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MOHAMED ECH CHAD

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Ranges and kernels of derivations

Mohamed ECH-CHAD*
Department of Mathematics, Faculty of Sciences Kénitra, Kénitra, Morocco

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Abstract: In this paper we establish some properties concerning the class of operators $A \in \mathcal{L}(\mathcal{H})$ that satisfy $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$, where $\overline{\mathcal{R}(\delta_A)}$ is the norm closure of the range of the inner derivation δ_A, defined on $\mathcal{L}(\mathcal{H})$ by $\delta_A(X) = AX - XA$. Here $\mathcal{H}$ stands for a Hilbert space; as a consequence, we show that the set $\{A \in \mathcal{L}(\mathcal{H}) \ / \ \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}\}$ is norm-dense. We also describe some classes of operators A, B for which we have $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$ ($\ker(\delta_{A^*,B^*})$ is the kernel of the generalized derivation δ_{A^*,B^*} defined on $\mathcal{L}(\mathcal{H})$ by $\delta_{A^*,B^*}(X) = A^*X - XB^*$).

Key words: Generalized derivation, p-hyponormal operator, log-hyponormal operator, range and kernel

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators acting on a complexe infinite dimensional Hilbert space $\mathcal{H}$. For A, $B \in \mathcal{L}(\mathcal{H})$ we define the generalized derivation $\delta_{A,B}$ associated with (A, B) by $\delta_{A,B}(X) = AX - XB$ for $X \in \mathcal{L}(\mathcal{H})$. If $A = B$, then $\delta_{A,A} = \delta_A$ is called the inner derivation implemented by $A \in \mathcal{L}(\mathcal{H})$. These concrete operators on $\mathcal{L}(\mathcal{H})$ occur in many settings in mathematical analysis and application, their properties, spectrum (see [7, 13, 20]), norm (see [23]), ranges, and kernels (see [4, 5, 8, 9, 15, 27]) have been much studied, and many of their problems remain also open (see [3, 18, 26]).

Let $\mathcal{N} = \bigcup_{A \in \mathcal{L}(\mathcal{H})} \mathcal{R}(\delta_A) \cap \{A\}'$, where $\mathcal{R}(\delta_A)$ denotes the range of δ_A and $\{A\}'$ is the commutant of A. In finite dimension, it is known that $\mathcal{N}$ is exactly the set of nilpotent operators. In infinite dimension the theorem of Kleinecke–Shirokov [19] confirms that any operator in $\mathcal{N}$ is quasinilpotent. However, an operator in $\overline{\mathcal{R}(\delta_A)} \cap \{A\}'$ is not necessarily quasinilpotent (Anderson [1] proved that there exists an operator A in $\mathcal{L}(\mathcal{H})$ such that $I \in \overline{\mathcal{R}(\delta_A)}$), where $\overline{\mathcal{R}(\delta_A)}$ is the normal closure of $\mathcal{R}(\delta_A)$.

In [2] Anderson proved the remarkable result that $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$ if A is normal or isometric. In the same direction, it should be noted that Bouali and Bouhafsi [6] showed that if A is a cyclic subnormal operator then $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$.

The purpose of the first section is to establish some properties of the class of operators $A \in \mathcal{L}(\mathcal{H})$ that satisfy $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$. As a consequence, we give a large class of operators $A \oplus B$ verifying $\overline{\mathcal{R}(\delta_{A \oplus B})} \cap \{A \oplus B\}' = \{0\}$, and we prove that the set $\{A \in \mathcal{L}(\mathcal{H}) \ / \ \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}\}$ is norm-dense in $\mathcal{L}(\mathcal{H})$.

