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Andoyer equations for noncollinear planar central configurations

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Abstract: In this article we obtain the Andoyer equations for noncollinear planar central configurations taking into account the center of mass of the system. We apply these equations to study two configurations. In the first one we prove that it is not possible to put a square central configuration and an equilateral triangle central configuration as a cocircular central configuration. In the second one we give the central configurations for the noncollinear planar 4–body problem with one pair of equal positive masses and two null masses.

Key words: Central configuration, n–body problem, planar central configuration, celestial mechanics

1. Introduction and statement of the main results

The classical Newtonian n–body problem consists of the study of a system formed by n punctual bodies with positives masses $m_1, \ldots, m_n$ interacting by Newton's gravitational law [11]. That is, if the position vectors are given by $r_1, \ldots, r_n$ in $\mathbb{R}^d$, $d = 2, 3$, the equations of motion are

$$\ddot{r}_i = F_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\,(r_i - r_j), \tag{1}$$

for $i = 1, \ldots, n$, where $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the bodies at r_i and r_j. In (1) we consider the gravitational constant $G = 1$. The vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^{nd}$ denotes a configuration of the n bodies and we assume that $r_i \neq r_j$, for $i \neq j$.

One integral of motion of system (1) is the *linear momentum*

$$P = \sum_{j=1}^{n} \frac{m_j}{M}\,\dot{r}_j, \tag{2}$$

where $M = m_1 + \ldots + m_n$ is the total mass. This implies that the center of mass of the system, which is given by

$$c = \sum_{j=1}^{n} \frac{m_j}{M}\,r_j,$$

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describes a uniform motion. Eq. (2) also implies that the total force on the system $F_T = \sum_{i=1}^{n} F_i$ equals the null vector.

Another integral of motion of system (1) is the *angular momentum*

$$J = \sum_{j=1}^{n} m_j r_j \wedge \dot{r}_i.$$

This implies that the torque of the system $\mathcal{T} = \sum_{i=1}^{n} r_i \wedge F_i$ equals the null vector.

Finally the last integral of motion of system (1) is the *energy*

$$E = \sum_{j=1}^{n} \frac{m_j \dot{r}_j^2}{2} - \sum_{i<j} \frac{m_i m_j}{r_{ij}}.$$

The above integrals of motion can be used in system (1) to reduce at most ten degrees of freedom. Thus the study of such a system via first integrals is not adequate for $n \geq 3$. Therefore, the knowledge of particular solutions became important in order to understand this problem.

An interesting class of particular solutions of (1) can be found in the literature as *homographic* solutions in which the shape of the configuration is preserved as time varies. The first homographic solutions are due to Euler [4] and Lagrange [7].

At a given instant $t = t_0$ the n bodies are in a *central configuration* if there exists $\lambda \neq 0$ such that $\ddot{r}_i = \lambda(r_i - c)$, for all $i = 1, \ldots, n$. Such configurations are closely related to homographic solutions. See [10, 12, 13, 15].

Note that to find a central configuration is essentially an algebraic problem since, from equations (1) and the definition, we must solve the following set of equations

$$\lambda(r_i - c) = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3} (r_i - r_j), \tag{3}$$

for $i = 1, \ldots, n$. Eqs. (3) are called equations of central configurations. A simple computation leads to

$$\lambda = -\frac{U}{2I}, \quad U = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{r_{ij}}, \quad I = \frac{1}{2M} \sum_{i<j} m_j m_i |r_i - r_j|^2,$$

where U is the Newtonian potential and I is the inertia moment of the system.

Two central configurations $(r_1, \ldots, r_n)$ and $(\bar{r}_1, \ldots, \bar{r}_n)$ of the n bodies are *related* if we can pass from one to the other through a dilation or a rotation (centered at the center of mass). Thus we can study the classes of central configurations defined by the above equivalence relation.

