

1-1-2017

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Recommended Citation

ASAR, ALİ OSMAN (2017) "Permutation groups with cyclic-block property and $\$MNFC\$$ -groups," *Turkish Journal of Mathematics*: Vol. 41: No. 4, Article 17. <https://doi.org/10.3906/mat-1508-66>
Available at: <https://journals.tubitak.gov.tr/math/vol41/iss4/17>

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Permutation groups with cyclic-block property and *MNFC*-groups

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Received: 20.08.2015

Accepted/Published Online: 07.10.2016

Final Version: 25.07.2017

Abstract: This work continues the investigation of perfect locally finite minimal non-*FC*-groups in totally imprimitive permutation p -groups. At present, the class of totally imprimitive permutation p -groups satisfying the cyclic-block property is known to be the only class of p -groups having common properties with a hypothetical minimal non-*FC*-group, because a totally imprimitive permutation p -group satisfying the cyclic-block property cannot be generated by a subset of finite exponent and every non-*FC*-subgroup of it is transitive, which are the properties satisfied by a minimal non-*FC*-group. Here a sufficient condition is given for the nonexistence of minimal non-*FC*-groups in this class of permutation groups. In particular, it is shown that the totally imprimitive permutation p -group satisfying the cyclic-block property that was constructed earlier and its commutator subgroup cannot be minimal non-*FC*-groups. Furthermore, some properties of a maximal p -subgroup of the finitary symmetric group on \mathbb{N}^* are obtained.

Key words: Finitary permutation, totally imprimitive, cyclic-block property, homogeneous permutation, *FC*-group

1. Introduction

Let Ω be a nonempty (infinite) set. A permutation g on Ω is called finitary if its support $\text{supp}(g)$ is finite. The set of all the finitary permutations on Ω forms a normal subgroup of the symmetric group $\text{Sym}(\Omega)$ and is called the restricted symmetric group on Ω . It is denoted by $\text{FSym}(\Omega)$. A subgroup of $\text{FSym}(\Omega)$ is called a finitary permutation group on Ω . Let G be a transitive finitary permutation group on Ω , where Ω is infinite. If G has no proper blocks or has a maximal proper block, then G is called primitive or almost primitive, respectively, and then G has a homomorphic image that is isomorphic to one of $\text{Alt}(\Omega)$ or $\text{FSym}(\Omega)$ by [10, p.261] (see also [9, Corollary 6.9]). Note that if Δ is a proper block for G , then there exists a $g \in G$ with $g(\Delta) \cap \Delta = \emptyset$ since two blocks are either equal or disjoint and then Δ must be finite since $\text{supp}(g)$ is finite. In the remaining case G is called totally imprimitive. In this case, G has an infinite ascending chain of proper blocks and their union is an infinite block for G , which must be equal to Ω since G is transitive. Thus, Ω and G are countably infinite. It is well known that a finitary permutation group G has only finite orbits if and only if one of the following holds:

G is solvable, hypercentral, an *FC*-group, or residually finite by [23, Theorems 1,2] or [10, Lemma 8.3D]. If G is locally solvable, then G is totally imprimitive and hyperabelian of height at most ω by [18, Theorem 2].

Let G be a totally imprimitive subgroup of $\text{FSym}(\Omega)$, where Ω is infinite. It is well known that set-wise stabilizers of finite sets are *FC*-groups and they are hypercentral when G is a p -group by [23, Theorem 1] or

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2010 *AMS Mathematics Subject Classification*: 20B 07 20B 35 20E 25 20F 15.

[10, Lemma 8.3D]. Subgroups having infinite orbits of G are non- FC -subgroups (NFC -subgroups for short). In general an NFC -group is called a minimal NFC -group ($MNFC$ -group for short) if every proper subgroup of this group is an FC -group.

