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Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications

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Abstract: Let a, b be two commutative generalized Drazin invertible elements in a Banach algebra; the expressions for the generalized Drazin inverse of the product ab and the sum $a+b$ were studied in some current literature on this subject. In this paper, we generalize these results under the weaker conditions $a^2b = aba$ and $b^2a = bab$. As an application of our results, we obtain some new representations for the generalized Drazin inverse of a block matrix with the generalized Schur complement being generalized Drazin invertible in a Banach algebra, extending some recent works.

Key words: Generalized Drazin inverse, Banach algebra, additive result, block matrix

1. Introduction

The generalized Drazin inverse in a Banach algebra was introduced in [10]. The expressions for the generalized Drazin inverse of the product and the sum were studied by many authors. For instance, in [10], for two commutative generalized Drazin invertible elements a, b in a Banach algebra, Koliha gave the expression of $(ab)^d$. Meanwhile, the representation of $(a+b)^d$ was obtained under the conditions $ab = ba = 0$ in a Banach algebra. Later, Djordjević and Wei [8] gave the expression of $(a+b)^d$ under the assumption $ab = 0$ in the context of the Banach algebra of all bounded linear operators on an arbitrary complex Banach space. In [1], Castro-González and Koliha obtained a formula for $(a+b)^d$ under the conditions $a^\pi b = b, ab^\pi = a, b^\pi aba^\pi = 0$, which are weaker than $ab = 0$ in Banach algebras. In [6], Deng and Wei derived necessary and sufficient conditions for the existence of $(P+Q)^d$ under the condition $PQ = QP$, where P, Q are bounded linear operators. Moreover, the expression of $(P+Q)^d$ was given. In [3], Cvetković-Ilić et al. extended the result of [6] to Banach algebras. More results on generalized Drazin inverse can be found in [2, 4, 7, 8, 12, 14].

In [13], Liu et al. deduced the explicit expressions for the Drazin inverses of the product ab and the sum $a+b$ under the conditions $a^2b = aba$ and $b^2a = bab$, where a and b are complex matrices. In [18], the corresponding results of [13] were studied for the pseudo Drazin inverse (which is a special case of generalized Drazin inverse [17]) in a Banach algebra. In this paper, we will further consider the results of [13] and [18] for the generalized Drazin inverse, which extend [10, Theorem 5.5] and [3, Theorem 2.1].

Another relevant topic is to establish a representation for the generalized Drazin inverse of a block matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in terms of its blocks under certain conditions. The generalized Schur complement $S = D - CA^dB$

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plays an important role in the representation for M^d. Here we list partially some conditions as follows:

(1) S is invertible, $A^\pi BC = 0$, $CA^\pi B = 0$, and $AA^\pi B = A^\pi BD$ (see [5]);

(2) S is invertible, $BCA^\pi = 0$, $CA^\pi B = 0$, and $CAA^\pi = DCA^\pi$ (see [5]);

(3) S is generalized Drazin invertible, $BCA^\pi = 0$, $DCA^\pi = 0$, $S^\pi CA = 0$, and $ABS^\pi = 0$ (see [16]);

(4) S is generalized Drazin invertible, $A^\pi B = 0$, and $S^\pi CA = 0$ (see [15]).

In this paper, we will extend the above results under weaker conditions as applications of our additive result.

2. Preliminaries

Throughout this paper, $\mathscr{A}$ denotes a complex Banach algebra with unity 1. For $a \in \mathscr{A}$, denote the spectrum and the spectral radius of a by $\sigma(a)$ and $r(a)$, respectively. $\mathscr{A}^{-1}$ and $\mathscr{A}^{qnil}$ stand for the sets of all invertible and quasinilpotent elements ($\sigma(a) = \{0\}$) in $\mathscr{A}$, respectively. The commutant of an element $a \in \mathscr{A}$ is defined by comm(a)=$\{b \in \mathscr{A} : ab = ba\}$. In addition, denote by C_n^k the binomial coefficient $\frac{n!}{k!(n-k)!}$ $(0 \leq k \leq n)$.

For the readers' convenience, we first recall the definitions of some generalized inverses. The generalized Drazin inverse [10] of $a \in \mathscr{A}$ (or Koliha–Drazin inverse of a) is the element $x \in \mathscr{A}$ that satisfies

$$xax = x, \quad ax = xa \ \text{ and } \ a - a^2x \in \mathscr{A}^{qnil}.$$

Such x, if it exists, is unique and will be denoted by a^d. It is well known that $a \in \mathscr{A}$ has a generalized Drazin inverse if and only if 0 is not an accumulation point of $\sigma(a)$. Let $\mathscr{A}^d$ denote the set of all generalized Drazin invertible elements in $\mathscr{A}$. If $a \in \mathscr{A}^d$, the spectral idempotent a^π of a corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. In this case, the resolvent $R(\lambda, a) = (\lambda 1 - a)^{-1}$ has a Laurent series

$$R(\lambda, a) = \sum_{n=1}^{\infty} \lambda^{-n} a^{n-1} a^\pi - \sum_{n=0}^{\infty} \lambda^n (a^d)^{n+1},$$

on some punctured disc $\{\lambda : 0 < |\lambda| < r\}, r > 0$ (see [10, Theorem 5.1]).

The group inverse of $a \in \mathscr{A}$ is the element $x \in \mathscr{A}$ that satisfies

$$axa = a, \quad xax = x \ \text{ and } \ ax = xa.$$

If the group inverse of a exists, it is unique and denoted by $a^\#$.

Let $p \in \mathscr{A}$ be an idempotent ($p^2 = p$). Then we can represent element $a \in \mathscr{A}$ as

$$a = \begin{bmatrix} a_1 & a_3 \\ a_4 & a_2 \end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = (1-p)a(1-p)$, $a_3 = pa(1-p)$, and $a_4 = (1-p)ap$.

It is well known that if $a \in \mathscr{A}^d$, then we have the following matrix representations:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \ \text{ and } \ a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathscr{A}p)^{-1}$, and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$.

Now we present two useful lemmas, which play an important role in the sequel.

