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RENYU ZHAO

WEI REN

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$\mathcal{VW}$-Gorenstein complexes

Renyu ZHAO*, Wei REN
College of Mathematics and Statistics, Northwest Normal University, Lanzhou, P.R. China

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Abstract: Let $\mathcal{V}, \mathcal{W}$ be two classes of modules. In this paper, we introduce and study $\mathcal{VW}$-Gorenstein complexes as a common generalization of $\mathcal{W}$-complexes, Gorenstein projective (resp., Gorenstein injective) complexes, and G_C-projective (resp., G_C-injective) complexes. It is shown that under certain hypotheses a complex X is $\mathcal{VW}$-Gorenstein if and only if each X^n is a $\mathcal{VW}$-Gorenstein module. This result unifies the corresponding results of the aforementioned complexes. As an application, the stability of $\mathcal{VW}$-Gorenstein complexes is explored.

Key words: $\mathcal{VW}$-Gorenstein complex, $\mathcal{VW}$-Gorenstein module, stability

1. Introduction

Inspired by Auslander and Bridger's notion of G-dimension for finitely generated modules over two-sided noetherian rings [1], Enochs and Jenda [8] introduced and studied Gorenstein projective and Gorenstein injective modules that developed Gorenstein homological algebra for modules. Later these modules and their analogues were further studied by many authors; see, e.g. [3–5, 9, 12, 14, 15, 17, 20, 21, 23, 25]. Recently, Zhao and Sun [29] introduced the notion of $\mathcal{VW}$-Gorenstein modules, where $\mathcal{V}$ and $\mathcal{W}$ are two classes of modules. This notion unifies the following notions: Gorenstein projective and Gorenstein injective modules [8], G_C-projective and G_C-injective modules [20], $\mathcal{W}$-Gorenstein modules [12, 21], and so on.

As an important abelian category, the category of complexes of modules has drawn wide attention and many results of the category of modules have been generalized to the category of complexes of modules, see for example [2, 5–7, 10, 11, 13, 18, 19, 22, 24–28]. Following this philosophy, in this paper we introduce the notion of $\mathcal{VW}$-Gorenstein complexes (see Definition 3.1). This provides us with an opportunity to study the Gorenstein projective and Gorenstein injective complexes [7], G_C-projective and G_C-injective complexes [26], and $\mathcal{W}$-Gorenstein complexes [19, 24] in a general setting. Some results in the literature for these complexes can be obtained as particular instances of the results on $\mathcal{VW}$-Gorenstein complexes. This paper is organized as follows.

In section 2, we collect some notations and terminology needed in our discussions.

In section 3, we introduce the notion of $\mathcal{VW}$-Gorenstein complexes, and then characterize the $\mathcal{VW}$-Gorenstein property of a complex via the $\mathcal{VW}$-Gorenstein property of all its terms, namely, we prove that for two classes $\mathcal{V}, \mathcal{W}$ of modules, if $\mathcal{V}, \mathcal{W}$ are closed under extensions, isomorphisms, and finite direct sums, and $\mathcal{V} \perp \mathcal{W}, \mathcal{V} \perp \mathcal{V}, \mathcal{W} \perp \mathcal{W}$, then for any complex X, X is $\mathcal{VW}$-Gorenstein if and only if each X^n is a $\mathcal{VW}$-

*Correspondence: renyuzhao@123.com
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Gorenstein module (Theorem 3.8). This result unifies [19, Corollary 4.8] [24, Theorem 3.12], [27, Theorems 1 and 2], [26, Theorems 4.6 and 4.7] and [28, Theorem 2.2, Proposition 2.8]. Consequently, some properties of $\mathcal{VW}$-Gorenstein complexes are obtained.

In section 4, as an application of our main result (Theorem 3.8), the stability of $\mathcal{VW}$-Gorenstein complexes is investigated.

2. Preliminaries

Throughout this article, the rings R and S are associative with identity. All modules are understood to be left R- or S-modules. Right R- or S-modules are identified with left modules over the opposite rings R^{op} or S^{op}. We denote by $\mathcal{C}$ the category of complexes of modules.

A complex

$$\cdots \longrightarrow X^{n+1} \xrightarrow{\delta^{n+1}} X^n \xrightarrow{\delta^n} X^{n-1} \longrightarrow \cdots$$

will be denoted by (X, δ) or simply X. The nth cycle (resp. boundary, homology) of X is denoted by $\mathrm{Z}^n(X)$ (resp., $\mathrm{B}^n(X)$, $\mathrm{H}^n(X)$). We will use subscripts to distinguish complexes. Hence if $\{X_i\}_{i\in I}$ is a family of complexes, X_i will be complex

$$\cdots \longrightarrow X_i^{n+1} \xrightarrow{\delta_i^{n+1}} X_i^n \xrightarrow{\delta_i^n} X_i^{n-1} \longrightarrow \cdots .$$

Given a module M, we will denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in 1 and 0th degrees. Given an $X \in \mathcal{C}$ and an integer m, $X[m]$ denotes the complex such that $X[m]^n = X^{n-m}$ and whose boundary operators are $(-1)^m\delta^{n-m}$. Given $X, Y \in \mathcal{C}$, we use $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to present the group of all morphisms from X to Y, and $\mathrm{Ext}^i_{\mathcal{C}}(X, Y)$ to denote the groups one gets from the right-derived functor of Hom for $i \geq 0$.

