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FATMA GÜRLER
CİHAN ÖZGÜR

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**f**-Biminimal immersions

Fatma GÜRLER, Cihan ÖZGÜR

Balkesir University, Department of Mathematics, Çağs, Balikesir, Turkey

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Abstract: In the present paper, we define **f**-biminimal immersions. We consider **f**-biminimal curves in a Riemannian manifold and **f**-biminimal submanifolds of codimension 1 in a Riemannian manifold, and we give examples of **f**-biminimal surfaces. Finally, we consider **f**-biminimal Legendre curves in Sasakian space forms and give an example.

Key words: **f**-Biminimal immersion, **f**-biminimal curve, **f**-biminimal surface, Legendre curve

1. Introduction and preliminaries

Let \((M, g)\) and \((N, h)\) be two Riemannian manifolds. A map \(\varphi : (M, g) \rightarrow (N, h)\) is called a harmonic map if it is a critical point of the energy functional

\[
E(\varphi) = \frac{1}{2} \int_{\Omega} \|d\varphi\|^2 \, dv_g,
\]

where \(\Omega\) is a compact domain of \(M\). The Euler–Lagrange equation gives the harmonic map equation

\[
\tau(\varphi) = tr(\nabla d\varphi) = 0,
\]

where \(\tau(\varphi) = tr(\nabla d\varphi)\) is called the tension field of the map \(\varphi\) [6]. The map \(\varphi\) is said to be biharmonic if it is a critical point of the bienergy functional

\[
E_2(\varphi) = \frac{1}{2} \int_{\Omega} \|\tau(\varphi)\|^2 \, dv_g,
\]

where \(\Omega\) is a compact domain of \(M\) [10]. In [10], Jiang obtained the Euler–Lagrange equation of \(E_2(\varphi)\). This gives us the biharmonic map equation

\[
\tau_2(\varphi) = tr(\nabla^2 \varphi - \nabla^2_{\varphi} \tau(\varphi) - tr(R^N(d\varphi, \tau(\varphi))d\varphi) = 0,
\]

(1.1)

which is the bitension field of \(\varphi\), and \(R^N\) is the curvature tensor of \(N\), defined by

\[
R^N(X, Y)Z = \nabla^X\nabla^Y Z - \nabla^Y\nabla^X Z - \nabla^Z_{[X, Y]} Z.
\]

An **f**-harmonic map with a positive function \(f : M \rightarrow \mathbb{R}\) is a critical point of \(f\)-energy

\*Correspondence: cozgur@balikesir.edu.tr

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where $\Omega$ is a compact domain of $M$. Using the Euler–Lagrange equation for the $f$-harmonic map, in [5] and [16] the $f$-harmonic map equation is obtained by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\text{grad} f) = 0,$$

(1.2)

where $\tau_f(\varphi)$ is called the $f$-tension field of the map $\varphi$. The map $\varphi$ is said to be $f$-biharmonic [13] if it is a critical point of the $f$-bienergy functional

$$E_{f}(\varphi) = \frac{1}{2} \int_{\Omega} f \|d\varphi\|^2 d\nu_g,$$

where $\Omega$ is a compact domain of $M$. The Euler–Lagrange equation for the $f$-biharmonic map is given by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla_{\text{grad} f}\tau(\varphi) = 0,$$

(1.3)

where $\tau_{2,f}(\varphi)$ is the $f$-bitension field of the map $\varphi$ [13]. If $f$ is a constant, an $f$-biharmonic map turns into a biharmonic map.

In [12], Loubeau and Montaldo defined and considered biminimal immersions. They studied biminimal curves in a Riemannian manifold, curves in a space form, and isometric immersions of codimension 1 in a Riemannian manifold.

An immersion $\varphi$ is called biminimal [12] if it is a critical point of the bienergy functional $E_2(\varphi)$ for variations normal to the image $\varphi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that $\varphi$ is a critical point of the $\lambda$-bienergy

$$E_{2,\lambda}(\varphi) = E_2(\varphi) + \lambda E(\varphi)$$

(1.4)

for any smooth variation of the map $\varphi_t : [-\epsilon, +\epsilon[ \ni \varphi_0 = \varphi$, such that $V = \frac{d\varphi_t}{dt} |_{t=0} = 0$ is normal to $\varphi(M)$. The Euler–Lagrange equation for a $\lambda$-biminimal immersion is

$$[\tau_{2,\lambda}(\varphi)]^\perp = [\tau_2(\varphi)]^\perp - \lambda[\tau(\varphi)]^\perp = 0$$

(1.5)

for some value of $\lambda \in \mathbb{R}$, where $[\cdot]^\perp$ denotes the normal component of $[\cdot]$. An immersion is called free biminimal if it is biminimal for $\lambda = 0$ [12].