Lemma 2.1 [1, Theorem 2.3] *Let $p^2 = p$, $x, y \in \mathscr{A}$ and let x and y have the representations*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p, \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p}. \tag{1}$$

(i) *If $a \in (p\mathscr{A}p)^d$ and $b \in ((1-p)\mathscr{A}(1-p))^d$, then $x, y \in \mathscr{A}^d$ and*

$$x^d = \begin{bmatrix} a^d & u \\ 0 & b^d \end{bmatrix}_p, \quad y^d = \begin{bmatrix} b^d & 0 \\ u & a^d \end{bmatrix}_{1-p}, \tag{2}$$

where

$$u = \sum_{n=0}^{\infty}(a^d)^{n+2}cb^nb^\pi + \sum_{n=0}^{\infty}a^\pi a^n c(b^d)^{n+2} - a^dcb^d. \tag{3}$$

(ii) *If $x \in \mathscr{A}^d$ [resp. $y \in \mathscr{A}^d$] and $a \in (p\mathscr{A}p)^d$, then $b \in ((1-p)\mathscr{A}(1-p))^d$, and x^d [resp. y^d] is given by* (2) *and* (3).

Lemma 2.2 [10, Theorem 5.5] *Let $a, b \in \mathscr{A}^d$ be such that ab=ba. Then $ab \in \mathscr{A}^d$ and $(ab)^d = a^db^d$.*

Next, the commuting property for the generalized Drazin inverse is investigated in a Banach algebra.

Theorem 2.3 *Let $a, b \in \mathscr{A}^d$ and $c \in \mathscr{A}$. If $ca = bc$, then $ca^d = b^dc$.*

Proof Suppose that $a, b \in \mathscr{A}^d$ and $ca = bc$, for any $n \in \mathbb{N}$, we have the following equations:

$$\begin{aligned} bb^dc - bb^dcaa^d &= bb^dc(1-aa^d) = (bb^d)^nc(1-aa^d) \\ &= (b^d)^n(b^nc)(1-aa^d) = (b^d)^n(ca^n)(1-aa^d), \end{aligned}$$

which imply

$$\|bb^dc - bb^dcaa^d\|^{\frac{1}{n}} = \|(b^d)^nca^n(1-aa^d)\|^{\frac{1}{n}} \le \|b^d\|\|c\|^{\frac{1}{n}}\|a^n(1-aa^d)\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

Thus, $bb^dc = bb^dcaa^d$, i.e. $b^dc = b^dcaa^d$.

On the other hand, we have that

$$\begin{aligned} ca^da - b^dcaa^da &= ca^da - b^dbca^da = (1-bb^d)ca^da = (1-bb^d)c(a^da)^n \\ &= (1-bb^d)(ca^n)(a^d)^n = (1-bb^d)(b^nc)(a^d)^n. \end{aligned}$$

Then we obtain

$$\|caa^d - b^dcaa^da\|^{\frac{1}{n}} = \|(1-bb^d)b^nc(a^d)^n\|^{\frac{1}{n}} \le \|b^n(1-bb^d)\|^{\frac{1}{n}}\|c\|^{\frac{1}{n}}\|a^d\| \xrightarrow{n\to\infty} 0.$$

Thus, $caa^d = b^dcaa^da$, i.e. $ca^d = b^dcaa^d$. Therefore, we deduce that $ca^d = b^dc$. □

Corollary 2.4 [10, Theorem 4.4] *Let $a \in \mathscr{A}^d$ and $c \in \mathscr{A}$. If $ca = ac$, then $ca^d = a^dc$.*

The following lemmas will also be useful.

Lemma 2.5 *Let $a, b \in \mathscr{A}^d$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

(i) $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a) \cap comm(a^d)$.

(ii) $\{ba, b^da, ba^d, b^da^d\} \subseteq comm(b) \cap comm(b^d)$.

Proof (i) By Corollary 2.4, it suffices to prove $\{ab, a^db, ab^d, a^db^d\} \subseteq comm(a)$.

Since $a^2b = aba$, then $(a^db)a = (a^d)^2aba = (a^d)^2a^2b = a(a^db)$.

Note that $bab^d = b^dba$, and we get $a(ab^d) = a^2b(b^d)^2 = aba(b^d)^2 = a(b^d)^2ba = (ab^d)a$, which implies $a(a^db^d) = (a^d)^2a(ab^d) = (a^d)^2(ab^d)a = (a^db^d)a$.

(ii) It is analogous to the proof of (i). □

Remark 2.6 *In Lemma 2.5, the conditions $a^2b = aba$ and $b^2a = bab$ are weaker than $ab = ba$. Indeed, it is clear that $ab = ba$ can imply $a^2b = aba$ and $b^2a = bab$. However, in general, the converse is false. The following example can illustrate this fact.*

Example 2.7 *Let $\mathscr{A}$ be the Banach algebra of all complex 3×3 matrices, and take*

$$a = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Clearly, $a^2b = aba$ and $b^2a = bab$. However, $ab \neq ba$.

Remark 2.8 *We have seen that if $a \in \mathscr{A}^d$, $b \in \mathscr{A}$, and $ab = ba$, then $a^db = ba^d$. However, under the conditions of Lemma 2.5, $a^db = ba^d$ may not be true, which can also be illustrated by the previous Example 2.7. Note that $a^3 = a$ and $b^3 = b$; then $a^d = a$ and $b^d = b$. However, $a^db \neq ba^d$.*

The next result was proved for complex matrices (see [13, Lemma 2.3]). Indeed, it is true in a Banach algebra.

Lemma 2.9 *Let $a, b \in \mathscr{A}$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

$$(a+b)^n = \sum_{i=0}^{n-1} C_{n-1}^i (a^{n-i}b^i + b^{n-i}a^i), \quad \text{where } n \in \mathbb{N}.$$

Next, we establish two crucial auxiliary results.

Lemma 2.10 *Let $a, b \in \mathscr{A}$ be such that $a^2b = aba$ and $b^2a = bab$. Then*

(i) $r(a+b) \leqslant r(a) + r(b)$.

