$$

A comparison between the approximation of the solution given by hat functions and the block pulse method is given in Table 1. In this example the Hurst parameter is $\frac{2}{3}$. You can see the exact and approximation solution of Example 1 for $t = 0.05$ with $n = 16$ and $m = 200$ in Figure 1 and the exact and approximation solution of it with $n = 64$ and $m = 500$ in Figure 2.

Table 1. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of Example 1 with n = 16 and 200 iterations.

t	$\bar{X}_E$ with method in [8]	$\bar{X}_E$ in our method	S_E	95% confidence interval for error mean	
				Lower	Upper
0.05	3.800000×10^{-5}	3.03682×10^{-7}	8.99901×10^{-8}	2.46586×10^{-8}	1.17196×10^{-7}
0.1	1.045000×10^{-4}	7.29227×10^{-7}	1.95298×10^{-7}	7.28581×10^{-8}	2.73684×10^{-7}
0.15	9.650000×10^{-4}	9.96107×10^{-7}	2.71468×10^{-7}	1.19682×10^{-7}	3.98834×10^{-7}
0.2	1.510000×10^{-4}	1.59443×10^{-6}	4.44693×10^{-7}	1.87610×10^{-7}	6.44890×10^{-7}

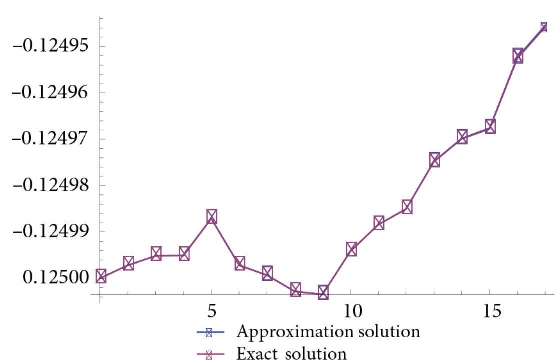

Figure 1. Exact and approximation solution of example 1 for $n = 16$, $m = 200$, and $t = 0.05$.

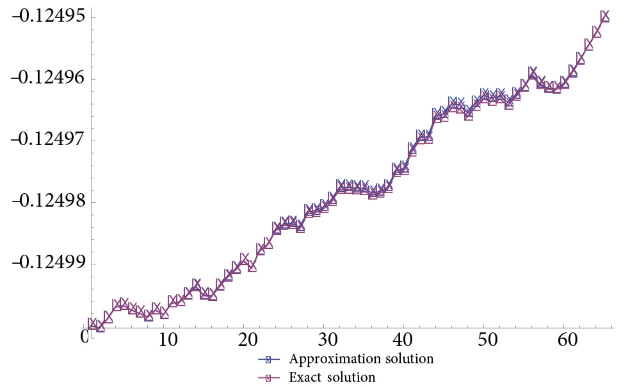

Figure 2. Exact and approximation solution of example 1 for $n = 64$, $m = 500$, and $t = 0.05$.

7.2. Example 2

Consider the following linear stochastic Itô–Volterra integral equation, which is caused by fractional Brownian motion:

$$X(t) = \frac{1}{12} - \int_0^t \frac{1}{8} Cos(s) \times X(s) ds - \int_0^t \frac{1}{16} X(s) dB^{(H)}(s), t \in (0, T), T < 1.$$

The exact solution of the above equation is:

$$X(t) = \frac{1}{12} exp(\frac{-1}{16} B^H(t) - \frac{1}{8} Sin(t) - \frac{1}{512} t^{2H}).$$

A comparison between the approximation of the solution given by hat functions and the block pulse method is given in Table 2. In this example the Hurst parameter is $\frac{2}{3}$. You can see the exact and approximation solution of this example for $t = 0.05$ in Figure 3.

Table 2. Error mean, $\bar{X}_E$, error standard deviation, S_E, and confidence interval for error mean of example 2 with n = 16 and 200 iterations.

t	$\bar{X}_E$ with method in [8]	$\bar{X}_E$ in our method	S_E	95% confidence interval for error mean	
				Lower	Upper
0.05	7.6300000×10^{-5}	8.70521×10^{-7}	3.05681×10^{-7}	3.99484×10^{-7}	7.13817×10^{-7}
0.1	1.4725000×10^{-4}	2.79501×10^{-6}	9.03227×10^{-7}	1.02205×10^{-6}	1.95084×10^{-6}
0.15	1.5430000×10^{-4}	4.53652×10^{-6}	1.57851×10^{-6}	1.82539×10^{-6}	3.44858×10^{-6}
0.2	2.6180000×10^{-4}	6.75661×10^{-6}	2.17244×10^{-6}	2.49057×10^{-6}	4.72450×10^{-6}

8. Conclusion

In this paper we numerically solved the linear stochastic Itô–Volterra integral equation driven by fractional Brownian motion, which was solved for simple Brownian motion in [10]. We used hat functions as basis functions for approximation, in which error analysis and the numerical examples showed the accuracy of the method such that the results signify that the efficiency of the suggested method is better than block pulse functions as basis functions used in [8].

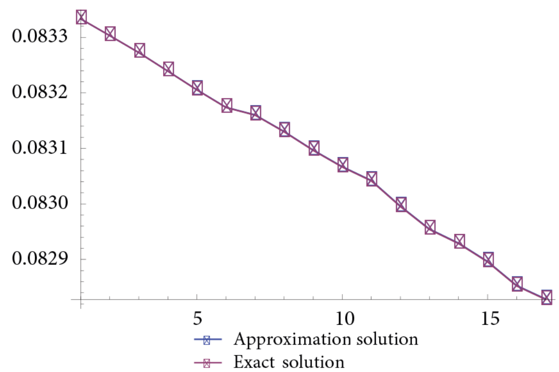

Figure 3. Exact and approximation solution of example 2 for $n = 16$, $m = 200$, and $t = 0.05$.

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