In [12], Loubeau and Montaldo studied biminimal immersions. In [9], Inoguchi and Lee completely classified biminimal curves in 2-dimensional space forms. In [8], Inoguchi studied biminimal curves and surfaces in contact 3-manifolds. In [13], Lu defined $f$-biharmonic maps between Riemannian manifolds. In [15], Ou considered $f$-biharmonic maps and $f$-biharmonic submanifolds. In [7], Güvenç and the second author studied $f$-biharmonic Legendre curves in Sasakian space forms. Motivated by the studies [12] and [13], in this paper, we define $f$-biminimal immersions. We consider $f$-biminimal curves in a Riemannian manifold. We also consider $f$-biminimal submanifolds of codimension 1 in a Riemannian manifold and give some examples of $f$-biminimal surfaces. Furthermore, we give an example for an $f$-biminimal Legendre curve in a Sasakian space form.

Now we give the following definition:
Definition 1.1 An immersion \( \varphi \) is called \( f \)-biminimal if it is a critical point of the \( f \)-bienergy functional \( E_{2f}(\varphi) \) for variations normal to the image \( \varphi(M) \subset N \), with fixed energy. Equivalently, there exists a constant \( \lambda \in \mathbb{R} \) such that \( \varphi \) is a critical point of the \( \lambda \cdot f \)-bienergy

\[
E_{2,\lambda,f}(\varphi) = E_{2f}(\varphi) + \lambda E_f(\varphi)
\]

for any smooth variation of the map \( \varphi_t \) defined above. Using the Euler–Lagrange equations for \( f \)-harmonic and \( f \)-biharmonic maps, an immersion is \( f \)-biminimal if

\[
[\tau_{2,\lambda,f}(\varphi)]^\perp = [\tau_{2,f}(\varphi)]^\perp - \lambda [\tau_f(\varphi)]^\perp = 0
\]

(1.6)

for some value of \( \lambda \in \mathbb{R} \). We call an immersion free \( f \)-biminimal if it is \( f \)-biminimal for \( \lambda = 0 \). If \( f \) is a constant, then the immersion is biminimal.

Remark 1.1 The notions of \( f \)-biharmonic submanifolds, biminimal submanifolds, and \( f \)-biminimal submanifolds are distinct. We will see details in the examples given in Section 4 and Section 5.

2. \( f \)-Biminimal curves

Let \( \gamma : I \subset \mathbb{R} \rightarrow (M^m,g) \) be a curve parametrized by arc length in a Riemannian manifold \((M^m,g)\). We recall the definition of Frenet frames:

Definition 2.1 [11] The Frenet frame \( \{E_i\}_{i=1,2,...,m} \) associated with a curve \( \gamma : I \subset \mathbb{R} \rightarrow (M^m,g) \) is the orthonormalization of the \( (m+1) \)–tuple

\[
\left\{ \frac{\nabla^{(k)} \gamma}{\| \gamma \|} \left( \frac{\partial}{\partial t} \right) \right\}_{k=0,1,...,m}
\]

described by

\[
E_1 = \frac{d\gamma}{\partial t} \left( \frac{\partial}{\partial t} \right),
\]

\[
\nabla^\gamma E_1 = k_1 E_2,
\]

\[
\nabla^\gamma E_i = -k_{i-1} E_{i-1} + k_i E_{i+1}, \quad 2 \leq i \leq m-1,
\]

\[
\nabla^\gamma E_m = -k_{m-1} E_{m-1},
\]

where the functions \( \{k_1, k_2, \tau, k_3, ..., k_{m-1}\} \) are called the curvatures of \( \gamma \). In addition \( E_1 = T = \gamma' \) is the unit tangent vector field to the curve.

First, we have the following proposition for an \( f \)-biminimal curve in a Riemannian manifold:

Proposition 2.1 Let \( M^m \) be a Riemannian manifold and \( \gamma : I \subset \mathbb{R} \rightarrow (M^m,g) \) be an isometric curve. Then \( \gamma \) is \( f \)-biminimal if and only if there exists a real number \( \lambda \) such that

\[
f \{ (k_1^\prime - k_1^3 - k_1 k_2^2) - k_1 g(R(E_1, E_2)E_1, E_2) \} + (f'' - \lambda f) k_1 + 2f'k' = 0.
\]

(2.1)
where $R$ is the curvature tensor of $(M^m, g)$ and $\{E_i\}_{i=1, 2, \ldots, m}$ is the Frenet frame of $\gamma$.