*Correspondence: m.echchad@yahoo.fr

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If H is a finite dimensional Hilbert space $< X,Y >= tr(XY^*)$ is an inner product on $\mathcal{L}(\mathcal{H})$ and we have the orthogonal direct sum decomposition $\mathcal{L}(\mathcal{H}) = \mathcal{R}(\delta_A)\bigoplus\{A^*\}'$. However, when H is infinite dimensional we do not have $\overline{\mathcal{R}(\delta_A)}\cap\{A^*\}' = \{0\}$ in general. The class of operators A that have the property $\overline{\mathcal{R}(\delta_A)}\cap\{A^*\}' = \{0\}$ includes the normal operators [2], isometries [25], the cyclic subnormal operators [16], the class of operators A such that $P(A)$ is normal for some quadratic polynomial P [16], and Jordan operators [22].

In [12] Elalami proved that $\overline{\mathcal{R}(\delta_{A,B})}\cap\ker(\delta_{A^*,B^*}) = \{0\}$ if A^* and B are hyponormal operators, where $\ker(\delta_{A^*,B^*})$ denotes the kernel of δ_{A^*,B^*}. In the second section we consider this problem; we show that $\overline{\mathcal{R}(\delta_{A,B})}\cap\ker(\delta_{A^*,B^*}) = \{0\}$ if $(P(A),P(B))$ and $(P(B),P(A))$ has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property for some quadratic polynomial P. Consequently, we extend the result of [16] to $\delta_{A,B}$. Using the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property we prove that $\overline{\mathcal{R}(\delta_{A,B})}\cap\ker(\delta_{A^*,B^*}) = \{0\}$ in each of the following cases:

(a) B is normal and A^* is p-hyponormal or log-hyponormal, $(0 < p \leq 1)$.

(b) A is normal and B is p-hyponormal or log-hyponormal, $(0 < p \leq 1)$.

An operator $A \in \mathcal{L}(\mathcal{H})$ is p-hyponormal, $0 < p \leq 1$, if $|A^*|^{2p} \leq |A|^{2p}$ (a 1-hyponormal operator is hyponormal and a $\frac{1}{2}-$hyponormal operator is semihyponormal). It is an immediate consequence of the Lowner–Heinz inequality that a p-hyponormal operator is q-hyponormal for all $0 < q \leq p$. An invertible operator $A \in \mathcal{L}(\mathcal{H})$ is log-hyponormal if $log|A^*|^{2p} \leq log|A|^{2p}$. An invertible p-hyponormal operator is log-hyponormal, but the converse is false; see [17, p. 169] for a reference. Log-hyponormal and p-hyponormal operators, which share a number of properties with hyponormal operators, have been considered by a number of authors in the recent past; see [11, 17, 24] for further references.

2. Commutants and derivation ranges

Definition 2.1 *A vector $x \in \mathcal{H}$ is cyclic for $A \in \mathcal{L}(\mathcal{H})$ if $\mathcal{H}$ is the smallest invariant subspace for A that contains x. The operator A is said to be cyclic if it has a cyclic vector.*

Definition 2.2 *Let $A \in \mathcal{L}(\mathcal{H})$. The operator A is said to be subnormal if there exists a normal operator N on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$, invariant under the operator N, and the restriction of N to $\mathcal{H}$ coincides with A.*

Consider the set $\mathcal{M}_{\mathcal{C}}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \ / \ \overline{\mathcal{R}(\delta_A)}\cap\{A\}' = \{0\}\}$.

Theorem 2.3 *Let A and B be in $\mathcal{M}_{\mathcal{C}}(\mathcal{H})$, such that $\sigma(A)\cap\sigma(B) = \emptyset$. Then $A\oplus B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H}\oplus\mathcal{H})$.*

Proof Assume that $A, B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$, and $\sigma(A)\cap\sigma(B) = \emptyset$. Let $C = A\oplus B \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$, and $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \in \overline{\mathcal{R}(\delta_C)}\cap\{C\}'$. Then there exists a net $(X_n)_n \subset \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ such that $X_n = \begin{pmatrix} X_n^1 & X_n^2 \\ X_n^3 & X_n^4 \end{pmatrix}$,

$$CX_n - X_nC \overset{\|\cdot\|}{\longrightarrow} D \quad \text{and} \quad CD = DC.$$