Taking into account this equivalence we have exactly five classes of central configurations in the 3−body problem. The finiteness of the number of central configurations performed by n bodies with positive masses is a question posed by Chazy in [2] and Wintner in [15], and reformulated to the planar case by Smale in [14]. For $n = 4$ this problem has an affirmative answer [6]. Alternatively, see [1] for a proof of the finiteness when $n = 4$ and a partial answer when $n = 5$. This question remains open when $n > 5$.

Consider the equations

$$f_{ij} = \sum_{\substack{k=1\\k\neq i,j}}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} = 0, \tag{4}$$

for $1 \le i < j \le n$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$. Note that Δ_{ijk} is twice the oriented area of the triangle formed by the bodies at r_i, r_j, and r_k (see [5]). These $n(n-1)/2$ equations are called Dziobek–Laura–Andoyer equations or simply Andoyer equations. We have the following theorem.

Theorem 1 *Consider n bodies with positive masses $m_1, m_2, \ldots, m_n$ and position vectors $r_1, r_2, \ldots, r_n$ in a planar noncollinear configuration. Then Eqs. (3) are equivalent to Eqs. (4).*

It is important to mention that Theorem 1 was proved in [5] and in [8] for the case where the center of mass is at the origin, that is $c = 0$.

It is well known that three bodies with arbitrary positive masses at the vertices of an equilateral triangle are in a central configuration. Another well-known result states that n bodies with equal positive masses at the vertices of a n–gon are in a central configuration, for $n > 3$.

As an application of Theorem 1 we prove that it is not possible to obtain a square central configuration (four bodies with equal positive masses at the vertices a square) and an equilateral triangle central configuration (three bodies with arbitrary positive masses at the vertices an equilateral triangle) as a cocircular central configuration.

Theorem 2 *Consider a configuration of seven bodies with cocircular position vectors $r_1, \ldots, r_7$ and positive masses $m_1, \ldots, m_7$. See Figure 1. Assume that r_1, r_2, r_3, r_4 are at the vertices of a square, $m = m_1 = m_2 = m_3 = m_4$ and r_5, r_6, r_7 are at the vertices of an equilateral triangle. Then there are no positive masses in order that this configuration performs a central configuration.*

For a deep study of cocircular central configurations see [3, 9].

As another application of Theorem 1 we have the following theorem.

Theorem 3 *Consider the noncollinear planar 4–body problem with position vectors r_1, r_2, r_3, r_4 and masses $m = m_1 = m_2 > 0$, $m_3 = m_4 = 0$. Then we have the following possibilities for a central configuration:*

1. *The position vectors r_1, r_2, and r_4 are at the vertices of an equilateral triangle and r_3 is either in the middle point between r_1 and r_2 or in a special position in the straight line through r_1 and r_2. See Figure 2a and 2b, respectively.*
2. *The four bodies are at the vertices of a rhombus with the bodies of equal masses in opposite vertices. See Figures 2c.*

This article is organized as follows. Theorem 1 is proved in Section 2, Theorem 2 is proved in Section 3, and Theorem 3 is proved in Section 4.