The structure of an imperfect $MNFC$ -group was determined in [6, 7] (see also [22, Theorem 8.13]). In this group the commutator subgroup is a divisible abelian q -group of finite rank (Chernikov q -group) and the commutator quotient is a finite p -group, where p, q are primes. On the other hand, it is still unknown whether or not a perfect $MNFC$ -group exists. If a perfect MNF -group exists, then it is a p -group for a prime p by [7, Theorem 2] and [14, Theorem], and it has a nontrivial representation in the group of finitary permutations on some infinite set by the characterizations given in [8, 15]. (Some partial results in this direction are contained in [1–5].)

Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$, where Ω is infinite. An element g of G is said to satisfy the cyclic-block property if the support of each cycle in the cycle decomposition of g is a block for G , and a subset Y of G satisfies the cyclic-block property if every element of Y satisfies this property. Now suppose in addition that G satisfies the cyclic-block property. By [4, Lemma 2.2] two blocks for G are either disjoint or one is contained in the other one. This implies that G must be a p -group for a prime p . Furthermore, every NFC -subgroup of G is transitive and a subset of finite exponent of G generates a subgroup of finite exponent and so cannot be equal to G (see Lemma 3.1(b) below). These are properties satisfied by a perfect $MNFC$ -group. (A perfect $MNFC$ -group cannot be generated by a subset of finite exponent (see [1, Remark 1.10]).) There are no known other types of p -groups that share common properties with a perfect $MNFC$ - p -group. For this reason it is a rather crucial step to settle the existence problem of $MNFC$ -groups in the class of permutation groups satisfying the cyclic-block property. In this work a new result (Theorem 1.1) is obtained in this direction. This result is a considerable generalization of [1, Theorem 1.5] (see below). In particular, if a group in this class is generated by homogeneous elements and satisfies $(*)$ (see below), then the group cannot be $MNFC$ (Corollary 1.2). Furthermore, Theorem 1.1 provides a short proof for [4, Theorem 1.2] (Corollary 1.3). (Another proof of [4, Theorem 1.2] is contained in [5, Theorem 1.6].) The group given in [4, Theorem 1.1] satisfies the cyclic-block property, it has an easily defined generating set, and all of its blocks of p -power size are easily described, but it is not known whether or not it contains an $MNFC$ -subgroup. (This group satisfying the cyclic-block property is a transitive subgroup of the maximal p -subgroup, denoted by W here, of $FSym(\mathbb{N}^*)$ constructed in [22]; see Proposition 2.1 for some properties of W .) [5] contains new properties of NFC -subgroups of a perfect totally imprimitive p -subgroup of $FSym(\Omega)$. Among other things it is shown there that the normalizer of an NFC -subgroup is self-normalizing and a self-normalizing subgroup is closed in the topology of point-wise convergence (see also [15]). It follows from [5] that a group of finitary permutations contains an $MNFC$ -subgroup if and only if the set of self-normalizing subgroups contains minimal elements.

Let G be a subgroup of $FSym(\Omega)$ and let $g \in G$. The minimum of the lengths of the cycles in the cycle decomposition of g is denoted by $m(g)$. g is called homogeneous if every cycle of g has equal length. An infinite subset Y of G is called ascending if Y has an infinite exponent and is not contained in a set stabilizer of a finite set. We say that Y satisfies the property $(*)$ if for every $y, z \in Y$, and for all cycles c_y, c_z in the cycle decompositions of y and z , respectively, the following holds. Put $supp(c_y) = \Delta$ and suppose that $\Delta \subseteq supp(c_z)$. Then

(*)

$$[c_z^{s(c_z, \Delta)}|_{\Delta}, c_y] = 1$$

where $s(c_z, \Delta)$ is the smallest positive integer such that $c_z^{s(c_z, \Delta)} \in G_{\{\Delta\}}$.

It is well known that this condition is equivalent to

$$c_z^{s(c_z, \Delta)}|_{\Delta} = c_y^k$$

for a $k \geq 1$ by [13, Lemma 1]. (The centralizer of a cycle is generated by the cycle itself and permutations disjoint with it.)