(ii) *If both a and b are quasinilpotent, then $a+b$ is quasinilpotent.*

Proof (i) Take any $\alpha > r(a)$ and $\beta > r(b)$. Let $a_1 = \frac{1}{\alpha}a$ and $b_1 = \frac{1}{\beta}b$. Then $r(a_1) < 1$ and $r(b_1) < 1$. From

Lemma 2.9, we have that

$$\begin{aligned}
\|(a+b)^{n+1}\| &= \|\sum_{i=0}^{n} C_n^i (a^{n+1-i}b^i + b^{n+1-i}a^i)\| \\
&= \|a\sum_{i=0}^{n} C_n^i a^{n-i}b^i + b\sum_{i=0}^{n} C_n^i b^{n-i}a^i\| \\
&\leq \|a\|\sum_{i=0}^{n} C_n^i \|a^{n-i}\|\|b^i\| + \|b\|\sum_{i=0}^{n} C_n^i \|b^{n-i}\|\|a^i\| \\
&= (\|a\|+\|b\|)\sum_{i=0}^{n} C_n^i \|a^i\|\|b^{n-i}\| \\
&= (\|a\|+\|b\|)\sum_{i=0}^{n} C_n^i \alpha^i\beta^{n-i}\|a_1^i\|\|b_1^{n-i}\|.
\end{aligned}$$

For each n, choose $n', n'' \in \mathbb{N}$ such that $n'+n''=n$ and $\|a_1^{n'}\|\|b_1^{n''}\| = \max_{0\leq i\leq n}\|a_1^i\|\|b_1^{n-i}\|$, then we have

$$\|(a+b)^{n+1}\| \leq (\|a\|+\|b\|)(\alpha+\beta)^n\|a_1^{n'}\|\|b_1^{n''}\|,$$

which implies

$$\begin{aligned}
r(a+b) &= \lim_{n\to\infty}(\|(a+b)^{n+1}\|^{\frac{1}{n+1}})^{\frac{n+1}{n}} = \lim_{n\to\infty}\|(a+b)^{n+1}\|^{\frac{1}{n}} \\
&\leq (\alpha+\beta)\lim_{n\to\infty}(\|a\|+\|b\|)^{\frac{1}{n}}\liminf_{n\to\infty}\|a_1^{n'}\|^{\frac{1}{n}}\|b_1^{n''}\|^{\frac{1}{n}} \\
&= (\alpha+\beta)\liminf_{n\to\infty}\|a_1^{n'}\|^{\frac{1}{n}}\|b_1^{n''}\|^{\frac{1}{n}}.
\end{aligned}$$

According to the proof of [9, Lemma 1.2.13], we obtain $r(a+b) \leq \alpha+\beta$, which yields $r(a+b) \leq r(a)+r(b)$.

(ii) This can be obtained by (i). □

Lemma 2.11 *Let $a, b \in \mathscr{A}$ be such that $a^2b = aba$ or $b^2a = bab$. Then*

(i) $r(ab) \leqslant r(a)r(b)$.

(ii) *If either a or b is quasinilpotent, then ab is quasinilpotent.*

Proof (i) Note the symmetry of $a^2b = aba$ and $b^2a = bab$, it suffices to prove the case $a^2b = aba$.

Assume $a^2b = aba$; then $(ab)^n = a^nb^n$ for $n \in \mathbb{N}$ by induction. Therefore,

$$\|(ab)^n\|^{\frac{1}{n}} = \|a^nb^n\|^{\frac{1}{n}} \leq \|a^n\|^{\frac{1}{n}}\|b^n\|^{\frac{1}{n}}.$$

Let $n \to \infty$; then we obtain that $r(ab) \leqslant r(a)r(b)$.

(ii) This follows from (i) directly. □

3. Main results

In this section, for $a, b \in \mathscr{A}^d$, we will investigate the representations of $(ab)^d$ and $(a+b)^d$ under the new conditions $a^2b = aba$ and $b^2a = bab$.

We start with a theorem that is an extension of [10, Theorem 5.5].

Theorem 3.1 *Let $a, b \in \mathscr{A}^d$ be such that $a^2b = aba$ and $b^2a = bab$. Then $ab \in \mathscr{A}^d$ and $(ab)^d = a^db^d$.*

Proof We consider the matrix representations of a and b relative to the idempotent $p = aa^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $a_1 \in (p\mathscr{A}p)^{-1}$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$.

The condition $a^2b = aba$ expressed in matrix form yields

$$\begin{bmatrix} a_1^2b_1 & a_1^2b_3 \\ a_2^2b_4 & a_2^2b_2 \end{bmatrix}_p = a^2b = aba = \begin{bmatrix} a_1b_1a_1 & a_1b_3a_2 \\ a_2b_4a_1 & a_2b_2a_2 \end{bmatrix}_p.$$

Thus, we have $a_1^2b_3 = a_1b_3a_2$, i.e. $b_3 = a_1^{-1}b_3a_2$, which implies $b_3 = a_1^{-n}b_3a_2^n$ for any $n \in \mathbb{N}$. Since $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$, then

$$\|b_3\|^{\frac{1}{n}} = \|a_1^{-n}b_3a_2^n\|^{\frac{1}{n}} \leqslant \|a_1^{-1}\|\|b_3\|^{\frac{1}{n}}\|a_2^n\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

Hence, $b_3 = 0$. Similarly, from $a_2b_4 = a_2^2b_4a_1^{-1}$, it follows that $a_2b_4 = 0$. In addition, we can get $a_1b_1 = b_1a_1$ and $a_2^2b_2 = a_2b_2a_2$. Then we have

$$b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p \text{ and } ab = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix}_p.$$

Next, we prove that $b_1 \in (p\mathscr{A}p)^d$ and $b_1^d = aa^db^daa^d$ by the definition of generalized Drazin inverse. Note that $b_1 = aa^dbaa^d = aa^db$ and $aa^db^daa^d = aa^db^d$ by Lemma 2.5(i). Therefore, we need to prove $b_1^d = aa^db^d$.

Let $v = aa^db^d$. Then we have

(1) $b_1v = aa^dbaa^db^d = aa^dbb^d = aa^db^daa^db = vb_1$.

(2) $vb_1v = aa^db^daa^dbaa^db^d = aa^db^dbaa^db^d = aa^dbab^da^db^d = aa^dba^dab^db^d = aa^db^d = v$.

(3) Note that $b_1 - b_1^2v = aa^db(1-bb^d)$. By induction and Lemma 2.5, we have that $(aa^db(1-bb^d))^n = aa^db^n(1-bb^d)$ for any $n \in \mathbb{N}$. Since $b(1-bb^d) \in \mathscr{A}^{qnil}$, then

$$\|(b_1 - b_1^2v)^n\|^{\frac{1}{n}} = \|aa^db^n(1-bb^d)\|^{\frac{1}{n}} \leq \|aa^d\|^{\frac{1}{n}}\|b^n(1-bb^d)\|^{\frac{1}{n}} \xrightarrow{n\to\infty} 0.$$

Thus $b_1 - b_1^2v \in (p\mathscr{A}p)^{qnil}$. Hence, $b_1^d = v$. Similarly, we have that $b_2^d = b^d(1-aa^d)$.