2. Proof of Theorem 1

In this section we prove Theorem 1. Suppose that

$$\lambda(r_i - c) = -\sum_{k\neq i} m_k R_{ik} (r_i - r_k)$$

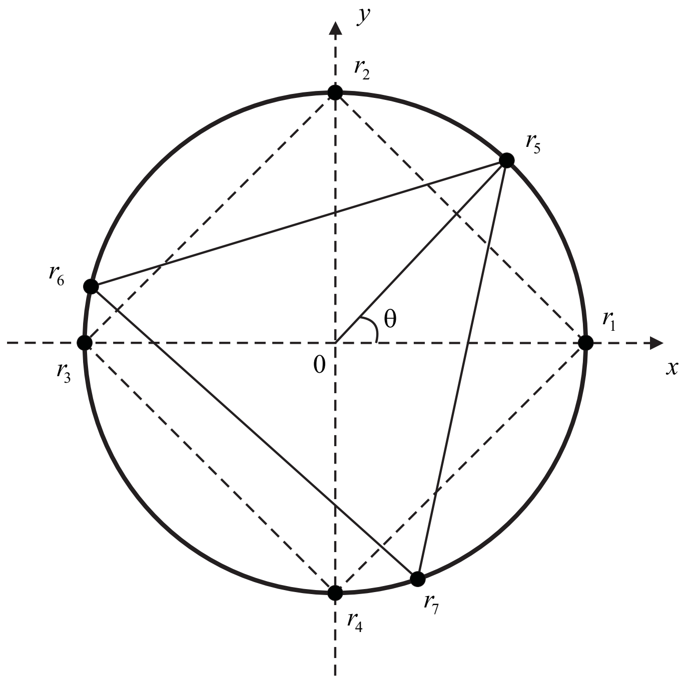

Figure 1. Coordinates for the proof of Theorem 2.

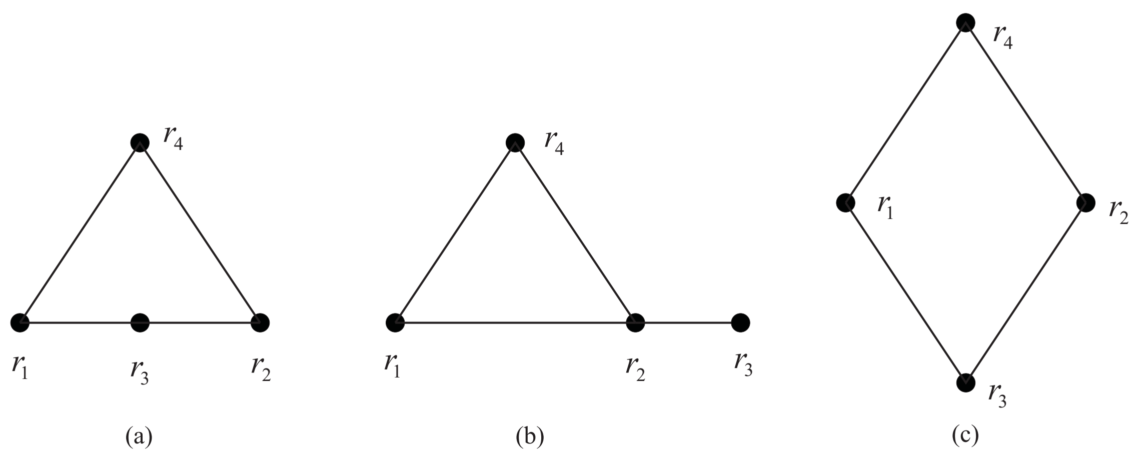

Figure 2. Three possibilities of a noncollinear planar 4–body central configuration with masses $m_1 = m_2 > 0$ and $m_3 = m_4 = 0$.

holds for $1 \leq i \leq n$. Taking $i, j \in \{1, 2, \dots, n\}$, $j \neq i$, we have

$$\lambda(r_i - c) = -\sum_{k \neq i,j} m_k R_{ik}(r_i - r_k) - m_j R_{ij}(r_i - r_j) \tag{5}$$

and

$$\lambda(r_j - c) = -\sum_{k \neq i,j} m_k R_{jk}(r_j - r_k) - m_i R_{ji}(r_j - r_i). \tag{6}$$

The difference between (6) and (5) gives

$$\lambda(r_i - r_j) = -\sum_{k\neq i,j} m_k[R_{ik}(r_i - r_k) - R_{jk}(r_j - r_k)] - [m_j R_{ij} - m_i R_{ij}](r_i - r_j). \qquad (7)$$

Taking the wedge product by $r_i - r_j$ in both sides of (7) we have

$$\begin{aligned} 0 &= -\sum_{k\neq i,j} m_k[R_{ik}(r_i - r_k)\wedge(r_i - r_j) - R_{jk}(r_j - r_k)\wedge(r_i - r_j)] \\ &= -\sum_{k\neq i,j} m_k[R_{ik}\Delta_{ikj} + R_{jk}\Delta_{jki}] \\ &= -\sum_{k\neq i,j} m_k[-R_{ik}\Delta_{ijk} + R_{jk}\Delta_{ijk}] \\ &= \sum_{k\neq i,j} m_k(R_{ik} - R_{jk})\Delta_{ijk} \\ &= f_{ij}. \end{aligned}$$

Therefore, $f_{ij} = 0$ for all $1 \leq i < j \leq n$.