Definition 2.1 ([16, Definition 2.1]) *An (R, S)-bimodule $C = {}_RC_S$ is called semidualizing if the following conditions are satisfied:*

(1) *${}_RC$ admits a resolution by finitely generated projective R-modules.*

(2) *C_S admits a resolution by finitely generated projective S^{op}-modules.*

(3) *The natural homothety map $\chi^R_C : {}_RR_R \longrightarrow \mathrm{Hom}_{S^{op}}(C, C)$ is an isomorphism.*

(4) *The natural homothety map $\chi^S_C : {}_SS_S \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism.*

(5) $\mathrm{Ext}^{\geqslant 1}_R(C, C) = 0$.

(6) $\mathrm{Ext}^{\geqslant 1}_{S^{op}}(C, C) = 0$.

Let $C = {}_RC_S$ be a semidualizing bimodule. We set

$\mathcal{P}_C(R)$ = the subcategory of R-modules $C \otimes_S P$ where ${}_SP$ is projective,
$\mathcal{I}_C(S)$ = the subcategory of S-modules $\mathrm{Hom}_R(C, I)$ where ${}_RI$ is injective.

Modules in $\mathcal{P}_C(R)$ and $\mathcal{I}_C(S)$ are called C-projective and C-injective, respectively. When $C = R$, we omit the subscript and recover the classes of projective and injective R-modules.

Let $\mathcal{W}$ be a class of modules. A complex X is called a $\mathcal{W}$-complex [13] if X is exact and $\mathrm{Z}^n(X) \in \mathcal{W}$ for all $n \in \mathbb{Z}$. We write $\widetilde{\mathcal{W}}$ for the class of $\mathcal{W}$-complexes. If $\mathcal{W}$ is the class of $\mathcal{P}(R)$ (resp., $\mathcal{I}(S)$, $\mathcal{P}_C(R)$, $\mathcal{I}_C(S)$), then X is just projective (resp., injective, C-projective, C-injective) complexes.

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Recall that a sequence $\mathbf{S}$ in $\mathcal{A}$ is $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{B})$-exact (resp., $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B},-)$-exact) if the sequence $\mathrm{Hom}_{\mathcal{A}}(\mathbf{S},B)$ (resp., $\mathrm{Hom}_{\mathcal{A}}(B,\mathbf{S})$) is exact for any $B \in \mathcal{B}$. For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$, we say $\mathcal{X} \perp \mathcal{Y}$ if $\mathrm{Ext}_{\mathcal{A}}^{\geqslant 1}(X,Y) = 0$ for any $X \in \mathcal{X}$ and any $Y \in \mathcal{Y}$. If $\mathcal{X} \perp \mathcal{X}$, $\mathcal{X}$ is called self-orthogonal.

Definition 2.2 ([29, Definition 3.1]) *Let $\mathcal{V}, \mathcal{W}$ be two classes of modules. A module M is called $\mathcal{V}\mathcal{W}$-Gorenstein, if there exists a $\mathrm{Hom}(\mathcal{V},-)$-exact and $\mathrm{Hom}(-,\mathcal{W})$-exact exact sequence of modules*

$$\cdots \longrightarrow V^1 \xrightarrow{\delta^1} V^0 \xrightarrow{\delta^0} W^{-1} \xrightarrow{\delta^{-1}} W^{-2} \longrightarrow \cdots$$

with each $V^i \in \mathcal{V}$ and each $W^i \in \mathcal{W}$, such that $M = \mathrm{Im}\delta^0$.

In the following, we denote by $\mathcal{G}(\mathcal{V}\mathcal{W})$ the class of $\mathcal{V}\mathcal{W}$-Gorenstein modules. This definition unifies the following notions: G_C-projective R-modules [20] (when $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$); G_C-injective S-modules [20] (when $\mathcal{V} = \mathcal{I}_C(S)$ and $\mathcal{W} = \mathcal{I}(S)$); $\mathcal{W}$-Gorenstein modules [12, 21] (in the case $\mathcal{V} = \mathcal{W}$), and of course Gorenstein projective and Gorenstein injective modules [8].

In what follows, we fix two classes $\mathcal{V}, \mathcal{W}$ of modules, and let C be a semidualizing (R,S)-bimodule.