According to the equation $a_1b_1 = b_1a_1$ and Lemma 2.2, we have that $a_1b_1 \in (p\mathscr{A}p)^d$ and $(a_1b_1)^d = a_1^{-1}b_1^d$. Observe that $a_2^2b_2 = a_2b_2a_2$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$; applying Lemma 2.11(ii) to the elements a_2, b_2, we get $a_2b_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$, i.e. $(a_2b_2)^d = 0$.

Finally, applying Lemma 2.1(i), we have $ab \in \mathscr{A}^d$ and

$$(ab)^d = \begin{bmatrix} (a_1b_1)^d & 0 \\ 0 & (a_2b_2)^d \end{bmatrix}_p = \begin{bmatrix} a_1^{-1}b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p = a^db^d.$$

□

Remark 3.2 *(1) From Lemma 2.2 and Corollary 2.4, we can see that* $(ab)^d = a^d b^d = b^d a^d$ *for commutative generalized Drazin invertible elements* $a, b \in \mathscr{A}$*. However, in general,* $(ab)^d \neq b^d a^d$ *under the conditions of Theorem 3.1. For example, let* a, b *be the same as the elements in Example 2.7. Clearly,*

$$ab = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (ab)^d.$$

However, $(ab)^d \neq b^d a^d$*.*

(2) In Theorem 3.1, if we replace $b^2 a = bab$ *with* $ba^2 = aba$*, then we can conclude that* $(ab)^d = a^d b^d = b^d a^d$*. The proof of the previous result is similar to the proof of Theorem 3.1 and so we omit the proof. The following example shows that the conditions* $a^2 b = aba$ *and* $ba^2 = aba$ *are weaker than* $ab = ba$*. Let* $\mathscr{A} = M_2(\mathbb{C})$ *and take*

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then we can get that $a^2 b = aba$ *and* $ba^2 = aba$*. However,* $ab \neq ba$*.*

Next, we present our main result, which recovers [3, Theorem 2.1].

Theorem 3.3 *Let* $a, b \in \mathscr{A}^d$ *be such that* $a^2 b = aba$ *and* $b^2 a = bab$*. Then the following conditions are equivalent:*

(i) $a + b \in \mathscr{A}^d$.

(ii) $1 + a^d b \in \mathscr{A}^d$.

(iii) $c = aa^d(a+b)bb^d \in \mathscr{A}^d$.

In this case,

$$\begin{aligned} (a+b)^d = {} & a^d(1+a^d b)^d + a^\pi b (a^d)^2((1+a^d b)^d)^2 + \sum_{n=0}^{\infty} (b^d)^{n+1}(-a)^n a^\pi \\ & + \sum_{n=0}^{\infty} (n+1) b^\pi a (b^d)^{n+2} (-a)^n a^\pi, \end{aligned} \tag{4}$$

$$\begin{aligned} (a+b)^d = {} & c^d + \sum_{n=0}^{\infty} (a^d)^{n+1}(-b)^n b^\pi + a^\pi b (c^d)^2 + \sum_{n=0}^{\infty} a^\pi b c^d (a^d)^{n+1}(-b)^n b^\pi \\ & + \sum_{n=0}^{\infty} a^\pi b (a^d)^{n+1}(-b)^n b^\pi c^d + \sum_{n=0}^{\infty} (n+1) a^\pi b (a^d)^{n+2}(-b)^n b^\pi \\ & + \sum_{n=0}^{\infty} (b^d)^{n+1}(-a)^n a^\pi + \sum_{n=0}^{\infty} (n+1) b^\pi a (b^d)^{n+2}(-a)^n a^\pi, \end{aligned} \tag{5}$$

$$(1+a^d b)^d = a^\pi + a^2 a^d (a+b)^d \text{ and } (aa^d(a+b)bb^d)^d = aa^d(a+b)^d bb^d. \tag{6}$$

Proof As in the proof of Theorem 3.1, we have that

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathscr{A}p)^{-1}$, and $a_2 \in ((1-p)\mathscr{A}(1-p))^{qnil}$. Moreover, we have $a_1b_1 = b_1a_1$, $a_2b_4 = 0$, $a_2^2b_2 = a_2b_2a_2$, $b_1^d = aa^db^d$, and $b_2^d = b^d(1-aa^d)$. From the condition $b^2a = bab$, it follows that $b_2^2a_2 = b_2a_2b_2$ and $b_2b_4 = 0$.

Let $p_1 = b_1b_1^d$ and $p_2 = b_2b_2^d$. Then $p_1p = pp_1 = p_1$ and $p_2(1-p) = (1-p)p_2 = p_2$ by Lemma 2.5. We now consider the matrix representations of b_1 and b_2 relative to idempotents p_1 and p_2, respectively. We have that

$$b_1 = \begin{bmatrix} b_1' & 0 \\ 0 & b_2' \end{bmatrix}_{p_1} \text{ and } b_2 = \begin{bmatrix} b_1'' & 0 \\ 0 & b_2'' \end{bmatrix}_{p_2},$$

where $b_1' \in (p_1\mathscr{A}p_1)^{-1}$, $b_1'' \in (p_2\mathscr{A}p_2)^{-1}$, $b_2' \in ((p-p_1)\mathscr{A}(p-p_1))^{qnil}$, and $b_2'' \in ((1-p-p_2)\mathscr{A}(1-p-p_2))^{qnil}$.