Now assume that $f_{ij} = 0$ for all $1 \leq i < j \leq n$. Then

$$\sum_{k\neq i,j} m_k(R_{ik} - R_{jk})(r_i - r_j)\wedge(r_i - r_k) = 0.$$

This equation can be written as

$$\sum_{k\neq i,j} m_k R_{ik}(r_i - r_j)\wedge(r_i - r_k) = \sum_{k\neq i,j} m_k R_{jk}(r_i - r_j)\wedge(r_i - r_k).$$

Inserting the index j in the sum on the left-hand side and the index i in the sum on the right-hand side we do not change the last equation, and so

$$\sum_{k\neq i} m_k R_{ik}(r_i - r_j)\wedge(r_i - r_k) = \sum_{k\neq j} m_k R_{jk}(r_i - r_j)\wedge(r_i - r_k),$$

which is equivalent to

$$(r_i - r_j)\wedge\sum_{k\neq i} m_k R_{ik}(r_i - r_k) = \sum_{k\neq j} m_k R_{jk}[r_i\wedge(r_j - r_k) + (r_j\wedge r_k)].$$

From the expression of F_i

$$(r_i - r_j)\wedge\frac{F_i}{m_i} = \sum_{k\neq j} m_k R_{jk}[r_i\wedge(r_j - r_k) + (r_j\wedge r_k)]$$

and adding $-r_j$ in the last term we do not change this equation, and so

$$(r_i - r_j)\wedge\frac{F_i}{m_i} = \sum_{k\neq j} m_k R_{jk}[r_i\wedge(r_j - r_k) + r_j\wedge(-r_j + r_k)].$$

Then

$$\begin{aligned}(r_i - r_j) \wedge \frac{F_i}{m_i} &= \sum_{k \neq j} m_k R_{jk}[r_i \wedge (r_j - r_k) - r_j \wedge (r_j - r_k)] \\ &= \sum_{k \neq j} m_k R_{jk}[(r_i - r_j) \wedge (r_j - r_k)] \\ &= (r_i - r_j) \wedge \frac{F_j}{m_j},\end{aligned}$$

which implies that

$$(r_i - r_j) \wedge \frac{F_i}{m_i} = (r_i - r_j) \wedge \frac{F_j}{m_j},$$

or equivalently

$$(r_i - r_j) \wedge (m_j F_i - m_i F_j) = 0. \tag{8}$$

Developing the expression we have

$$r_i \wedge m_j F_i - r_i \wedge m_i F_j - r_j \wedge m_j F_i + r_j \wedge m_i F_j = 0,$$

that is

$$m_j r_i \wedge F_i - m_i r_i \wedge F_j - m_j r_j \wedge F_i + m_i r_j \wedge F_j = 0.$$

Taking the summation in j from 1 to n we get

$$M r_i \wedge F_i - m_i r_i \wedge \sum_{j=1}^{n} F_j - \left(\sum_{j=1}^{n} m_j r_j \right) \wedge F_i + m_i \sum_{j=1}^{n} r_j \wedge F_j = 0, \tag{9}$$

where M is the total mass. Note that with the above definitions equation (9) is

$$M r_i \wedge F_i - m_i r_i \wedge F_T - M c \wedge F_i + m_i \mathcal{T} = 0.$$

Thus, $M(r_i - c) \wedge F_i = 0$, which implies that $(r_i - c)$ and F_i are parallel, that is, $F_i = \lambda_i (r_i - c)$, or equivalently $\ddot{r}_i = (\lambda_i / m_i)(r_i - c)$. From (8) we have

$$\left(\frac{F_i}{m_i} - \frac{F_j}{m_j} \right) \wedge (r_i - r_j) = 0,$$

so

$$\left(\frac{\lambda_i}{m_i}(r_i - c) - \frac{\lambda_j}{m_j}(r_j - c) \right) \wedge (r_i - r_j) = 0$$

or equivalently

$$-\frac{\lambda_i}{m_i}(r_i - c) \wedge (r_j - c) - \frac{\lambda_j}{m_j}(r_j - c) \wedge (r_i - c) = 0,$$

which implies that

$$\left(\frac{\lambda_i}{m_i} - \frac{\lambda_j}{m_j} \right)(r_j - c) \wedge (r_i - c) = 0, \qquad \forall \ i, j.$$