**Proof** Using equation (1.2), Definition 2.1, and $\tau(\gamma) = k_1 E_2$ (see [12]), the $f$-tension field of $\gamma$ is

$$
\tau_f(\gamma) = f k_1 E_2 + f' E_1. \tag{2.5}
$$

From Definition 2.1, we have

$$
\nabla_T^2 T = -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3, \tag{2.6}
$$

$$
\nabla_T \nabla_T \nabla_T T = -3k_1 k_1' E_1 + (k_1'' - k_1'' - k_1) E_2 + (k_1' k_2 + (k_1 k_2)) E_3 \tag{2.7}
$$

and

$$
\nabla_{grad} \tau(\gamma) = f' \left\{-k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3 \right\}. \tag{2.8}
$$

Using equations (2.6), (2.7), and (2.8) in equation (1.3), its $f$-bitension field is

$$
\tau_{2,f}(\gamma) = f \left\{(-3k_1 k_1') E_1 + (k_1'' - k_1'' - k_1) E_2 + (k_1' k_2 + (k_1 k_2)) E_3 \right\}
+ (k_1 k_2) E_4 - k_1 R(E_1, E_2) E_1
+ f'' k_1 E_2 + 2f' \left\{-k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3 \right\}. \tag{2.9}
$$

By the use of equations (2.5) and (2.9) in equation (1.6), we find

$$
f \left\{(k_1'' - k_1'' - k_1) E_2 + (k_1' k_2 + (k_1 k_2)) E_3 \right\}
+ (k_1 k_2) E_4 - k_1 \left[R(E_1, E_2)E_1\right] \right\}
+ f'' k_1 E_2 + 2f' \left\{k_1' E_2 + k_1 k_2 E_3 \right\} - \lambda \left\{k_1 E_2 \right\} = 0. \tag{2.10}
$$

Then taking the scalar product of equation (2.10) with $E_2, E_3, E_4, \text{ and } E_j$, $5 \leq j \leq m$, respectively, we obtain the desired results. □

Now we investigate $f$-biminimality conditions for a surface or a three-dimensional Riemannian manifold with a constant sectional curvature. We have the following corollary:
Corollary 2.1 1) A curve $\gamma$ on a surface of Gaussian curvature $G$ is $f$-biminimal if and only if its signed curvature $k$ satisfies the equation

$$f \left( k'' - k^3 + kG \right) + \left( f'' - \lambda f \right) k + 2f'k' = 0$$  \hspace{0.5cm} (2.11)

for some $\lambda \in \mathbb{R}$.

2) A curve $\gamma$ on Riemannian 3-manifold $M$ of constant sectional curvature $c$ is $f$-biminimal if and only if its curvature $k$ and torsion $\tau$ satisfy the system

$$f \left( k'' - k^3 - k\tau^2 + kc \right) + \left( f'' - \lambda f \right) k + 2f'k' = 0$$

$$f \left( k'\tau + (k\tau)' \right) + 2f'k\tau = 0$$  \hspace{0.5cm} (2.12)

for some $\lambda \in \mathbb{R}$.

Proof 1) Since $\gamma$ is a curve on a surface, if $\gamma$ is $f$-biminimal then by the use of equation (2.1), we obtain

$$f \left\{ k'' - k^3 - k\left(R(T, N)T, N\right) \right\} + \left( f'' - \lambda f \right) k + 2f'k' = 0.$$ \hspace{0.5cm} (2.13)

Then we have

$$g(R(T, N)T, N) = -G.$$ \hspace{0.5cm} (2.14)

Finally, substituting equation (2.14) into equation (2.13), we obtain

$$f \left\{ k'' - k^3 + kG \right\} + \left( f'' - \lambda f \right) k + 2f'k' = 0.$$ \hspace{0.5cm} (2.15)

2) Since $\gamma$ is a curve on a Riemannian 3-manifold, the Frenet frame of $\gamma$ is $\{ T, N = B_2, B = B_3 \}$, and then equations (2.1) and (2.2) turn into

$$f \left\{ k'' - k^3 - k\tau^2 - k\left(R(T, N)T, N\right) \right\} + \left( f'' - \lambda f \right) k + 2f'k' = 0$$ \hspace{0.5cm} (2.15)

and

$$f \left\{ k'\tau + (k\tau)' - k\left(R(T, N)T, B\right) \right\} + 2f'k\tau = 0.$$ \hspace{0.5cm} (2.16)