3. $\mathcal{V}\mathcal{W}$-Gorenstein complexes

In this section we extend the notion of $\mathcal{V}\mathcal{W}$-Gorenstein modules to that of complexes and characterize such complexes. We start with the following definition.

Definition 3.1 *A complex X is called a $\mathcal{V}\mathcal{W}$-Gorenstein complex, if there exists a $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}},-)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-,\widetilde{\mathcal{W}})$-exact exact sequence*

$$\boldsymbol{G} = \ \cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 \xrightarrow{\sigma_0} W_{-1} \xrightarrow{\sigma_{-1}} W_{-2} \longrightarrow \cdots$$

with each $V_i \in \widetilde{\mathcal{V}}$ and each $W_j \in \widetilde{\mathcal{W}}$, such that $X = \mathrm{Im}\sigma_0$. The exact sequence $\boldsymbol{G}$ is called a complete $\widetilde{\mathcal{V}}\widetilde{\mathcal{W}}$-resolution.

In the following, we denote by $\mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$ the class of $\mathcal{V}\mathcal{W}$-Gorenstein complexes.

Remark 3.2 *Here are some special cases of $\mathcal{V}\mathcal{W}$-Gorenstein complexes.*

(1) *If $\mathcal{V} = \mathcal{W}$, then $\mathcal{V}\mathcal{W}$-Gorenstein complexes are exactly the $\mathcal{W}$-Gorenstein complexes in [19] or completely $\mathcal{W}$-resolved complexes in [24]. In particular, if they are the class of projective (resp., injective) modules, then $\mathcal{V}\mathcal{W}$-Gorenstein complexes coincide with Gorenstein projective (resp., injective) complexes in [7].*

(2) *If $\mathcal{V} = \mathcal{P}(R)$ and $\mathcal{W} = \mathcal{P}_C(R)$, then $\mathcal{V}\mathcal{W}$-Gorenstein complexes are just G_C-projective complexes in [26].*

(3) *If $\mathcal{V} = \mathcal{I}_C(S)$ and $\mathcal{W} = \mathcal{I}(S)$, then $\mathcal{VW}$-Gorenstein complexes are just G_C-injective complexes of S-modules in [26].*

To characterize the $\mathcal{VW}$-Gorenstein complexes, we need some preparations.

Lemma 3.3 ([13, Lemma 3.1]) *Let X be a complex and M a module. Then we have the following natural isomorphisms:*

(1) $\mathrm{Hom}_{\mathcal{C}}(\overline{M}[n], X) \cong \mathrm{Hom}(M, X^{n+1})$.

(2) $\mathrm{Hom}_{\mathcal{C}}(X, \overline{M}[n]) \cong \mathrm{Hom}(X^n, M)$.

(3) $\mathrm{Ext}^1_{\mathcal{C}}(\overline{M}[n], X) \cong \mathrm{Ext}^1(M, X^{n+1})$.

(4) $\mathrm{Ext}^1_{\mathcal{C}}(X, \overline{M}[n]) \cong \mathrm{Ext}^1(X^n, M)$.

Let $\mathcal{F}$ be a class of modules. Recall that a complex X is a $\#$-$\mathcal{F}$ complex [18] if all terms X^n are in $\mathcal{F}$ for $n \in \mathbb{Z}$. We let $\widetilde{\#\mathcal{F}}$ denote the subcategory of $\#$-$\mathcal{F}$ complexes.

Lemma 3.4 *([19, Lemma 4.4]) Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be classes of modules. If $\mathcal{Z} \perp \mathcal{Z}$, then the following statements hold:*

(1) *$\mathcal{X} \perp \mathcal{Z}$ if and only if $\widetilde{\#\mathcal{X}} \perp \widetilde{\mathcal{Z}}$.*

(2) *$\mathcal{Z} \perp \mathcal{Y}$ if and only if $\widetilde{\mathcal{Z}} \perp \widetilde{\#\mathcal{Y}}$.*

By [29, Proposition 3.5] and Lemma 3.4, we have

Corollary 3.5 *Let X be a complex. If $\mathcal{V} \perp \mathcal{V}$, $\mathcal{W} \perp \mathcal{W}$, $\mathcal{V} \perp \mathcal{W}$ and each X^n is a $\mathcal{VW}$-Gorenstein module, then $\mathrm{Ext}^1_{\mathcal{C}}(V, X) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(X, W) = 0$ for any $V \in \widetilde{\mathcal{V}}, W \in \widetilde{\mathcal{W}}$.*

Lemma 3.6 *Let*

$$\cdots \longrightarrow U_1 \xrightarrow{\sigma_1} U_0 \xrightarrow{\sigma_0} U_{-1} \longrightarrow \cdots$$

be a $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact sequence of complexes; then for any $n \in \mathbb{Z}$, the sequence

$$\cdots \longrightarrow U_1^n \xrightarrow{\sigma_1^n} U_0^n \xrightarrow{\sigma_0^n} U_{-1}^n \longrightarrow \cdots$$

is $\mathrm{Hom}(\mathcal{V}, -)$-exact and $\mathrm{Hom}(-, \mathcal{W})$-exact.