Since the configuration is noncollinear, we have

$$\frac{\lambda_i}{m_i} = \frac{\lambda_j}{m_j} = \lambda,$$

for all i, j. Therefore,

$$\lambda(r_i - c) = \sum_{j \neq i} m_j R_{ij} \,(r_i - r_j),$$

for all $i = 1, 2, \ldots, n$ and thus the equation of central configuration is satisfied.

3. Proof of Theorem 2

In this section we prove Theorem 2 using Theorem 1. Without loss of generality, consider the following coordinates (see Figure 1)

$$r_1 = (1, 0), \quad r_2 = (0, 1), \quad r_3 = (-1, 0), \quad r_4 = (0, -1), \quad r_5 = (\cos(\theta), \sin(\theta)),$$

$$r_6 = \left(\cos\left(\theta + \frac{2\pi}{3}\right), \sin\left(\theta + \frac{2\pi}{3}\right)\right), \quad r_7 = \left(\cos\left(\theta + \frac{4\pi}{3}\right), \sin\left(\theta + \frac{4\pi}{3}\right)\right).$$

Take Andoyer equations (4) according to Theorem 1. For the 7-body problem there are 21 equations in (4). In particular, equation $f_{56} = 0$ must be satisfied, that is

$$f_{56} = m_1 \,(R_{51} - R_{61})\, \Delta_{561} + m_2 \,(R_{52} - R_{62})\, \Delta_{562} + m_3 \,(R_{53} - R_{63})\, \Delta_{563} + m_4 \,(R_{54} - R_{64})\, \Delta_{564} + m_7 \,(R_{57} - R_{67})\, \Delta_{567} = 0.$$

From our assumptions $m_1 = m_2 = m_3 = m_4 = m$ and $R_{57} = R_{67}$ the last equation can be written as

$$f_{56} = m\big[\,(R_{51} - R_{61})\, \Delta_{561} + (R_{52} - R_{62})\, \Delta_{562}\, (R_{53} - R_{63})\, \Delta_{563} + (R_{54} - R_{64})\, \Delta_{564}\big] = 0.$$

Define

$$g(\theta) = (R_{51} - R_{61})\, \Delta_{561} + (R_{52} - R_{62})\, \Delta_{562}\, (R_{53} - R_{63})\, \Delta_{563} + (R_{54} - R_{64})\, \Delta_{564}.$$

Thus a necessary condition to get a central configuration is to take θ as a root of g, that is

$$\begin{aligned} g(\theta) = {} & \frac{\frac{\sqrt{3}}{2} + \sin(x) - \cos\left(x + \frac{\pi}{6}\right)}{(-2\cos(x) + 2)^{-3/2}} - \frac{\frac{\sqrt{3}}{2} + \sin(x) - \cos\left(x + \frac{\pi}{6}\right)}{\left(2 - 2\sin(x + \frac{\pi}{6})\right)^{-3/2}} + \\ & \frac{\frac{\sqrt{3}}{2} - \cos(x) - \sin\left(x + \frac{\pi}{6}\right)}{(-2\sin(x) + 2)^{-3/2}} + \frac{\frac{\sqrt{3}}{2} - \cos(x) - \sin\left(x + \frac{\pi}{6}\right)}{\left(-2\cos(x + \frac{\pi}{6}) + 2\right)^{-3/2}} + \\ & \frac{\frac{\sqrt{3}}{2} - \sin(x) + \cos\left(x + \frac{\pi}{6}\right)}{(2\cos(x) + 2)^{-3/2}} - \frac{\frac{\sqrt{3}}{2} - \sin(x) + \cos\left(x + \frac{\pi}{6}\right)}{\left(2 + 2\sin(x + \frac{\pi}{6})\right)^{-3/2}} + \\ & \frac{\frac{\sqrt{3}}{2} + \cos(x) + \sin\left(x + \frac{\pi}{6}\right)}{(2\sin(x) + 2)^{-3/2}} - \frac{\frac{\sqrt{3}}{2} + \cos(x) + \sin\left(x + \frac{\pi}{6}\right)}{\left(2 + 2\cos(x + \frac{\pi}{6})\right)^{-3/2}}. \end{aligned}$$