Note that $p_1a_1(p-p_1) = b_1b_1^da_1(p-b_1b_1^d) = b_1a_1b_1^d(p-b_1b_1^d) = b_1a_1(b_1^d - b_1^db_1b_1^d) = 0$. Similarly, $(p-p_1)a_1p_1 = 0$ and $p_2a_2(1-p-p_2) = 0$. Thus, we get the following matrix representations:

$$a_1 = \begin{bmatrix} a_1' & 0 \\ 0 & a_2' \end{bmatrix}_{p_1} \text{ and } a_2 = \begin{bmatrix} a_1'' & 0 \\ a_4'' & a_2'' \end{bmatrix}_{p_2}.$$

Note that $a_2^2b_2 = a_2b_2a_2$ and $b_2^2a_2 = b_2a_2b_2$; as in the proof of Theorem 3.1, we have that $b_1''a_1'' = a_1''b_1''$, $(b_2'')^2a_2'' = b_2''a_2''b_2''$ and $(a_2'')^2b_2'' = a_2''b_2''a_2''$. Moreover, $(a_1'')^d = p_2a_2^d = 0$ and $(a_2'')^d = a_2^d(1-p-p_2) = 0$, which imply a_1'' and a_2'' are quasinilpotent. Besides these, $b_2''a_4'' = a_2''a_4'' = 0$.

Next, we will prove that $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^d$. Observe that

$$a_2 + b_2 = \begin{bmatrix} a_1'' + b_1'' & 0 \\ a_4'' & a_2'' + b_2'' \end{bmatrix}_{p_2}.$$

Since $a_1'' + b_1'' = b_1''(p_2 + (b_1'')^{-1}a_1'')$ and a_1'' is quasinilpotent, we have that $a_1'' + b_1''$ is invertible in subalgebra $p_2\mathscr{A}p_2$ and

$$(a_1'' + b_1'')^{-1} = (b_1'')^{-1}(p_2 + (b_1'')^{-1}a_1'')^{-1} = (b_1'')^{-1}(p_2 + \sum_{n=1}^{\infty}(b_1'')^{-n}(-a_1'')^n).$$

Note that $(b_1'')^{-1} = b_2^d = b^d(1-aa^d)$. By induction, we can obtain that $(b_1'')^{-n} = (b^d)^n(1-aa^d)$ for any $n \in \mathbb{N}$. In addition, we verify that

$$a_1'' = p_2a_2p_2 = b_2b_2^da_2b_2b_2^d = b_2b_2^da_2 = (ba^\pi)(b^da^\pi)(a^\pi a) = bb^da^\pi a,$$

which implies $(-a_1'')^n = bb^d(-a)^na^\pi$ for any $n \in \mathbb{N}$ by induction. Note that $a^\pi bb^da^\pi = bb^da^\pi$ and $p_2 = b_2b_2^d =$

$ba^\pi b^d a^\pi = bb^d a^\pi$; then we get

$$\begin{aligned}(a_1''+b_1'')^{-1} &= b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty}(b^d)^n a^\pi (bb^d(-a)^n a^\pi)\\ &= b^d a^\pi (bb^d a^\pi + \sum_{n=1}^{\infty}(b^d)^n bb^d(-a)^n a^\pi)\\ &= b^d a^\pi bb^d a^\pi + b^d a^\pi \sum_{n=1}^{\infty}(b^d)^n(-a)^n a^\pi\\ &= b^d a^\pi + \sum_{n=1}^{\infty}(b^d)^{n+1}(-a)^n a^\pi\\ &= \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi.\end{aligned}$$

Applying Lemma 2.10(ii) to the element a_2'', b_2'', we have that $a_2''+b_2''$ is quasinilpotent, i.e. $(a_2''+b_2'')^d = 0$. Lemma 2.1(i) ensures that $a_2+b_2 \in ((1-p)\mathscr{A}(1-p))^d$ and

$$(a_2+b_2)^d = \begin{bmatrix} (a_1''+b_1'')^{-1} & 0 \\ x & 0 \end{bmatrix}_{p_2},$$

where $x = a_4''(a_1''+b_1'')^{-2}$. Note that

$$a_4'' = (1-p-p_2)a_2p_2 = (b^\pi a^\pi)(a^\pi a)(bb^d a^\pi) = b^\pi a a^\pi bb^d a^\pi = b^\pi abb^d a^\pi.$$

Because $a^\pi(b^d)^n a^\pi = (b^d)^n a^\pi$ for any $n \in \mathbb{N}$, then

$$\begin{aligned}x &= b^\pi abb^d a^\pi (\sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi)^2\\ &= b^\pi abb^d a^\pi (\sum_{n=0}^{\infty}(n+1)(b^d)^{n+2}(-a)^n a^\pi)\\ &= b^\pi a \sum_{n=0}^{\infty}(n+1)(b^d)^{n+2}(-a)^n a^\pi.\end{aligned}$$

Therefore, we can obtain

$$(a_2+b_2)^d = \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi + b^\pi a \sum_{n=0}^{\infty}(n+1)(b^d)^{n+2}(-a)^n a^\pi.$$

Since

$$a+b = \begin{bmatrix} a_1+b_1 & 0 \\ b_4 & a_2+b_2 \end{bmatrix}_p,$$

by Lemma 2.1, we have that $a+b \in \mathscr{A}^d$ if and only if $a_1+b_1 \in (p\mathscr{A}p)^d$. In this case, we have

$$(a+b)^d = \begin{bmatrix} (a_1+b_1)^d & 0 \\ y & (a_2+b_2)^d \end{bmatrix}_p,$$

where $y = b_4((a_1+b_1)^d)^2$.

(i) $\Leftrightarrow$ (ii) From

$$1+a^d b = \begin{bmatrix} p+a_1^{-1}b_1 & 0 \\ 0 & 1-p \end{bmatrix}_p,$$

it follows that $1+a^db \in \mathscr{A}^d$ if and only if $p+a_1^{-1}b_1 \in (p\mathscr{A}p)^d$. By Lemma 2.2, we have that $a_1+b_1 = a_1(p+a_1^{-1}b_1) \in (p\mathscr{A}p)^d$ if and only if $p+a_1^{-1}b_1 \in (p\mathscr{A}p)^d$. Hence, $a+b \in \mathscr{A}^d$ if and only if $1+a^db \in \mathscr{A}^d$. In this case, we have

$$(1+a^db)^d = \begin{bmatrix} (p+a_1^{-1}b_1)^d & 0 \\ 0 & 1-p \end{bmatrix}_p .$$

Moreover, we deduce that

$$(a_1+b_1)^d = a_1^{-1}(p+a_1^{-1}b_1)^d = a^d((1+a^db)^d-(1-p)) = a^d(1+a^db)^d.$$

By a straightforward computation, we obtain that the equation (4) holds.