Proof Let $V \in \mathcal{V}$, $W \in \mathcal{W}$, and $n \in \mathbb{Z}$. Then $\overline{V}[n-1] \in \widetilde{\mathcal{V}}$ and $\overline{W}[n] \in \widetilde{\mathcal{W}}$. Thus we have the following exact sequences:

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(\overline{V}[n-1], U_1) \to \mathrm{Hom}_{\mathcal{C}}(\overline{V}[n-1], U_0) \to \mathrm{Hom}_{\mathcal{C}}(\overline{V}[n-1], U_{-1}) \to \cdots,$$

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(U_{-1}, \overline{W}[n]) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(U_0, \overline{W}[n]) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(U_1, \overline{W}[n]) \longrightarrow \cdots.$$

Using the adjointness property (1) and (2) from Lemma 3.3, we get exact sequences

$$\cdots \longrightarrow \mathrm{Hom}(V, U_1^n) \longrightarrow \mathrm{Hom}(V, U_0^n) \longrightarrow \mathrm{Hom}(V, U_{-1}^n) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \mathrm{Hom}(U^n_{-1}, W) \longrightarrow \mathrm{Hom}(U^n_0, W) \longrightarrow \mathrm{Hom}(U^n_1, W) \longrightarrow \cdots .$$

Now the result follows. □

In what follows, we always assume that $\mathcal{V}, \mathcal{W}$ satisfy the conditions

$$(P): \quad \begin{array}{l} \mathcal{V}, \mathcal{W} \text{ are closed under extensions, isomorphisms, and finite direct sums, and} \\ \mathcal{V} \perp \mathcal{W}, \mathcal{V} \perp \mathcal{V}, \mathcal{W} \perp \mathcal{W} \quad \text{and} \quad \mathcal{V}, \mathcal{W} \subseteq \mathcal{G}(\mathcal{VW}). \end{array}$$

Remark 3.7 *The following are some pairs $(\mathcal{V}, \mathcal{W})$ that satisfy the conditions (P).*

(1) *Let $\mathcal{V} = \mathcal{W} = \mathcal{P}(R)$ (resp. $\mathcal{I}(R)$). Then the pair $(\mathcal{V}, \mathcal{W})$ satisfies the conditions (P).*

(2) *Let $\mathcal{V} = \mathcal{P}(R), \mathcal{W} = \mathcal{P}_C(R)$. Then the pair $(\mathcal{V}, \mathcal{W})$ satisfies the conditions (P) by [12, Remark 2.3(4)], [16, Proposition 5.2], and [23, Proposition 2.6].*

(3) *Let $\mathcal{V} = \mathcal{I}_C(S), \mathcal{W} = \mathcal{I}(S)$. Then the pair $(\mathcal{V}, \mathcal{W})$ satisfies the conditions (P) by [12, Remark 2.3(4)], [16, Proposition 5.2], and [26, Lemma 3.1].*

We are now in a position to present our main result, which generalizes [19, Corollary 4.8], [24, Theorem 3.12], [26, Theorems 4.6 and 4.7], and of course [27, Theorems 1, 2] and [28, Theorem 2.2, Proposition 2.8].

Theorem 3.8 *Let X be a complex. Then X is $\mathcal{VW}$-Gorenstein if and only if each X^n is a $\mathcal{VW}$-Gorenstein module.*

Proof $\Longrightarrow$) Assume that X is a $\mathcal{VW}$-Gorenstein complex. Then there exists an exact sequence of complexes

$$\boldsymbol{G} = \ \cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 \xrightarrow{\sigma_0} W_{-1} \xrightarrow{\sigma_{-1}} W_{-2} \longrightarrow \cdots$$

with each $V_i \in \widetilde{\mathcal{V}}$ and each $W_j \in \widetilde{\mathcal{W}}$, such that $X = \mathrm{Im}\sigma_0$ and $\boldsymbol{G}$ remains exact after applying $\mathrm{Hom}_{\mathcal{C}}(V, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, W)$ for any $V \in \widetilde{\mathcal{V}}$ and $W \in \widetilde{\mathcal{W}}$. Now for any but fixed $n \in \mathbb{Z}$, by Lemma 3.6, we have the following $\mathrm{Hom}(\mathcal{V}, -)$-exact and $\mathrm{Hom}(-, \mathcal{W})$-exact exact sequence:

$$\cdots \longrightarrow V^n_1 \xrightarrow{\sigma^n_1} V^n_0 \xrightarrow{\sigma^n_0} W^n_{-1} \xrightarrow{\sigma^n_{-1}} W^n_{-2} \longrightarrow \cdots$$

with $X^n = \mathrm{Im}\sigma^n_0$, and $V^n_i \in \mathcal{V}$, $W^n_{-i-1} \in \mathcal{W}$ for any $i \geq 0$ since $\mathcal{V}, \mathcal{W}$ are closed under extensions. Hence X^n is a $\mathcal{VW}$-Gorenstein module.