Since $M$ has constant sectional curvature we have

$$g(R(T, N)T, N) = -c$$ \hspace{0.5cm} (2.17)

and

$$g(R(T, N)T, B) = 0.$$ \hspace{0.5cm} (2.18)

Finally, substituting equations (2.17) and (2.18) into equations (2.15) and (2.16), respectively, we get

$$f \left\{ k'' - k^3 - k\tau^2 + kc \right\} + \left( f'' - \lambda f \right) k + 2f'k' = 0$$

and

$$f \left\{ k'\tau + (k\tau)' \right\} + 2f'k\tau = 0.$$ This completes the proof. \hspace{0.5cm} $\square$
Remark 2.1 In Proposition 2.1 and Corollary 2.1, if we take \( f \) as a constant, we obtain Proposition 2.2 and Corollary 2.4 in [12].

Now assume that \( M^2 \subset \mathbb{R}^3 \) is a surface of revolution obtained by rotating the arc length parametrized curve \( \alpha(u) = (h(u), 0, g(u)) \) in the \( xz \)-plane around the \( z \)-axis. Then it can be easily seen that the Gaussian curvature \( G \) of the surface of revolution is

\[
G = -\frac{h''(u)}{h(u)}. \tag{2.19}
\]

The Gaussian curvature \( G \) depends only on \( u \); that is, \( G \) is constant along any parallel. This implies that if the Gaussian curvature is constant along a curve, then either the curve is a parallel or the curve lies in a part of the surface with constant Gaussian curvature [4]. From equation (2.19) and equation (2.11), it is easy to see that if a parallel of \( M \) is \( f \)-biminimal then \( f \) is a constant, which means that the parallel is biminimal. Biminimal curves in a surface of revolution was studied by Aykut in [1]. Hence, we can state the following result:

**Proposition 2.2** An \( f \)-biminimal parallel in a surface of revolution is biminimal.

### 3. Codimension-1 \( f \)-biminimal submanifolds

Let \( \varphi : M^m \rightarrow N^{m+1} \) be an isometric immersion of codimension 1. We shall denote by \( B, \eta, A, \Delta, \) and \( H_1 = H\eta \) the second fundamental form, the unit normal vector field, the shape operator, the Laplacian, and the mean curvature vector field of \( \varphi \) (\( H \) the mean curvature function), respectively.

Then we have the following proposition:

**Proposition 3.1** Let \( \varphi : M^m \rightarrow N^{m+1} \) be an isometric immersion of codimension 1 and \( H_1 = H\eta \) its mean curvature vector. Then \( \varphi \) is \( f \)-biminimal if and only if

\[
\Delta H - H \| B \|^2 + H \text{Ricci}(\eta, \eta) + \left( \frac{\Delta f}{f} - \lambda \right) H + 2\text{grad}\ln f(H) = 0 \tag{3.1}
\]

for some value of \( \lambda \) in \( \mathbb{R} \).

**Proof** Assume that \( \varphi \) is \( f \)-biminimal. Let \( \{e_i\}, 1 \leq i \leq m \) be a local geodesic orthonormal frame at \( p \in M \). Then using equation (1.2), the \( f \)-tension field of \( \varphi \) is

\[
\tau_f(\varphi) = fmH\eta + d\varphi(\text{grad} f) \tag{3.2}
\]

and using equation (1.3) and the definitions of \( \tau(\varphi) \) and \( \tau_2(\varphi) \) in [12], its \( f \)-bitension field is

\[
\tau_{2,f}(\varphi) = f \left\{ m(\Delta H)\eta + 2m \sum_{i=1}^{m} e_i(H)v_i^\varphi \eta - mH\Delta v^\varphi \eta \right. \\
left. + mH \sum_{i=1}^{m} R^N(d\varphi(e_i), \eta)d\varphi(e_i) \right\} + \Delta f(mH\eta) + 2m\nabla^\varphi_{\text{grad} f} H\eta. \tag{3.3}
\]

Then taking the scalar product of equations (3.2) and (3.3) with \( \eta \), respectively, we find

\[
g(\tau_f(\varphi), \eta) = fmH \tag{3.4}
\]
and
\[ g(\tau_2 f(\varphi), \eta) = f \left\{ m(\Delta H) + 2m \sum_{i=1}^{m} e_i(H)g(\nabla_{e_i}^\varphi \eta, \eta) - mHg(\Delta^\varphi \eta, \eta) \right\} \]
\[ -mHg(\sum_{i=1}^{m} R^N(d\varphi(e_i), \eta)d\varphi(e_i), \eta) \right\} + \Delta f(mH) + 2mg(\nabla_{\text{grad} f}^\varphi H\eta, \eta). \tag{3.5} \]