Let Δ be a block for G and put $\Sigma = \{x(\Delta) : x \in G\}$. Then the kernel of the natural permutation representation of G into $Sym(\Sigma)$ is denoted by $Ker_G(\Delta)$ and is called the kernel subgroup of G with respect to Δ . Since $Ker_G(\Delta)$ fixes $x(\Delta)$ for every $x \in G$ it follows that $Ker_G(\Delta)$ is isomorphic to a subgroup of the direct product of copies of a finite group, and so $Ker_G(\Delta)$ is an FC -group of finite exponent.

For a nonempty subset X of G , $exp(X)$ denotes the maximum of the set $\{|x| : x \in X\}$ if it exists; otherwise, it is equal to ∞ .

Theorem 1.1 *Let G be a perfect totally imprimitive p -subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that G contains an ascending subset X satisfying the cyclic-block property such that the following properties hold.*

(a) *X satisfies (*). Thus for all $x, y \in X$ and for all cycles c_x, c_y in the cycle decompositions of x and y , respectively, the following holds. If $supp(c_x) \subseteq supp(c_y)$, then*

$$[c_y^{s(c_y, supp(c_x))}|_{supp(c_x)}, c_x] = 1,$$

where $supp(c_x)$ and $supp(c_y)$ are blocks for G , which is equivalent to

$$c_y^{s(c_y, supp(c_x))}|_{supp(c_x)} = c_x^{q(c_x)}$$

for a $q(c_x) \geq 1$.

(b) *For every $x \in X$ there exists a $y \in X$ so that $m(x) < m(y)$.*

Then G cannot be an $MNFC$ -group.

Theorem 1.1 is a considerable generalization of [1, Theorem 1.5]. In [1, Theorem 1.5] if F is a finite subgroup of G and $supp(F) \subseteq \Delta$ for a finite block Δ , then there exists $y \in G \setminus G_{\{\Delta\}}$ so that $y^{s(y, \Delta)} \in C_G(F)$. In particular $[F^y, F] = 1$ since $supp(F^y) \cap \Delta = 1$. This leads to the existence of an ascending subgroup H of G for a given $a \in G$ with $\langle a^G \rangle$ nonabelian so that $\langle a^H \rangle$ is abelian, which gives a contradiction. On the other hand, in Theorem 1.1, there is information only about the centralizer of a cycle, namely c_x of $x \in X$, but X is required to satisfy the additional property called the cyclic-block property. (Also in the proof of Theorem 1.1 $\langle c_x^G \rangle$ is not abelian, but there will exist an ascending subgroup, say X^* of G , so that $\langle c_x^{X^*} \rangle$ is abelian, which gives a contradiction.) It is not known yet whether condition (b) of Theorem 1.1 is indispensable.

Corollary 1.2 *Let G be a perfect totally imprimitive p -subgroup of $FSym(\Omega)$, where Ω is infinite. Suppose that G contains an ascending subset X of homogeneous elements satisfying the cyclic-block property and the (*) condition. Then G cannot be an MNFC-group.*

Corollary 1.3 *The totally imprimitive p -subgroup of $FSym(\mathbb{N}^*)$ given in [4, Theorem 1.1] and its commutator subgroup cannot be MNFC-groups.*

For definitions, notations, and basic properties the reader is referred to [9, 10, 21, 22].

Question. Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$ satisfying the cyclic-block property where Ω is infinite. Does G contain a minimal non-FC subgroup?

2. A finitary permutation group with cyclic-block property

In this section the finitary permutation p -group given in [4] and satisfying the cyclic-block property is described briefly for the convenience of the reader. This group is a subgroup of the example given in [23] by Wiegold.