$\Longleftarrow$) Suppose that each X^n is a $\mathcal{VW}$-Gorenstein module. Then for any $n \in \mathbb{Z}$, there exists an exact sequence of modules

$$0 \longrightarrow Y^n \longrightarrow V^n \longrightarrow X^n \longrightarrow 0,$$

where $V^n \in \mathcal{V}$, and $Y^n \in \mathcal{G}(\mathcal{VW})$ by [29, Corollary 4.6]. Thus we have the following short exact sequence:

$$0 \longrightarrow \oplus_{n\in\mathbb{Z}} \overline{Y^n}[n-1] \longrightarrow \oplus_{n\in\mathbb{Z}} \overline{V^n}[n-1] \longrightarrow \oplus_{n\in\mathbb{Z}} \overline{X^n}[n-1] \longrightarrow 0.$$

Put $V_0 = \bigoplus_{n\in\mathbb{Z}} \overline{V^n}[n-1]$. It is easy to see that $V_0 \in \widetilde{\mathcal{V}}$. On the other hand, there is an obvious (degreewise split) short exact sequence

$$0 \longrightarrow X[-1] \xrightarrow{\binom{1}{-\delta}} \oplus_{n\in\mathbb{Z}} \overline{X^n}[n-1] \xrightarrow{(\delta, 1)} X \longrightarrow 0,$$

where δ is the differential of X. Now let $\beta: V_0 \longrightarrow X$ be the composite

$$\oplus_{n\in\mathbb{Z}}\overline{V^n}[n-1] \longrightarrow \oplus_{n\in\mathbb{Z}}\overline{X^n}[n-1] \longrightarrow X.$$

Then β is epic since it is the composite of two epimorphisms. Denote Kerβ by K_0. Then by the Snake Lemma, we have a short exact sequence

$$0 \longrightarrow \oplus_{n\in\mathbb{Z}}\overline{Y^n}[n-1] \longrightarrow K_0 \longrightarrow X[-1] \longrightarrow 0.$$

Since both $\oplus_{n\in\mathbb{Z}}\overline{Y^n}[n-1]$ and $X[-1]$ are $\mathcal{VW}$-Gorenstein in each degree, K_0 is degreewise $\mathcal{VW}$-Gorenstein by [29, Corollary 3.8]. Thus $\mathrm{Ext}^1_{\mathcal{C}}(V, K_0) = 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(X, W) = 0$ for any $V \in \widetilde{\mathcal{V}}, W \in \widetilde{\mathcal{W}}$ by Corollary 3.5. This implies that $0 \longrightarrow K_0 \longrightarrow V_0 \longrightarrow X \longrightarrow 0$ is both $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact. Continuously using the method above, we can construct a $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact exact sequence of complexes

$$\cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 \xrightarrow{\beta} X \longrightarrow 0, \tag{†}$$

where each V_i is a $\mathcal{V}$-complex.

Dually, we can construct a both $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} W_{-1} \xrightarrow{\sigma_{-1}} W_{-2} \longrightarrow \cdots 0, \tag{‡}$$

where each W_i is a $\mathcal{W}$-complex.

Finally, assembling the sequences (†) and (‡) together, we get a both $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact exact sequence of complexes

$$\cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 \xrightarrow{\sigma_0} W_{-1} \xrightarrow{\sigma_{-1}} W_{-2} \longrightarrow \cdots$$

with $V_i \in \widetilde{\mathcal{V}}$, $W_i \in \widetilde{\mathcal{W}}$ such that $X = \mathrm{Im}\sigma_0$. Hence X is a $\mathcal{VW}$-Gorenstein complex. □

Based on Theorem 3.8, we have the following results.

Corollary 3.9 *The class of $\mathcal{VW}$-Gorenstein complexes is closed under extensions.*

Proof Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

be a short exact sequence of complexes, where X and Z are $\mathcal{VW}$-Gorenstein. Then for any $n \in \mathbb{Z}$, in the exact sequence

$$0 \longrightarrow X^n \longrightarrow Y^n \longrightarrow Z^n \longrightarrow 0,$$

X^n and Z^n are $\mathcal{VW}$-Gorenstein modules by Theorem 3.8. Thus Y^n is a $\mathcal{VW}$-Gorenstein module by [29, Corollary 3.8]. Therefore, Y is $\mathcal{VW}$-Gorenstein by Theorem 3.8. □

Corollary 3.10 *Let*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \qquad (*)$$

be a short exact sequence of complexes.