(i) $\Leftrightarrow$ (iii) From $a_1 \in (p\mathscr{A}p)^{-1}$, we have $a_1' \in (p_1\mathscr{A}p_1)^{-1}$ and $a_2' \in ((p-p_1)\mathscr{A}(p-p_1))^{-1}$. Note that $a_2'b_2' = b_2'a_2'$ and b_2' is quasinilpotent; then $a_2'+b_2' = a_2'((p-p_1)+(a_2')^{-1}b_2')$ is invertible in subalgebra $(p-p_1)\mathscr{A}(p-p_1)$ and $(a_2'+b_2')^{-1} = \sum\limits_{n=0}^{\infty}(a^d)^{n+1}(-b)^nb^\pi$, which is similar to the proof of the expression for $(a_1''+b_1'')^{-1}$. Since

$$a_1+b_1 = \begin{bmatrix} a_1'+b_1' & 0 \\ 0 & a_2'+b_2' \end{bmatrix}_{p_1},$$

we have $a_1+b_1 \in (p\mathscr{A}p)^d$ if and only if $a_1'+b_1' \in (p_1\mathscr{A}p_1)^d$. In this case,

$$(a_1+b_1)^d = (a_1'+b_1')^d + (a_2'+b_2')^{-1}.$$

The following matrix representations

$$c = aa^d(a+b)bb^d = \begin{bmatrix} (a_1+b_1)b_1b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p \text{ and } (a_1+b_1)b_1b_1^d = \begin{bmatrix} a_1'+b_1' & 0 \\ 0 & 0 \end{bmatrix}_{p_1}$$

yield the equality $c = a_1'+b_1'$. Therefore, we conclude that $a+b \in \mathscr{A}^d$ if and only if $c \in \mathscr{A}^d$. In this case, we have

$$y = a^\pi b((c^d)^2 + \sum_{n=0}^{\infty} c^d(a^d)^{n+1}(-b)^nb^\pi + \sum_{n=0}^{\infty}(a^d)^{n+1}(-b)^nb^\pi c^d + \sum_{n=0}^{\infty}(n+1)(a^d)^{n+2}(-b)^nb^\pi),$$

and the equation (5) holds. Finally, the equation (6) can be obtained by an elemental computation. □

Next, we consider some specializations of our main result.

Corollary 3.4 [3, Theorem 2.1] *Let $a, b \in \mathscr{A}^d$ be such that $ab = ba$ Then $a+b \in \mathscr{A}^d$ if and only if $1+a^db \in \mathscr{A}^d$. In this case,*

$$(a+b)^d = a^d(1+a^db)^dbb^d + \sum_{n=0}^{\infty} b^\pi(-b)^n(a^d)^{n+1} + \sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^na^\pi. \tag{7}$$

Proof Only the expression for $(a+b)^d$ needs a proof. It follows directly from (6) that $(aa^d(a+b)bb^d)^d = a^d(1+a^db)^dbb^d$. Note that $a^\pi a^d = 0$ and $b^\pi b^d = 0$; then the equation (7) holds by (5). □

Corollary 3.5 *Let $a,b\in\mathscr{A}^d$ be such that $a^2b=aba$ and $b^2a=bab$.*

(i) *If $1\notin\sigma(-a^db)$ (or $\sigma(a^db)=\{0\}$), then $a+b\in\mathscr{A}^d$,*

$$(a+b)^d=a^d(1+a^db)^{-1}+a^\pi b(a^d)^2(1+a^db)^{-2}+\sum_{n=0}^{\infty}(b^d)^{n+1}(-a)^n a^\pi+\sum_{n=0}^{\infty}(n+1)b^\pi a(b^d)^{n+2}(-a)^n a^\pi,$$

and

$$(1+a^db)^{-1}=a^\pi+a^2a^d(a+b)^d.$$

(ii) *If $\sigma(b)=\{0\}$, then $a+b\in\mathscr{A}^d$ and*

$$(a+b)^d=a^d(1+a^db)^{-1}+a^\pi b(a^d)^2(1+a^db)^{-2}=\sum_{n=0}^{\infty}(a^d)^{n+1}(-b)^n+\sum_{n=0}^{\infty}(n+1)a^\pi b(a^d)^{n+2}(-b)^n.$$

Proof (i) This follows from Theorem 3.3 directly.

(ii) Since $\sigma(b)=\{0\}$, then $b\in\mathscr{A}^{qnil}$, i.e. $b^d=0$, which implies $aa^d(a+b)bb^d=0$. Thus, we have that $a+b\in\mathscr{A}^d$ by Theorem 3.3. To show that $1+a^db\in\mathscr{A}^{-1}$, it suffices to prove that $a^db\in\mathscr{A}^{qnil}$. From Lemma 2.5(i), it follows that $(a^d)^2b=a^dba^d$, which yields $a^db\in\mathscr{A}^{qnil}$ by Lemma 2.11(ii). The expressions of $(a+b)^d$ can be obtained by the equations (4) and (5). □

4. Applications to block matrices

Let

$$x=\begin{bmatrix} a & b \\ c & d \end{bmatrix}_p\in\mathscr{A} \tag{8}$$

relative to idempotent $p\in\mathscr{A}$, $a\in(p\mathscr{A}p)^d$, and let $s=d-ca^db\in((1-p)\mathscr{A}(1-p))^d$ be the generalized Schur complement of a in x.

In this section, we get some representations for the generalized Drazin inverse of a block matrix x with applications of our previous result.

For future reference we state two known results.

Lemma 4.1 [1, Example 4.5] *Let $a,b\in\mathscr{A}^d$. If $ab=0$, then $a+b\in\mathscr{A}^d$ and*

$$(a+b)^d=\sum_{n=0}^{\infty}(b^d)^{n+1}a^na^\pi+\sum_{n=0}^{\infty}b^\pi b^n(a^d)^{n+1}.$$

Lemma 4.2 [11, Lemma 2.1] *Let x be defined as in (8). Then the following statements are equivalent:*

(i) *$x\in\mathscr{A}^d$ and $x^d=r$, where*

$$r=\begin{bmatrix} a^d+a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix}; \tag{9}$$

(ii) $a^\pi bs^d=a^dbs^\pi$, $s^\pi ca^d=s^dca^\pi$, *and* $y=\begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix}\in\mathscr{A}^{qnil}$.