By use of the Weingarten formula, we have
\[ \nabla_{\text{grad} f}^\varphi H\eta = (\text{grad} f(H))\eta + H\nabla_{\text{grad} f}^\varphi \eta \]
\[ = (\text{grad} f(H))\eta + H(-A_\eta \text{grad} f + \nabla_{\text{grad} f}^\perp \eta) \]
\[ = (\text{grad} f(H))\eta - HA_\eta \text{grad} f. \]

Hence, taking the scalar product of the above equation with \( \eta \), we obtain
\[ g(\nabla_{\text{grad} f}^\varphi H\eta, \eta) = \text{grad} f(H). \tag{3.6} \]

Moreover, we have
\[ g(\nabla_{e_i}^\varphi \eta, \eta) = \frac{1}{2} e_i g(\eta, \eta) = 0 \tag{3.7} \]
and
\[ g(\sum_{i=1}^{m} R^N(d\varphi(e_i), \eta)d\varphi(e_i), \eta) = -\text{Ricci}(\eta, \eta). \tag{3.8} \]

Using the definition of the Laplacian, we get
\[ g(\Delta^\varphi \eta, \eta) = \sum_{i=1}^{m} g(-\nabla_{e_i}^\varphi \nabla_{e_i}^\varphi \eta + \nabla_{\nabla_{e_i}^\varphi e_i}^\varphi \eta, \eta) \]
\[ = \sum_{i=1}^{m} g(\nabla_{e_i}^\varphi \eta, \nabla_{e_i}^\varphi \eta) = \|B\|^2. \tag{3.9} \]

By use of equations (3.6), (3.7), (3.8), and (3.9) in equation (3.5), we have
\[ g(\tau_2 f(\varphi), \eta) = f \left\{ m(\Delta H) - mH \|B\|^2 + m\text{Ricci}(\eta, \eta) \right\} \]
\[ + \Delta f(mH) + 2mg(\text{grad} f(H)). \tag{3.10} \]

Finally, substituting equations (3.4) and (3.10) in equation (1.6), we obtain (3.1).

Conversely, assume that (3.1) holds on \( M^m \). If we take the product of equation (3.1) with \( mf \) we have
\[ mf \Delta H - mf H \|B\|^2 + mf H\text{Ricci}(\eta, \eta) \]
\[ + (mf \Delta f - mf \lambda) H + 2m\text{grad} f(H) = 0. \tag{3.11} \]
It is easy to see that
\[(\tau_2, f(\varphi)) = f \left\{ m(\Delta H) - mH\|B\|^2 - mHRicci(\eta, \eta) \right\} + \Delta f(mH) + 2m\text{grad}f(H) \] (3.12)
and
\[(\tau_f(\varphi)) = fmH. \] (3.13)

In view of equations (3.12) and (3.13), equation (3.11) turns into
\[(\tau_2, f(\varphi)) - \lambda (\tau_f(\varphi)) = 0, \]
which means that \(M^m\) is \(f\)-biminimal. This proves the proposition. 

**Corollary 3.1** Let \(\varphi : M^m \rightarrow N^{m+1}(c)\) be an isometric immersion of a Riemannian manifold \(N^{m+1}(c)\) of constant curvature \(c\). Then \(\varphi\) is \(f\)-biminimal if and only if there exists a real number \(\lambda\) such that
\[
\Delta H - \left( m^2H^2 - s + m(m-2)c - \frac{\Delta f}{f} + \lambda \right) H - 2\text{grad} \ln f(H) = 0, \] (3.14)
where \(H\) is the mean curvature function and \(s\) the scalar curvature of \(M^m\). In addition, let \(\varphi : M^2 \rightarrow N^3(c)\) be an isometric immersion from a surface to a three-dimensional space form. Then \(\varphi\) is \(f\)-biminimal if and only if
\[
\Delta H - 2 \left( 2H^2 - G - \frac{1}{2} \frac{\Delta f}{f} + \frac{1}{2} \lambda \right) H - \text{grad} \ln f(H) = 0 \] (3.15)
for some \(\lambda \in \mathbb{R}\).