For each $k, n \geq 1$ define

$$x_{k,n} = \prod_{i=1}^{p^{k-1}} (i + (n-1)p^k, i + (n-1)p^k + p^{k-1}, \dots, i + (n-1)p^k + (p-1)p^{k-1}).$$

Each $x_{k,n}$ is a disjoint product of p^{k-1} cycles, each of which has length p .

For each $k \geq 1$ define

$$T_k = \{x_{k,n}; n \geq 1\} \text{ and } T_k^* = \langle T_i : 1 \leq i \leq k \rangle.$$

Wiegold’s group, denoted here by W , is defined as $W = \langle T_k : k \geq 1 \rangle$. T_k is a set of pairwise disjoint permutations of order p and it is easy to check that $T_k^* \triangleleft W$ and T_{k+1}^*/T_k^* is elementary abelian for every $k \geq 0$, where $T_0 = 1$. W is a totally imprimitive p -subgroup of $FSym(\mathbb{N}^*)$ since every element of every T_k^* has finite support.

For $p = 2$,

$$T_1 = \{(1, 2), (3, 4), (5, 6), \dots\}, T_2 = \{(1, 3)(2, 4), (5, 7)(6, 8), (9, 11)(10, 12), \dots\}$$

$$T_3 = \{(1, 5)(2, 6)(3, 7)(4, 8), (9, 13)(10, 14)(11, 15)(12, 16), \dots\}.$$

For all $k, n \geq 1$ the sets

$$\Delta_{k,n} = \{1 + (n-1)p^k, 2 + (n-1)p^k, \dots, p^k + (n-1)p^k\}$$

are blocks for W and $|\Delta_{k,n}| = p^k$. We may show that each $\Delta_{k,1}$ is a block. We may put $\Delta_k = \Delta_{k,1}$ when no confusion arises. Thus, $\Delta_k = \{1, 2, \dots, p^k\}$ for $k \geq 1$. It suffices to show that $T_k^*(1) = \Delta_k$ for all $k \geq 1$ by [10, Theorem 1.6A(i)]. For $k = 1$ $T_1(1) = \{1, 2, \dots, p\} = \Delta_1$. Assume that $T_k^*(1) = \Delta_k$. Now

$$\begin{aligned} x_{k+1,1} &= (1, 1 + p^k, 1 + 2p^k, \dots, 1 + (p-1)p^k) \cdots (p^k, p^k + p^k, p^k + 2p^k, \dots, p^k + (p-1)p^k) \\ &= (1, 1 + p^k, 1 + 2p^k, \dots, 1 + (p-1)p^k) \cdots (p^k, 2p^k, 3p^k, \dots, p^{k+1}). \end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned} \langle x_{k+1,1} \rangle(\Delta_k) &= \{1, 2, \dots, p^k\} \cup \{1 + p^k, 2 + p^k, \dots, 2p^k\} \cup \dots \cup \{1 + (p-1)p^k, 2 + (p-1)p^k, \dots, p^{k+1}\} \\ &= \Delta_{k+1} \end{aligned}$$

since the sets in the union are pairwise disjoint and are contained in Δ_{k+1} . In particular, it is easy to see that $x_{k+1,1}$ permutes the sets

$$\{1, 2, \dots, p^k\}, \{1 + p^k, 2 + p^k, \dots, 2p^k\}, \dots, \{1 + (p-1)p^k, 2 + (p-1)p^k, \dots, p^{k+1}\}$$

among themselves. Since $\langle x_{k+1,1} \rangle(\Delta_k) = \langle x_{k+1,1} \rangle(T_k^*(1)) = (\langle x_{k+1,1} \rangle T_k^*)(1) = T_{k+1}^*(1)$ it follows that $T_{k+1}^*(1) = \Delta_{k+1}$, which was to be shown. It can be shown that the finite blocks of p -power size for W consist of

$$\Delta_{k,n} = \{1 + (n-1)p^k, 2 + (n-1)p^k, \dots, p^k + (n-1)p^k\}$$

for $k, n \geq 1$.