Note that, in Lemma 4.2, if $y=0$, then we can check that $xrx=x$, and so that we have the following corollary.

Corollary 4.3 *Let x be defined as in (8). If $a^\pi bs^d=a^dbs^\pi$, $s^\pi ca^d=s^dca^\pi$, and $y=\begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix}=0$, then $x\in\mathscr{A}^\#$ and*

$$x^\# = \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix}.$$

Remark 4.4 *For item (ii) of Lemma 4.2, we can see that $a^\pi bs^d=a^dbs^\pi$ is equivalent to $a^\pi bs^d=a^dbs^\pi=0$. Moreover, $s^\pi ca^d=s^dca^\pi$ is equivalent to $s^\pi ca^d=s^dca^\pi=0$. Now, we drop any one of the four equations $a^\pi bs^d=0$, $a^dbs^\pi=0$, $s^\pi ca^d=0$, $s^dca^\pi=0$ and replace the quasinilpotency by the generalized Drazin invertibility of y. Here, we only give the one of the four cases. Similarly, we can prove the others.*

Theorem 4.5 *Let x be defined as in (8). If $a^\pi bs^d=0$, $s^\pi ca^d=0$, $s^dca^\pi=0$, and $y=\begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & ss^\pi \end{bmatrix}\in\mathscr{A}^d$, then $x\in\mathscr{A}^d$ and*

$$x^d = \begin{bmatrix} a^\pi & -a^dbs^\pi \\ 0 & s^\pi \end{bmatrix} y^d + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} p & a^dbs^\pi \\ 0 & 1-p \end{bmatrix} y^n y^\pi, \tag{10}$$

where r is defined as in (9).

Proof From the condition $s^\pi ca^d=0$, we have $s^\pi ca^\pi+ss^dc=c$ and $s^\pi s+ss^dd=d$. Then we can write

$$x = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi ca^\pi & s^\pi s \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^db \\ ss^dc & ss^dd \end{bmatrix} := y+z.$$

The equations $a^\pi a^d=0$ and $a^\pi bs^d=0$ imply $yz=0$.

To show that $z\in\mathscr{A}^d$, we consider the following decomposition:

$$z = \begin{bmatrix} 0 & aa^dbs^\pi \\ 0 & ss^dds^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dc & ss^ddss^d \end{bmatrix} := z_1+z_2.$$

Clearly, $z_1z_2=0$ and $z_1^2=0$.

Next, we will prove that $z_2\in\mathscr{A}^d$. Let $z_2=\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, where $a_2=a^2a^d$, $b_2=aa^dbss^d$, $c_2=ss^dc$, and $d_2=ss^ddss^d$. It is clear that a_2 is group invertible, $a_2^\#=a^d$, and $a_2^\pi=a^\pi$. Note that $s_2:=d_2-c_2a_2^\#b_2=ss^ddss^d-ss^dca^dbss^d=s^2s^d$, which gives s_2 is group invertible, $s_2^\#=s^d$, and $s_2^\pi=s^\pi$. Furthermore, we can deduce that $a_2^\pi b_2s_2^\#=0$, $a_2^\#b_2s_2^\pi=0$, $s_2^\pi c_2a_2^\#=0$, $s_2^\#c_2a_2^\pi=s^dca^\pi=0$, and $y_2:=\begin{bmatrix} a_2a_2^\pi & a_2^\pi b_2 \\ s_2^\pi c_2a_2^\pi & s_2s_2^\pi \end{bmatrix}=0$. By Corollary 4.3, we obtain that z_2 is group invertible and $z_2^\#=r$, where r is defined as in (9).

It follows directly from Lemma 4.1 that $z \in \mathscr{A}^d$ and $z^d = r + r^2 z_1$. By a direct computation, we have $z^\pi = \begin{bmatrix} a^\pi & -a^d b s^\pi \\ 0 & s^\pi \end{bmatrix}$ and $zz^\pi = 0$. Thus, z is group invertible.

Finally, we deduce that $x \in \mathscr{A}^d$ by Lemma 4.1 again. In addition, the equation (10) holds. □

In the following result, we give a new representation for the generalized Drazin inverse of block matrix x in (8) in terms of a^d and s^d.

Theorem 4.6 *Let x be defined as in (8). If $aa^\pi bc = 0, ca^\pi bc = 0$, $a^\pi bca^\pi b = 0$, $s^\pi ca = 0$, $a^2 a^\pi b + bca^\pi b = aa^\pi bd$, and $caa^\pi b + dca^\pi b = ca^\pi bd$, then $x \in \mathscr{A}^d$ and*

$$\begin{aligned} x^d =& w + \sum_{n=1}^{\infty} w^{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1} a^\pi & 0 \end{bmatrix} - 2 \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & ca^{n-1} a^\pi b \end{bmatrix} \\ &+ \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} \left(w^2 + \sum_{n=1}^{\infty} w^{n+2} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1} a^\pi & 0 \end{bmatrix} \right), \end{aligned} \tag{11}$$

where

$$w^k = r^k \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+k} \begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix}, \quad k \in \mathbb{N}, \tag{12}$$

and r is defined as in (9).

Proof Since $aa^d b + a^\pi b = b$, then

$$x = \begin{bmatrix} a & aa^d b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} := x_1 + x_2.$$

By a computation, the hypotheses imply $x_1^2 x_2 = x_1 x_2 x_1$ and $x_2^2 x_1 = x_2 x_1 x_2$.

We must show that $x_1 \in \mathscr{A}^d$. Let

$$x_1 = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d b \\ caa^d & d \end{bmatrix} := x_1' + x_1'';$$

then $x_1' x_1'' = 0$.