**Proof** Let \(\{e_i\}, 1 \leq i \leq m\) be a local geodesic orthonormal frame of \(M^m\), \(\{k_1, k_2, ..., k_m\}\) its principal curvatures, and \(B\) its second fundamental form. Then using the proof of Corollary 3.2. in [12], we have
\[
\|B\|^2 = m^2H^2 - s + m(m-1)c
\]
and
\[Ricci(\eta, \eta) = mc. \]

By use of Proposition 3.1, we obtain
\[
\Delta H - \left( m^2H^2 - s + m(m-2)c - \frac{\Delta f}{f} + \lambda \right) H - 2\text{grad} \ln f(H) = 0. \] (3.16)

For \(\varphi : M^2 \rightarrow N^3(c)\), substituting \(m = 2\) into equation (3.16), we get the result. 

**Remark 3.1** In Proposition 3.1 and Corollary 3.1, if we take \(f\) as a constant, we obtain Proposition 3.1 and Corollary 3.2 in [12].
4. Examples of $f$-biminimal surfaces

In the present section, we give some examples of $f$-biminimal surfaces. To obtain examples of free $f$-biminimal surfaces, similar to Theorem 2.3 in [15], we state the following theorem:

**Theorem 4.1** $\varphi : (M^2, g) \to (N^n, h)$ is a free $f$-biminimal map if and only if $\varphi : (M^2, f^{-1}g) \to (N^n, h)$ is a free biminimal map.

**Proof** Using equation (1.6), $\varphi : (M^2, g) \to (N^n, h)$ is a free $f$-biminimal map if and only if

$$[\tau_2, f(\varphi, g)]^\perp = f[\tau_2(\varphi, g)]^\perp + \Delta f[\tau(\varphi)]^\perp + 2\left[\nabla_{\text{grad} f} \tau(\varphi, g)\right]^\perp = 0,$$

which is equivalent to

$$[\tau_2(\varphi, g)]^\perp + (\Delta \ln f + \|\text{grad} \ln f\|^2) [\tau(\varphi)]^\perp + 2\left[\nabla_{\text{grad} \ln f} \tau(\varphi)\right]^\perp = 0.$$

Furthermore, by Corollary 1 in [14], the relationship between the bitension field $[\tau_2(\varphi, g)]^\perp$ and that of map $\varphi : (M^2, \bar{g} = F^{-2}g) \to (N^n, h)$ is given by

$$[\tau_2(\varphi, \bar{g})]^\perp = F^4 [\tau_2(\varphi, g)]^\perp + (\Delta \ln F^2 + \|\text{grad} \ln F^2\|^2) [\tau(\varphi)]^\perp + 2\left[\nabla_{\text{grad} \ln F^2} \tau(\varphi)\right]^\perp = 0.$$ 

Then map $\varphi : (M^2, \bar{g} = F^{-2}g) \to (N^n, h)$ is free biminimal if and only if

$$[\tau_2(\varphi, g)]^\perp + (\Delta \ln F^2 + \|\text{grad} \ln F^2\|^2) [\tau(\varphi)]^\perp + 2\left[\nabla_{\text{grad} \ln F^2} \tau(\varphi)\right]^\perp = 0. \quad (4.1)$$

Substituting $F^2 = f$ into equation (4.1), we obtain the result. \hfill \Box

**Examples**

1. Let us consider the cone on a free biminimal curve on $S^2$ with

$$\varphi : (S^2, d\theta^2) \to (\mathbb{R}^3 \setminus \{0\} = \mathbb{R}^+ \times_{S^2} S^2, dt^2 + t^2 d\theta^2).$$

Then it is a free biminimal surface [12], where $\times_{S^2}$ denotes the warped product. Hence, from Theorem 4.1, $\varphi : (S^2, f d\theta^2) \to (\mathbb{R}^3 \setminus \{0\} = \mathbb{R}^+ \times_{S^2} S^2, dt^2 + t^2 d\theta^2)$ is a free $f$-biminimal surface.

2. Let $\beta : I \to \mathbb{R}^2$ be the logarithmic spiral whose curvature $k = \frac{1}{\sqrt{2}s}$ and $\alpha : I \to \mathbb{R}^3$ be a helix of the cylinder on the plane curve $\beta$ with its Frenet frame $\{T, N, B\}$. Then the envelope $S$ of $\alpha$ parametrized by $X : (\mathbb{R}^2, g) \to (\mathbb{R}^3, \bar{g}), \; X(u, s) = \alpha(s) + u(B + T)$ is a free biminimal surface [12]. Hence, from Theorem 4.1, $X : (\mathbb{R}^2, f g) \to (\mathbb{R}^3, \bar{g})$ is a free $f$-biminimal surface.