Define

$$u_k = x_{k,1} x_{k-1,1} \cdots x_{1,1}$$

for all $k \geq 1$. Then $u_k \in T_k^*$ and $u_k = (a_1, a_2, \dots, a_{p^k})$, where $1 \leq a_i \leq p^k$ by [4, Lemma 3.2(a)]. Next define

$$v_k = u_k^{x_{k+1,1}} \cdots u_k^{x_{k+1,1}^{p-1}}.$$

Then $v_k = u_k^{x_{k+1,1}} \times \cdots \times u_k^{x_{k+1,1}^{p-1}}$, i.e. a product of disjoint cycles since $\text{supp}(u_k) = \Delta_k$ and $x_{k+1,1}$ sends each $1 \leq i \leq p^{k+1}$ to $i + p^k \pmod{(p^{k+1})}$. (Always $c = a \times b$ means that a, b are disjoint permutations.) Put $g_k = u_k \times v_k$ for every $k \geq 1$ and define $G = \langle g_k : k \geq 1 \rangle$. Then G satisfies the cyclic-block property by [4, Theorem 1.1]. We see from the definitions that $\{g_k : k \geq 1\}$ is an ascending set of homogeneous elements of G . Furthermore, it follows from the definition that

$$u_{k+1}^p = (x_{k+1,1} u_k)^p = x_{k+1,1}^p u_k^{x_{k+1,1}^{p-1}} \cdots u_k^{x_{k+1,1}} u_k = u_k \times u_k^{x_{k+1,1}} \times \cdots \times u_k^{x_{k+1,1}^{p-1}} = g_k$$

for every $k \geq 1$. Hence, it follows that the g_k satisfy (*) as can be seen from the proof of Corollary 1.2. It can also be shown easily that $G \leq W'$. Indeed,

$$g_1 = x_{1,1} x_{1,1}^{x_{2,1}} \cdots x_{1,1}^{x_{2,1}^{p-1}} = x_{1,1}^p [x_{1,1}, x_{2,1}] \cdots [x_{1,1}, x_{2,1}^{p-1}] \in W'$$

since $x_{1,1}^p = 1$. Assume that $g_k \in W'$ for a $k \geq 1$. Now

$$g_{k+1} = u_k u_k^{x_{k+1,1}} \cdots u_k^{x_{k+1,1}^{p-1}} = u_k^p [u_k, x_{k+1,1}] \cdots [u_k, x_{2,1}^{p-1}].$$

Since $u_k^p = g_{k-1} \in W'$ it follows that $g_{k+1} \in W'$, which completes the induction, and so $G \leq W'$.

As was indicated above, each u_k is a cycle of length p^k with $\text{supp}(u_k) = \Delta_k$ by [4, Lemma 3.2(a)].

Hence, $\text{supp}(u_k^{x_{k+1,1}^i}) = x_{k+1,1}^{-i}(\Delta_k)$ for every $i \geq 1$ and hence

$$\text{supp}(v_k) = \bigcup_{i=1}^{p^k-1} \text{supp}(u_k^{x_{k+1,1}^i}) = \Delta_{k+1} \setminus \Delta_k.$$

For $p = 2$

$$u_1 = (1, 2); u_2 = (1, 4, 2, 3); u_3 = (1, 8, 4, 6, 2, 7, 3, 5)$$

and

$$u_4 = (1, 16, 8, 12, 4, 14, 6, 10, 2, 15, 7, 11, 3, 13, 5, 9).$$

Hence,

$$g_1 = (1, 2)(3, 4); g_2 = (1, 4, 2, 3)(5, 8, 6, 7); g_3 = (1, 8, 4, 6, 2, 7, 3, 5)(9, 16, 12, 14, 10, 15, 11, 13).$$

Finally, it follows from [4, Theorem 1.1, Lemmas 2.2 and 3.4] that G satisfies the cyclic-block property, any two blocks for G are either disjoint or one is contained in the other one, and the blocks for G are the blocks for W . Thus, the set of the blocks of the same p -power size for G form a block system for G and hence also for W .