(1) *If* $Y, Z \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$ *and* $(*)$ *is* $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$*-exact, then* $X \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$*.*

(2) *If* $X, Y \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$ *and* $(*)$ *is* $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$*-exact, then* $Z \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$*.*

Proof (1) Let $n \in \mathbb{Z}$. Then there is an exact sequence $0 \longrightarrow X^n \longrightarrow Y^n \longrightarrow Z^n \longrightarrow 0$ with $Y^n, Z^n \in \mathcal{G}(\mathcal{V}\mathcal{W})$ by Theorem 3.8, and it is $\mathrm{Hom}(\mathcal{V}, -)$-exact by Lemma 3.6. By [29, Proposition 3.5], $\mathrm{Ext}^1(Z^n, W) = 0$ for any $W \in \mathcal{W}$. Thus

$$0 \longrightarrow X^n \longrightarrow Y^n \longrightarrow Z^n \longrightarrow 0$$

is $\mathrm{Hom}(-, \mathcal{W})$-exact. Hence $X^n \in \mathcal{G}(\mathcal{V}\mathcal{W})$ by [29, Proposition 3.10], and so $X \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$ by Theorem 3.8.

(2) The proof is similar to that of (1). □

Let $\mathcal{A}$ be an abelian category with enough projective objects and injective objectives. Recall that a class $\mathcal{X}$ of objects of $\mathcal{A}$ is said to be projectively resolving (resp., injectively resolving) if it is closed under extensions and kernels of epimorphisms (resp., cokernels of monomorphisms), and it contains all projective (resp., injective) objects of $\mathcal{A}$.

Corollary 3.11 (1) *The class of Gorenstein projective (resp.,* G_C*-projective) complexes is projectively resolving.*

(2) *The class of Gorenstein injective (resp.,* G_C*-injective) complexes is injectively resolving.*

Proof If P is a projective complex, then each P^n is projective. Thus each P^n is G_C-projective by [23, Proposition 2.6], and so P is a G_C-projective complex by [28, Theorem 4.7]. Dually, by [28, Lemma 3.1] and [28, Theorem 4.6], every injective complex is a G_C-projective complex. Then applying Corollaries 3.9 and 3.10 we can get the results. □

When $\mathcal{V} = \mathcal{W}$, the next two results follow readily from the definition of the $\mathcal{W}$-Gorenstein complexes. However, in the case of $\mathcal{V}\mathcal{W}$-Gorenstein, they are not so trivial.

Corollary 3.12 $\mathcal{V}$*-complexes and* $\mathcal{W}$*-complexes are* $\mathcal{V}\mathcal{W}$*-Gorenstein complexes.*

Proof Let $V \in \widetilde{\mathcal{V}}, W \in \widetilde{\mathcal{W}}$. Then each $V^n \in \mathcal{V}$ and $W^n \in \mathcal{W}$ since $\mathcal{V}, \mathcal{W}$ are closed under extensions. Thus V^n and W^n are $\mathcal{V}\mathcal{W}$-Gorenstein modules since $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}(\mathcal{V}\mathcal{W})$. Hence V, W are $\mathcal{V}\mathcal{W}$-Gorenstein complexes by Theorem 3.8. □

Corollary 3.13 *Every cokernel in a complete* $\widetilde{\mathcal{V}}\widetilde{\mathcal{W}}$*-resolution is in* $\mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$*.*

Proof Let

$$\cdots \longrightarrow V_1 \xrightarrow{\sigma_1} V_0 \xrightarrow{\sigma_0} W_{-1} \xrightarrow{\sigma_{-1}} W_{-2} \longrightarrow \cdots$$

be a complete $\widetilde{\mathcal{V}}\widetilde{\mathcal{W}}$-resolution. Set $X_i = \mathrm{Coker}\sigma_{i+1}$ for $i \in \mathbb{Z}$. Then X_0 is a $\mathcal{V}\mathcal{W}$-Gorenstein complex. Thus each $X_i \in \mathcal{G}(\widetilde{\mathcal{V}}\widetilde{\mathcal{W}})$ by Corollaries 3.12 and 3.10. □

4. Stability of $\mathcal{VW}$-Gorenstein complexes

As an application of Theorem 3.8, in this section, we investigate the stability of $\mathcal{VW}$-Gorenstein complexes. This issue was initiated by Sather-Wagstaff et al. [21]. They proved that if R is a (commutative) ring, then an R-module M is a Gorenstein projective (resp., Gorenstein injective) module if and only if there exists an exact sequence of the Gorenstein projective (resp., Gorenstein injective) R-modules

$$G = \cdots \longrightarrow G^1 \xrightarrow{\delta^1} G^0 \xrightarrow{\delta^0} G^{-1} \longrightarrow \cdots$$

such that the complexes $\mathrm{Hom}_R(H,G)$ and $\mathrm{Hom}_R(G,H)$ are exact for each Gorenstein projective (resp., Gorenstein injective) R-module H and $M = \mathrm{Im}\delta^0$. This was developed by Bouchiba [4] and Xu and Ding [25], respectively. They showed that over any ring R, a left R-module M is a Gorenstein projective (resp., Gorenstein injective) module if and only if there exists an exact sequence of the Gorenstein projective (resp., Gorenstein injective) left R-modules