3. The circular cylinder $\varphi : D = \{(u, v) \in (0, 2\pi) \times \mathbb{R}\} \to \mathbb{R}^3$ with $\varphi(u, v) = (r \cos u, r \sin u, v)$ is an $f$-biminimal surface for $f(u) = C_1 e^{-\sqrt{-1} - \lambda u} + C_2 e^{-\sqrt{-1} - \lambda^2 u}$, where $C_1$ and $C_2$ are real constants. It is easy to see that this surface with $f(u) = C_1 e^{-\sqrt{-1} - \lambda u} + C_2 e^{-\sqrt{-1} - \lambda^2 u}$ is not an $f$-biharmonic surface because if $\varphi$ is $f$-biharmonic, then using Theorem 3.2 of [15] we get $\lambda = 0$. Then the function $f$ is indefinite, so this surface can not be $f$-biharmonic and free $f$-biminimal. Moreover, using Proposition 3.1 of [12], we obtain that $\varphi$ cannot be biminimal unless $\lambda = -\frac{1}{r^2}$. This shows that the $f$-biharmonicity, biminimality, and $f$-biminimality of $\varphi$ are different.
5. \(f\)-Biminimal Legendre curves in Sasakian space forms

Let \((M^{2m+1}, \varphi, \xi, \eta, g)\) be a contact metric manifold. If the Nijenhuis tensor of \(\varphi\) equals \(-2d\eta \otimes \xi\), then \((M^{2m+1}, \varphi, \xi, \eta, g)\) is called a Sasakian manifold [2]. If a Sasakian manifold has constant \(\varphi\)-sectional curvature \(c\), then it is called a Sasakian space form. The curvature tensor of a Sasakian space form is given by

\[
R(X, Y)Z = \frac{c + 3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c - 1}{4} \{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\
+ 2g(X, \varphi Y)\varphi Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
+ g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\xi\}
\]  
(5.1)

for all \(X, Y, Z \in TM\) [3].

A submanifold of a Sasakian manifold is called an integral submanifold if \(\eta(X) = 0\) for every tangent vector \(X\). A 1-dimensional integral submanifold of a Sasakian manifold is called a Legendre curve of \(M\). Hence, a curve \(\gamma : I \rightarrow M = (M^{2m+1}, \varphi, \xi, \eta, g)\) is called a Legendre curve if \(\eta(T) = 0\), where \(T\) is the tangent vector field of \(\gamma\) [3].

We can state the following theorem:

**Theorem 5.1** Let \(\gamma : (a, b) \rightarrow M\) be a nongeodesic Legendre Frenet curve of osculating order \(r\) in a Sasakian space form \(M = (M^{2m+1}, \varphi, \xi, \eta, g)\). Then \(\gamma\) is \(f\)-biminimal if and only if the following three equations hold:

\[
k_1'' - k_1^3 - k_1 k_2^2 + \frac{(c + 3)}{4} k_1 + 2k_1' \frac{f'}{f} + k_1'' \frac{f''}{f} - \lambda k_1 + \frac{3(c - 1)}{4} [k_1 g(\varphi T, E_2)^2]^\perp = 0, \\
k_1' k_2 + (k_1 k_2)' + 2k_1 k_2 \frac{f'}{f} + \frac{3(c - 1)}{4} [k_1 g(\varphi T, E_2)g(\varphi T, E_3)]^\perp = 0, \\
\]

and

\[
k_1 k_2 k_3 + \frac{3(c - 1)}{4} [k_1 g(\varphi T, E_2)g(\varphi T, E_4)]^\perp = 0.
\]

**Proof** Let \(M = (M^{2m+1}, \varphi, \xi, \eta, g)\) be a Sasakian space form and \(\gamma : (a, b) \rightarrow M\) a Legendre Frenet curve of osculating order \(r\). Differentiating \(\eta(T) = 0\) and using Definition 2.1, we obtain

\[
\eta(E_2) = 0.
\]  
(5.2)

Then using equations (5.1) and (5.2), we have

\[
R(T, \nabla_T T)T = -k_1 \left(\frac{c + 3}{4} E_2 - 3k_1 \frac{(c - 1)}{4} g(\varphi T, E_2)\varphi T\right).
\]  
(5.3)

By use of equations (2.5), (2.9), and (5.3) in equation (1.6), we find

\[
\left(k_1'' - k_1^3 - k_1 k_2^2 + \frac{(c + 3)}{4} k_1 + 2k_1' \frac{f'}{f} + k_1'' \frac{f''}{f} - \lambda k_1\right) E_2 + \left(k_1' k_2 + (k_1 k_2)' + 2k_1 k_2 \frac{f'}{f}\right) E_3
\]
\[ + (k_1k_2k_3)E_4 + \frac{3(c - 1)}{4} [k_1g(\varphi T, E_2)\varphi T] \perp = 0. \]  

(5.4)

Then taking the scalar product of equation (5.4) with \( E_2, E_3, \) and \( E_4, \) respectively, we obtain the desired results.