We end this section with a characterization of W .

Proposition 2.1 *W is a transitive maximal p -subgroup of $FSym(\mathbb{N}^*)$, $Z(W) = 1$, self-normalizing, and W/W' is infinite elementary abelian.*

Proof Put $W_k = \langle x_{1,1}, x_{2,1}, \dots, x_{k,1} \rangle$ for every $k \geq 1$. Then $W = \bigcup_{k=1}^{\infty} W_k$ and also $\mathbb{N}^* = \bigcup_{k=1}^{\infty} \Delta_k$, where $\Delta_k = \{1, 2, \dots, p^k\}$ for every $k \geq 1$. It is easy to see that each W_k is transitive on Δ_k , which implies that W is transitive on Ω , and then $Z(W) = 1$ by [10, Lemma 8.3C(ii)].

First we show that W_k is a Sylow p -subgroup of $Sym(\Delta_k)$ for every $k \geq 1$. Note that $supp(W_k) = \Delta_k$. Put

$$P_k = (\dots (\langle x_{1,1} \rangle \wr \langle x_{2,1} \rangle) \wr \dots \wr \langle x_{k,1} \rangle).$$

Then P_k is isomorphic to a Sylow p -subgroup of $Sym(\Delta_k)$ by [11, Proposition 19.10] since each $\langle x_{k,i} \rangle$ has order p . It will suffice to show that $W_k \cong P_k$. For $k = 1$ the assertion holds since $\langle x_{1,1} \rangle = \langle (1, 2, \dots, p) \rangle$ is a Sylow p -subgroup of $Sym(\{1, 2, \dots, p\})$. Suppose that the assertion holds for $k \geq 1$. Then $W_k \cong \langle x_{1,1} \rangle \wr \langle x_{2,1} \rangle \wr \dots \wr \langle x_{k,1} \rangle$, and by identifying these two groups, W_k becomes a Sylow p -subgroup of $Sym(\Delta_k)$. Thus, we get $P_{k+1} \cong W_k \wr \langle x_{k+1,1} \rangle$. Let B_k be the base subgroup; that is, $B_k = \prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b$, where each $(W_k)_b$ is equal to W_k . Then P_{k+1} is isomorphic to $B_k \langle x_{k+1,1} \rangle$ and so $|P_{k+1}| = p|B_k| = p|W_k|^p$. However, we have seen above that $x_{k+1,1}$ permutes the sets $\{1, 2, \dots, p^k\}, \{1+p^k, 2+p^k, \dots, 2p^k\}, \dots, \{1+(p-1)p^k, 2+(p-1)p^k, \dots, p^{k+1}\}$ among themselves and induces a cycle of length p on them. Also, W_k is a Sylow p -subgroup of $Sym(\Delta_k)$. Clearly then $x_{k+1,1}^{-i} W_k x_{k+1,1}^i$ are Sylow p -subgroups on the corresponding sets $x_{k+1,1}^{-i}(\Delta_k)$ for $i = 1, \dots, p$. Thus, $x_{k+1,1}^{-i} W_k x_{k+1,1}^i$ and $x_{k+1,1}^{-j} W_k x_{k+1,1}^j$ have disjoint supports for $i \neq j$ and so they commute. Therefore, we get

$$W_{k+1} = (W_k \times x_{k+1,1}^{-1} W_k x_{k+1,1} \times \dots \times x_{k+1,1}^{-(p-1)} W_k x_{k+1,1}^{p-1}) \langle x_{k+1,1} \rangle.$$

This gives $|W_{k+1}| = |W_k|^p |x_{k+1,1}| = p|W_k|^p$ and hence $|W_{k+1}| = |P_{k+1}|$. This implies that W_{k+1} is a Sylow p -subgroup of $Sym(\Delta_{k+1})$ since $W_{k+1} \leq Sym(\Delta_{k+1})$, which completes the induction. Clearly it follows from this that W is a Sylow p -subgroup of $FSym(\mathbb{N}^*)$ since $FSym(\mathbb{N}) = \bigcup_{k=1}^{\infty} Sym(\Delta_k)$.