In order to prove that $x_1'' \in \mathscr{A}^d$, we can write $x_1'' := \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, where $a_1 = a^2 a^d, b_1 = aa^d b, c_1 = caa^d$ and $d_1 = d$. Obviously, a_1 is group invertible, $a_1^\# = a^d$, and $a_1^\pi = a^\pi$. Besides these, we can obtain that $s_1 := d_1 - c_1 a_1^\# b_1 = d - ca^d b = s \in \mathscr{A}^d$. Moreover, we clearly have that $a_1^\pi b_1 s_1^d = a^\pi aa^d bs^d = 0$, $s_1^\pi c_1 a_1^\# = s^\pi ca^d = 0$, and $s_1^d c_1 a_1^\pi = s^d caa^d a^\pi = 0$. Let $y_1 := \begin{bmatrix} a_1 a_1^\pi & a_1^\pi b_1 \\ s_1^\pi c_1 a_1^\pi & s_1 s_1^\pi \end{bmatrix}$, then $y_1 = \begin{bmatrix} 0 & 0 \\ 0 & ss^\pi \end{bmatrix} \in \mathscr{A}^{qnil}$. Therefore, according to Theorem 4.5, we have that $x_1'' \in \mathscr{A}^d$ and $(x_1'')^d = w$, where

$$w = r \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} + \sum_{n=1}^{\infty} r^{n+1} \begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix}.$$

Observe that $\sigma(x_1') \subseteq \sigma(aa^\pi) \cup \{0\}$ and $aa^\pi \in \mathscr{A}^{qnil}$; then $x_1' \in \mathscr{A}^{qnil}$, i.e. $(x_1')^d = 0$. Applying Lemma 4.1, we deduce that $x_1 \in \mathscr{A}^d$ and

$$x_1^d = w + \sum_{n=1}^{\infty} w^{n+1} \begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1}a^\pi & 0 \end{bmatrix}.$$

From the equality $x_2^2 = 0$, it follows that $x_2^d = 0$, which yields $x_1 x_1^d (x_1 + x_2) x_2 x_2^d = 0 \in \mathscr{A}^d$. Applying Theorem 3.3, we obtain that $x \in \mathscr{A}^d$ and

$$x^d = x_1^d - (x_1^d)^2 x_2 + x_1^\pi x_2 (x_1^d)^2 - 2x_1^\pi x_2 (x_1^d)^3 x_2.$$

Note that $x_2(x_1^d)^3 x_2 = x_2^2 (x_1^d)^3 = 0$ by Lemma 2.5. Then

$$x^d = x_1^d - (x_1^d)^2 x_2 + x_1^\pi x_2 (x_1^d)^2 = x_1^d - 2(x_1^d)^2 x_2 + x_2 (x_1^d)^2. \tag{13}$$

Next, we prove the expression of x^d. Note that, for $n \in \mathbb{N}$,

$$\begin{bmatrix} 0 & a^d b s^n s^\pi \\ 0 & s^n s^\pi \end{bmatrix} r = 0 \quad \text{and} \quad \begin{bmatrix} p & a^d b s^\pi \\ 0 & 1-p \end{bmatrix} r = r;$$

then the equation (12) holds. By substituting the expression of x_1^d into the equation (13) and using the following equalities

$$\begin{bmatrix} a^n a^\pi & 0 \\ ca^{n-1}a^\pi & 0 \end{bmatrix} r = 0 \quad \text{and} \quad w \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} = 0,$$

we can get the equation (11). □

From Theorem 4.6, we can obtain the following corollary, which recovers [5, Theorem 8] for a 2×2 operator matrix.

Corollary 4.7 *Let x be defined as in (8). If $a^\pi bc = 0$, $ca^\pi b = 0$, $aa^\pi b = a^\pi bd$, and $s = d - ca^d b$ is invertible, then $x \in \mathscr{A}^d$ and*

$$x^d = \left(r - \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r^2 \right) \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^\pi a^n & 0 \end{bmatrix} \right), \tag{14}$$

where r is defined as in (9) with $s^d = s^{-1}$.

Proof As in the proof of Theorem 4.6. Note that $x_1 x_2 = x_2 x_1$; then $x_1^d x_2 = x_2 x_1^d$. Thus $x^d = x_1^d - (x_1^d)^2 x_2$. By a computation, we can get the equation (14). □

Remark 4.8 *Theorem 4.6 generalizes [15, Theorem 2.3], where an expression for x^d is given under the conditions $a^\pi b = 0$ and $s^\pi ca = 0$. Indeed, $a^\pi b = 0$ and $s^\pi ca = 0$ can imply the conditions of Theorem 4.6. However, in general, the converse is false. The following example can illustrate this fact.*

Example 4.9 *Let $\mathscr{A}$ be the Banach algebra of all complex 3×3 matrices, and take*

$$x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } c = d = 0.$$

Obviously,

$$a^d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } a^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that the conditions of Theorem 4.6 hold. However, $a^\pi b \neq 0$.

Following the same strategy as in the proof of Theorem 4.6, we derive another formula for x^d. Here we omit the proof.

Theorem 4.10 *Let x be defined as in (8). If $bca^\pi b = 0, dca^\pi b = 0$, $ca^\pi bca^\pi = 0$, $s^\pi ca = 0$, $d^2ca^\pi + cbca^\pi = dcaa^\pi$, and $abca^\pi + bdca^\pi = bcaa^\pi$, then $x \in \mathscr{A}^d$ and*

$$\begin{aligned} x^d =& w + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & a^{n-1}a^\pi b \\ 0 & 0 \end{bmatrix} w^{n+1} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & ca^{n-1}a^\pi b \end{bmatrix} w^{n+2} \\ & - 2\left(w^2 + \sum_{n=1}^{\infty} \begin{bmatrix} a^n a^\pi & a^{n-1}a^\pi b \\ 0 & 0 \end{bmatrix} w^{n+2}\right) \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}, \end{aligned} \tag{15}$$

where w^k is defined as in (12) for $k \in \mathbb{N}$, and r is defined as in (9).

Now, we state a special case of Theorem 4.10, which also generalizes [5, Theorem 9] for a 2×2 operator matrix.

Corollary 4.11 *Let x be defined as in (8). If $bca^\pi = 0$, $ca^\pi b = 0$, $caa^\pi = dca^\pi$, and $s = d - ca^d b$ is invertible, then $x \in \mathscr{A}^d$ and*

$$x^d = r + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2} - \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & ca^n a^\pi b \end{bmatrix} r^{n+3},$$

where r is defined as in (9) with $s^d = s^{-1}$.

Remark 4.12 *Theorem 4.10 extends [16, Theorem 3.2], where the generalized Drazin inverse of x is considered in the case that $bca^\pi = 0$, $dca^\pi = 0$, $s^\pi ca = 0$, and $abs^\pi = 0$. In fact, Example 4.9 can also illustrate that the conditions of Theorem 4.10 are weaker than those of [16, Theorem 3.2].*

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