Let us recall some notions about the Sasakian space form \( \mathbb{R}^{2m+1}(-3) \) [3]:

Let us take \( M = \mathbb{R}^{2m+1} \) with the standard coordinate functions \( (x_1, ..., x_m, y_1, ..., y_m, z) \), the contact structure \( \eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i) \), the characteristic vector field \( \xi = 2 \frac{\partial}{\partial z} \), and the tensor field \( \varphi \) given by

\[
\varphi = \begin{bmatrix}
0 & \delta_{ij} & 0 \\
-\delta_{ij} & 0 & 0 \\
0 & y_j & 0
\end{bmatrix}.
\]

The Riemannian metric is \( g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{m} ((dx_i)^2 + (dy_i)^2) \). Then \( (M^{2m+1}, \varphi, \xi, \eta, g) \) is a Sasakian space form with constant \( \varphi \)-sectional curvature \( c = -3 \) and it is denoted by \( \mathbb{R}^{2m+1}(-3) \). The vector fields

\( X_i = 2 \frac{\partial}{\partial y_i}, \ X_{i+m} = \varphi X_i = 2(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}), \ 1 \leq i \leq m, \ \xi = 2 \frac{\partial}{\partial z}, \)  

(5.5)

form a g-orthonormal basis and the Levi-Civita connection is calculated as

\[
\nabla_{X_i}X_j = \nabla_{X_{i+m}}X_j + \delta_{ij} \xi, \ \nabla_{X_{i+m}}X_j = -\delta_{ij} \xi,
\]

\[
\nabla_{X_i} \xi = \nabla_{X_{i+m}} \xi = \nabla_{\xi}X_{i+m} = X_i
\]

(see [2]).

Now let us produce an example of \( f \)-biminimal Legendre curves in \( \mathbb{R}^5(-3) \):

**Example** Let \( \gamma = (\gamma_1, ..., \gamma_5) \) be a unit speed Legendre curve in \( \mathbb{R}^5(-3) \). The tangent vector field of \( \gamma \) is

\[
T = \frac{1}{2} \left\{ \gamma_3'X_1 + \gamma_4'X_2 + \gamma_1'X_3 + \gamma_2'X_4 + (\gamma_5' - \gamma_1'\gamma_3 - \gamma_2'\gamma_4)\xi \right\}.
\]

Using the above equation, since \( \gamma \) is a unit speed Legendre curve, we have \( \eta(T) = 0 \) and \( g(T, T) = 1 \); that is,

\[
\gamma_5' = \gamma_1'\gamma_3 + \gamma_2'\gamma_4
\]

and

\[
(\gamma_1')^2 + ... + (\gamma_5')^2 = 4.
\]

For a Legendre curve, we can use the Levi-Civita connection and equation (5.5) to write

\[
\nabla_T T = \frac{1}{2} \left( \gamma_3''X_1 + \gamma_4''X_2 + \gamma_1''X_3 + \gamma_2''X_4 \right),
\]

(5.6)

\[
\varphi T = \frac{1}{2} \left( -\gamma_1'X_1 - \gamma_2'X_2 + \gamma_3'X_3 + \gamma_4'X_4 \right).
\]

(5.7)
Equations (5.6) and (5.7) and φT ⊥ E2 hold if and only if
\[ \gamma_1'' \gamma_3'' + \gamma_2'' \gamma_4'' = \gamma_3'' \gamma_1'' + \gamma_4'' \gamma_2''. \]

Finally, we can give the following explicit example:

Let us take \( \gamma(t) = (\sin 2t, -\cos 2t, 0, 0, 1) \) in \( \mathbb{R}^5 \). Using the above equations and Theorem 5.1, \( \gamma \) is an \( f \)-biminimal Legendre curve with osculating order \( r = 2, k_1 = 2, f = e^t, \varphi T \perp E_2 \). We can easily check that the conditions of Theorem 5.1 are verified. Using Theorem 3.1 of [7], the curve \( \gamma \) is not \( f \)-biharmonic. For \( \lambda \neq -4 \), it is easy to see that \( \gamma \) is not biminimal. Hence, the biminimality and \( f \)-biminimality of \( \gamma \) are different unless \( \lambda = -4 \).

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References

[1] Aykut DB. Some special curves on surfaces. MSc, Balikesir University, Balikesir, Turkey, 2015.