A simple calculation shows that

$$AX_n^1 - X_n^1A \overset{\|\cdot\|}{\longrightarrow} D_1 \quad \text{and} \quad AD_1 = D_1A,$$

$$BX_n^4 - X_n^4 B \xrightarrow{\|\cdot\|} D_4 \quad \text{and} \quad BD_4 = D_4 B,$$

$$BX_n^3 - X_n^3 A \xrightarrow{\|\cdot\|} D_3 \quad \text{and} \quad BD_3 = D_3 A,$$

$$AX_n^2 - X_n^2 B \xrightarrow{\|\cdot\|} D_2 \quad \text{and} \quad AD_2 = D_2 B.$$

Hence $D_1 \in \overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$, $D_4 \in \overline{\mathcal{R}(\delta_B)} \cap \{B\}' = \{0\}$, $D_3 \in \overline{\mathcal{R}(\delta_{B,A})} \cap \ker(\delta_{B,A})$, and $D_2 \in \overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A,B})$. Since $\sigma(A) \cap \sigma(B) = \emptyset$, it follows from Rosemblem's theorem [21] that $D_2 = D_3 = 0$. Thus $A \oplus B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H} \oplus \mathcal{H})$. □

Theorem 2.4 *Let A, $B \in \mathcal{L}(\mathcal{H})$, with B similar to A and $A \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$. Then $B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$.*

Proof Let A, $B \in \mathcal{L}(\mathcal{H})$, such that $A \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$ and there exists an invertible operator $S \in \mathcal{L}(\mathcal{H})$ verifying $B = S^{-1}AS$. Then for all $X \in \mathcal{L}(\mathcal{H})$,

$$S^{-1}(AX - XA)S = B(S^{-1}XS) - (S^{-1}XS)B.$$

Thus $S^{-1}\overline{\mathcal{R}(\delta_A)}S = \overline{\mathcal{R}(\delta_B)}$. Hence

$$\begin{aligned} \overline{\mathcal{R}(\delta_B)} \cap \{B\}' &= \left[S^{-1}\overline{\mathcal{R}(\delta_A)}S\right] \cap \left[S^{-1}\{A\}'S\right] \\ &= S^{-1}\left[\overline{\mathcal{R}(\delta_A)} \cap \{A\}'\right]S \\ &= \{0\}. \end{aligned}$$

This completes the proof. □

Corollary 2.5 *Let $A \in \mathcal{L}(\mathcal{H})$. If A is similar to a normal, isometric, or cyclic subnormal operator then*

$$\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}.$$

Proof Anderson proved that $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$ if A is normal or isometric [2], and in [6] Bouali and Bouhafsi showed that if A is cyclic subnormal then $\overline{\mathcal{R}(\delta_A)} \cap \{A\}' = \{0\}$. □

Corollary 2.6 *Let A, $B \in \mathcal{L}(\mathcal{H})$, with $\sigma(A) \cap \sigma(B) = \emptyset$. If A and B are similar to normal, isometric, or cyclic subnormal operators, all combinations are allowed; then*

$$\overline{\mathcal{R}(\delta_{A \oplus B})} \cap \{A \oplus B\}' = \{0\}.$$

Definition 2.7 *[14] we shall say that a certain property (P) of operators acting on a Hilbert space $\mathcal{H}$ is a bad-property, or b-property, if:*

(i) *Whenever A satisfies (P), then for $\alpha \in \mathbb{C}$, with $\alpha \neq 0$, and $\beta \in \mathbb{C}$, the operator $\alpha A + \beta$ satisfies (P);*

(ii) *If B is similar to A, and A satisfies (P), then B also satisfies (P);*

(iii) *If A and B satisfy (P), such that $\sigma(A)\cap\sigma(B)=\emptyset$, then $A\oplus B$ satisfies (P).*

Theorem 2.8 *$\mathcal{M}_{\mathcal{C}}(\mathcal{H})$ is norm-dense in $\mathcal{L}(\mathcal{H})$.*

Proof Using [14], theorem 3.5.1, it is sufficient to establish that the property $A\in\mathcal{M}_{\mathcal{C}}(\mathcal{H})$ is a b-property.