Note that, under the assumptions, we just need consider $\theta \in (0, \pi/6)$. Thus, in order to prove the theorem we need to show that g has no root in such interval.

It is clear that g is a differentiable function in the interval $(0,\pi/6)$. Taking the limits for g in this interval we get

$$\lim_{\theta\to 0^+} g(\theta) = +\infty, \qquad \lim_{\theta\to \frac{\pi}{6}^-} g(\theta) = 0.$$

Taking the derivative of g with respect to θ we get $g'(\theta) < -1$ for all values of $\theta \in (0,\pi/6)$. Therefore, the function g has no root in the interval $(0,\pi/6)$. In short, Theorem 2 is proved.

4. Proof of Theorem 3

In this section we prove Theorem 3. For the 4-body problem there are 6 equations in (4), which can be written as: $f_{12} = 0$ is trivially satisfied,

$$f_{13} = m(R_{12} - R_{23})\Delta_{132} = 0, \quad f_{14} = m(R_{12} - R_{24})\Delta_{142} = 0,$$

$$f_{23} = m(R_{12} - R_{13})\Delta_{123} = 0, \quad f_{24} = m(R_{12} - R_{14})\Delta_{124} = 0,$$

$$f_{34} = m\left[(R_{13} - R_{14})\Delta_{134} + (R_{23} - R_{24})\Delta_{234}\right] = 0.$$

For noncollinear central configurations we have the following cases:

Case i) Either $\Delta_{132} = 0$ and $\Delta_{124} \neq 0$ or $\Delta_{132} \neq 0$ and $\Delta_{124} = 0$.

Case ii) $\Delta_{132} \neq 0$ and $\Delta_{124} \neq 0$.

Case i) Without loss of generality, consider $\Delta_{132} = 0$ and $\Delta_{124} \neq 0$. The study of the other possibility is analogous. As $\Delta_{132} = 0$ it means that r_3 is on the straight line through r_1 and r_2. From $f_{14} = 0$, $f_{24} = 0$ and the assumption in this case we have $r_{12} = r_{24}$ and $r_{12} = r_{14}$. Therefore, r_1, r_2, and r_4 are at the vertices of an equilateral triangle. To study the position vector r_3 we take, without loss of generality, the following coordinates

$$r_1 = (-1,0), \quad r_2 = (1,0), \quad r_3 = (x_3,0), x_3 \geq 0, \quad r_4 = (0,\sqrt{3}).$$

Thus, $f_{34} = 0$ has the form

$$\frac{1}{(x_3+1)^2} + \frac{x_3 - 1}{\sqrt{(x_3-1)^2}} - \frac{x_3}{4} = 0,$$

whose solutions satisfying $x_3 \geq 0$ are: $x_3 = 0$ and $x_3 = 2.39681$ with five decimals. See Figures 2a and 2b. Hence we have proved item 1 of Theorem 3.

Case ii) $\Delta_{132} \neq 0$ and $\Delta_{124} \neq 0$. From equations $f_{13} = 0$, $f_{14} = 0$, $f_{23} = 0$, and $f_{24} = 0$ we have $r_{12} = r_{23}$, $r_{12} = r_{24}$, $r_{12} = r_{13}$, and $r_{12} = r_{14}$, respectively. From these equalities equation $f_{34} = 0$ is satisfied. Therefore, the position vectors r_1, r_2, r_3, and r_4 are at the vertices of a rhombus. See Figure 2c. We have proved item 2 of Theorem 3.

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