$$G = \cdots \longrightarrow G^1 \xrightarrow{\delta^1} G^0 \xrightarrow{\delta^0} G^{-1} \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_R(G,H)$ (resp., $\mathrm{Hom}_R(H,G)$) is exact for any projective (resp., injective) left R-module H and $M = \mathrm{Im}\delta^0$. For more details, see [4]. For $\mathcal{VW}$-Gorenstein complexes, we have the following result, which is a generalization of [19, Corollary 4.5], [25, Theorem 3.1] and its dual.

Theorem 4.1 *Let X be a complex. Then X is $\mathcal{VW}$-Gorenstein if and only if there exists a $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}},-)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-,\widetilde{\mathcal{W}})$-exact exact sequence of $\mathcal{VW}$-Gorenstein complexes*

$$\cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that $X = \mathrm{Im}\sigma_0$.

Proof $\Longrightarrow$) It is obvious.

$\Longleftarrow$) Suppose that there exists an exact sequence of $\mathcal{VW}$-Gorenstein complexes

$$\cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

that is both $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}},-)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-,\widetilde{\mathcal{W}})$-exact such that $X = \mathrm{Im}\sigma_0$. Let $n \in \mathbb{Z}$; then by Lemma 3.6, we have the following both $\mathrm{Hom}(\mathcal{V},-)$-exact and $\mathrm{Hom}(-,\mathcal{W})$-exact exact sequence of modules:

$$\cdots \longrightarrow G_1^n \xrightarrow{\sigma_1^n} G_0^n \xrightarrow{\sigma_0^n} G_{-1}^n \longrightarrow \cdots$$

with $X^n = \mathrm{Im}\sigma_0^n$, where all G_i^n are $\mathcal{VW}$-Gorenstein by Theorem 3.8. Thus X^n is a $\mathcal{VW}$-Gorenstein module by [29, Theorem 4.2]. Therefore, X is $\mathcal{VW}$-Gorenstein by Theorem 3.8. □

As special cases of Theorem 4.1, we have the following result, which shows that an iteration of the procedure used to define the G_C-projective (resp., G_C-injective) complexes yields exactly the G_C-projective (resp., G_C-injective) complexes.

Corollary 4.2 *Let X be a complex. Then the following are equivalent:*

(1) *X is a G_C-projective (resp., G_C-injective) complex.*

(2) *There exists an exact sequence of G_C-projective (resp., G_C-injective) complexes*

$$\boldsymbol{G} = \cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that the complexes $\mathrm{Hom}_{\mathcal{C}}(H, \boldsymbol{G})$ and $\mathrm{Hom}_{\mathcal{C}}(\boldsymbol{G}, H)$ are exact for each G_C-projective (resp., G_C-injective) complex H and $X = \mathrm{Im}\sigma_0$.

(3) *There exists an exact sequence of G_C-projective (resp., G_C-injective) complexes*

$$\boldsymbol{G} = \cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_{\mathcal{C}}(\boldsymbol{G}, H)$ (resp., $\mathrm{Hom}_{\mathcal{C}}(H, \boldsymbol{G})$) is exact for each G_C-projective (resp., G_C-injective) complex H and $X = \mathrm{Im}\sigma_0$.

(4) *There exists an exact sequence of G_C-projective (resp., G_C-injective) complexes*

$$\boldsymbol{G} = \cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that the complex $\mathrm{Hom}_{\mathcal{C}}(\boldsymbol{G}, H)$ (resp., $\mathrm{Hom}_{\mathcal{C}}(H, \boldsymbol{G})$) is exact for each C-projective (resp., C-injective) complex H and $X = \mathrm{Im}\sigma_0$.

By Theorem 4.1, we get immediately the following equivalent characterization of $\mathcal{VW}$-Gorenstein complexes. It shows that the $\mathcal{VW}$-Gorenstein complexes possess the symmetry just as the $\mathcal{W}$-Gorenstein complexes do.

Corollary 4.3 *Let X be a complex. Then X is $\mathcal{VW}$-Gorenstein if and only if there exists a $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact exact sequence of complexes*

$$\cdots \longrightarrow U_1 \xrightarrow{\sigma_1} U_0 \xrightarrow{\sigma_0} U_{-1} \longrightarrow \cdots$$

with each $U_i \in \widetilde{\mathcal{V}} \cup \widetilde{\mathcal{W}}$, such that $X = \mathrm{Im}\sigma_0$.

Proof $\Longrightarrow$) It is trivial.