(i) If $A\in\mathcal{M}_{\mathcal{C}}(\mathcal{H})$, then for $\alpha\in\mathbb{C}$, with $\alpha\neq 0$, and $\beta\in\mathbb{C}$,

$$\overline{\mathcal{R}(\delta_{\alpha A+\beta})}\cap\{\alpha A+\beta\}'=\overline{\mathcal{R}(\delta_A)}\cap\{A\}'=\{0\}.$$

Thus $\alpha A+\beta\in\mathcal{M}_{\mathcal{C}}(\mathcal{H})$. This proves the first condition.

(ii) By theorem 2.4, $A\in\mathcal{M}_{\mathcal{C}}(\mathcal{H})$ is invariant for similarity. The second condition is then verified.

(iii) By theorem 2.3, the third condition of the b-property is fulfilled. This completes the proof.

□

Remark 2.9 *In [16], theorem 2, Ho shows that $N=\{A\in\mathcal{L}(\mathcal{H})\ \ /\ \ I\notin\overline{\mathcal{R}(\delta_A)}\}$ is norm-dense in $\mathcal{L}(\mathcal{H})$. Clearly $\mathcal{M}_{\mathcal{C}}(\mathcal{H})\subset N$. Theorem 2.8 generalizes Ho's result.*

3. Ranges and kernels of generalized derivations

Definition 3.1 *Let A, B be in $\mathcal{L}(\mathcal{H})$. The pair (A,B) is said to possess the Fuglede–Putnam property $(F-P)_{\mathcal{L}(\mathcal{H})}$ if; $AT=TB$ and $T\in\mathcal{L}(\mathcal{H})$ implies $A^*T=TB^*$.*

Lemma 3.2 *Let A, $X\in\mathcal{L}(\mathcal{H})$ such that $P\geq 0$ and $PX+XP=0$. Then $PX=XP=0$.*

Proof Assume that $PX+XP=0$. Then $P^2X=XP^2$, and since $P\in\{P^2\}''$ ($\{P^2\}''$ is the bicommutant of P^2), it follows that $PX=XP$. Thus $PX=XP=0$. □

Lemma 3.3 *Let A, $B\in\mathcal{L}(\mathcal{H})$. If (A,B) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property, then*

$$\overline{\mathcal{R}(\delta_{A,B})}\cap\ker(\delta_{A,B})=\{0\}.$$

Proof In the proof of theorem 1 [27], Yusun shows that $\|\delta_{A,B}(X)+T\|\geq\|T\|$ for all $X\in\mathcal{L}(\mathcal{H})$ and $T\in\ker(\delta_{A,B})$, if (A,B) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property. □

Theorem 3.4 *Let A, B be in $\mathcal{L}(\mathcal{H})$. If $(P(A),P(B))$ and $(P(B),P(A))$ have the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property for some quadratic polynomial P then*

$$\overline{\mathcal{R}(\delta_{A,B})}\cap\ker(\delta_{A^*,B^*})=\{0\}.$$

Proof Since for all $(\alpha,\beta)\in\mathbb{C}^2$, with $\alpha\neq 0$,

$$\mathcal{R}(\delta_{\alpha A+\beta,\alpha B+\beta})=\mathcal{R}(\delta_{A,B})\ \text{ and }\ \ker(\delta_{\alpha A+\beta,\alpha B+\beta})=\ker(\delta_{A,B})$$

we may assume without loss of generality that (A^2, B^2) and (B^2, A^2) have the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property. Let $T^* \in \overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*})$. Then there exists a sequence $(X_n)_n$ in $\mathcal{L}(\mathcal{H})$ such that:

$$AX_n - X_nB \xrightarrow{\|\cdot\|} T^* \quad \text{and} \quad TA = BT.$$

This implies that

$$A^2X_n - X_nB^2 \xrightarrow{\|\cdot\|} AT^* + T^*B \quad \text{and} \quad TA^2 = B^2T.$$

Since (B^2, A^2) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property, it follows that $A^2T^* = T^*B^2$. Hence $A^2(AT^* + T^*B) = (AT^* + T^*B)B^2$. Consequently,

$$AT^* + T^*B \in \overline{\mathcal{R}(\delta_{A^2,B^2})} \cap \ker(\delta_{A^2,B^2}).$$

Using lemma 3.3 we have $AT^* + T^*B = 0$. By multiplication right by T, and using $BT = TA$, we obtain $AP + PA = 0$ with $P = T^*T$. It follows from lemma 3.2 that $AP = PA = 0$. On the other hand, $A(X_nT) - (X_nT)A \xrightarrow{\|\cdot\|} T^*T = P$; and by multiplication of right and left by P, we get $P^3 = 0$. Since P is self-adjoint, then $P = 0$, and this necessarily implies $T = 0$. Thus $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$. □

Corollary 3.5 *[16] Let $A \in \mathcal{L}(\mathcal{H})$. If $P(A)$ is normal for some quadratic polynomial P, then $\overline{\mathcal{R}(\delta_A)} \cap \{A^*\}' = \{0\}$.*

Corollary 3.6 *Let $A,\ B \in \mathcal{L}(\mathcal{H})$. If $P(A)$ and $P(B)$ are normal operators for some quadratic polynomial P, then $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$.*

Proposition 3.7 *Let A, B be in $\mathcal{L}(\mathcal{H})$, such that (B, A) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property. If $T \in \overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*})$, then $T^*T \in \overline{\mathcal{R}(\delta_B)} \cap \{B\}'$ and $TT^* \in \overline{\mathcal{R}(\delta_A)} \cap \{A\}'$.*

Proof Assume that $T \in \overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*})$. Then there exists a sequence $(X_n)_n$ of elements of $\mathcal{L}(\mathcal{H})$ such that

$$AX_n - X_nB \xrightarrow{\|\cdot\|} T \quad \text{and} \quad BT^* = T^*A.$$

Since right and left multiplication are continuous with respect to the norm topology, it follows that

$$B(T^*X_n) - (T^*X_n)B = T^*(AX_n - X_nB) \xrightarrow{\|\cdot\|} T^*T,$$

and

$$A(X_nT^*) - (X_nT^*)A = (AX_n - X_nB)T^* \xrightarrow{\|\cdot\|} TT^*.$$

Hence $T^*T \in \overline{\mathcal{R}(\delta_B)}$ and $TT^* \in \overline{\mathcal{R}(\delta_A)}$. On the other hand, (B, A) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property; then $TB = AT$. Consequently we get $T^*T \in \overline{\mathcal{R}(\delta_B)} \cap \{B\}'$ and $TT^* \in \overline{\mathcal{R}(\delta_A)} \cap \{A\}'$. □

Corollary 3.8 *Let* A, B *be in* $\mathcal{L}(\mathcal{H})$, *such that* (B,A) *has the* $(F-P)_{\mathcal{L}(\mathcal{H})}$ *property. If* $A \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$ *or* $B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$, *then* $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$.

Corollary 3.9 *Let* A, B *in* $\mathcal{L}(H)$, *then* $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$ *in one of the follwing conditions:*

(1) *B is normal and A^* is p-hyponormal or log-hyponormal,* $(0 < p \leq 1)$.

(2) *A is normal and B is p-hyponormal or log-hyponormal,* $(0 < p \leq 1)$.

Proof (1). Assume that B is normal and A^* is p-hyponormal or log-hyponormal. Then B is p-hyponormal and A^* is p-hyponormal or log-hyponormal. It follows from lemma 2.1 [10] that (B,A) has the $(F-P)_{\mathcal{L}(\mathcal{H})}$ property. Since B is normal, $B \in \mathcal{M}_{\mathcal{C}}(\mathcal{H})$ [2]. Using the corollary 3.8 we obtain $\overline{\mathcal{R}(\delta_{A,B})} \cap \ker(\delta_{A^*,B^*}) = \{0\}$. We obtain (2) in the same way. □

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