$\Longleftarrow$) This follows from Corollary 3.12 and Theorem 4.1. □

Finally, we give a sufficient and necessary condition for the cycles of an exact complex to be $\mathcal{VW}$-Gorenstein, which is a generalization of [19, Proposition 4.16]

Proposition 4.4 *Let X be an exact complex. Then X is $\mathcal{VW}$-Gorenstein and is both $\mathrm{Hom}(\mathcal{V}, -)$-exact and $\mathrm{Hom}(-, \mathcal{W})$-exact if and only if $\mathrm{Z}^n(X)$ is $\mathcal{VW}$-Gorenstein for any $n \in \mathbb{Z}$.*

Proof $\Longleftarrow$) Assume that $\mathrm{Z}^n(X)$ is $\mathcal{VW}$-Gorenstein for any $n \in \mathbb{Z}$. Then each X^n is $\mathcal{VW}$-Gorenstein by [29, Corollary 3.8] since X is exact. Hence X is $\mathcal{VW}$-Gorenstein by Theorem 3.8. The last assertions follow from [29, Propostion 3.5].

$\Longrightarrow$) Suppose that X is an exact $\mathcal{VW}$-Gorenstein complex with $\mathrm{Hom}(M, X)$ and $\mathrm{Hom}(X, N)$ are exact for any $M \in \mathcal{V}$ and $N \in \mathcal{W}$. Then by Corollary 4.3, there is a both $\mathrm{Hom}_{\mathcal{C}}(\widetilde{\mathcal{V}}, -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \widetilde{\mathcal{W}})$-exact exact sequence

$$\cdots \longrightarrow U_1 \xrightarrow{\sigma_1} U_0 \xrightarrow{\sigma_0} U_{-1} \longrightarrow \cdots$$

with each $U_i \in \widetilde{\mathcal{V}} \cup \widetilde{\mathcal{W}}$, such that $X = \mathrm{Im}\sigma_0$. Let $X_i = \mathrm{Im}\sigma_i$ for $i \in \mathbb{Z}$. Then X_i is exact for any $i \in \mathbb{Z}$ since $X_0 = X$ and each U_i are exact. Thus, for any but fixed $n \in \mathbb{Z}$, by [19, Lemma 4.15(1)], we have the following exact sequence:

$$\cdots \longrightarrow \mathrm{Z}^n(U_1) \xrightarrow{\mathrm{Z}^n(\sigma_1)} \mathrm{Z}^n(U_0) \xrightarrow{\mathrm{Z}^n(\sigma_0)} \mathrm{Z}^n(U_{-1}) \longrightarrow \cdots \qquad (\star)$$

with $\mathrm{Z}^n(U_i) \in \mathcal{V} \cup \mathcal{W}$ for all $i \in \mathbb{Z}$, such that $\mathrm{Z}^n(X) = \mathrm{Im}\mathrm{Z}^n(\sigma_0)$. By [29, Propostion 4.5], to end the proof, we need only to show that the sequence $(\star)$ is $\mathrm{Hom}(\mathcal{V}, -)$-exact and $\mathrm{Hom}(-, \mathcal{W})$-exact.

Let $M \in \mathcal{V}, N \in \mathcal{W}$. By Corollary 3.13, each X_i is $\mathcal{VW}$-Gorenstein, and so $X_i^n \in \mathcal{G}(\mathcal{VW})$ for each $i \in \mathbb{Z}$ by Theorem 3.8. Thus

$$0 \longrightarrow \mathrm{Hom}(M, X_i) \longrightarrow \mathrm{Hom}(M, U_{i-1}) \longrightarrow \mathrm{Hom}(M, X_{i-1}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(X_{i-1}, N) \longrightarrow \mathrm{Hom}(U_{i-1}, N) \longrightarrow \mathrm{Hom}(X_i, N) \longrightarrow 0$$

are exact by [29, Proposition 3.5]. By the hypothesis, $\mathrm{Hom}(M, X)$ and $\mathrm{Hom}(X, N)$ are exact. Since $U_i \in \widetilde{\mathcal{V}} \cup \widetilde{\mathcal{W}}$ and $\mathcal{V} \perp \mathcal{V}, \mathcal{W} \perp \mathcal{W}, \mathcal{V} \perp \mathcal{W}$, $\mathrm{Hom}(M, U_{i-1})$ and $\mathrm{Hom}(U_{i-1}, N)$ are exact. Thus $\mathrm{Hom}(M, X_i)$ and $\mathrm{Hom}(X_i, N)$ are exact for any $i \in \mathbb{Z}$. Hence $\mathrm{Ext}^i(M, \mathrm{Z}^n(X_i)) = 0, \mathrm{Ext}^i(\mathrm{Z}^n(X_i), N) = 0$ for any $i \in \mathbb{Z}$ since each $X_i^n \in \mathcal{G}(\mathcal{VW})$. Thus the sequence $(\star)$ is $\mathrm{Hom}(\mathcal{V}, -)$-exact and $\mathrm{Hom}(-, \mathcal{W})$-exact, as desired. □

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