Next we show that W is self-normalizing. Assume not. Then there exists a subgroup Y of $FSym(\mathbb{N}^*)$ with $W < Y$ and Y/W is abelian. Also, Y is transitive since W is. Moreover, $Y' \leq W$ and so Y' is a p -group, but then Y is a p -group by [20, Lemma 2.1], which is a contradiction.

Finally, we show that $W/W' = \prod_{k=1}^{\infty} \langle x_{k,1}W' \rangle$, as a direct product. This will be the case if we can show that $W_k/W'_k = \prod_{i=1}^k \langle x_{i,1}W'_k \rangle$, as a direct product, for every $k \geq 1$. For $k = 1$ this is trivial. Suppose that the assertion holds for $k \geq 1$. We have seen above that

$$W_{k+1} \cong W_k \wr \langle x_{k+1,1} \rangle = \left(\prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b \right) \langle x_{k+1,1} \rangle = B_k \langle x_{k+1,1} \rangle,$$

which implies that $W_{k+1}/W'_{k+1} \cong B_k \langle x_{k+1,1} \rangle / (B_k \langle x_{k+1,1} \rangle)'$. We can now apply [17, Corollary 4.5] to $B_k \langle x_{k+1,1} \rangle$. This gives

$$(B_k \langle x_{k+1,1} \rangle)' = M$$

where $M = \{f \in B_k : \pi(f) \in W'_k\}$ and $\pi(f) = \prod_{b \in \langle x_{k+1,1} \rangle} f(b)$. Next define $x_{i,1}^*(1) = x_{i,1}$ and $x_{i,1}^*(b) = 1$ for $b \neq 1$ for $1 \leq i \leq k$. Each $x_{i,1}^* \in B_k = \prod_{b \in \langle x_{k+1,1} \rangle} (W_k)_b$. We claim that $x_{1,1}^*M, \dots, x_{k,1}^*M$ are linearly independent over \mathbb{Z}_p , the field of p elements. Assume if possible that there exists an $f = (x_{1,1}^*)^{s_1} \cdots (x_{r,1}^*)^{s_r}$, where $1 \leq r \leq k$ and $1 \leq s_i < p$ so that $f \in M$. Then $f = (x_{1,1}^{s_1} \cdots x_{r,1}^{s_r}, 1, \dots, 1)$ and $\pi(f) = x_{1,1}^{s_1} \cdots x_{r,1}^{s_r} \in W'_k$, but since $W_k/W'_k = \langle x_{1,1}W'_k \rangle \times \cdots \times \langle x_{k,1}W'_k \rangle$ by the induction hypothesis it follows that $x_{1,1}^{s_1}W'_k = \cdots = x_{r,1}^{s_r}W'_k = 1$, which means that $x_{i,1}^{s_i} \in W'_k$ and then $p|s_i$ since $|x_{i,1}| = p$, which is impossible since $1 \leq s_i < p$ for every $i \geq 1$. Consequently it follows that $x_{1,1}^*M, \dots, x_{k,1}^*M$ are linearly independent in $B_k \langle x_{k+1,1} \rangle / M$. Then also $x_{1,1}^*M, \dots, x_{k,1}^*M, x_{k+1,1}^*M$ are linearly independent in $B_k \langle x_{k+1,1} \rangle / M$ since $\langle x_{k+1,1}^* \rangle \cap B_k = 1$. Therefore,

$$B_k \langle x_{k+1,1} \rangle / M = \langle x_{1,1}^*M \rangle \times \cdots \times \langle x_{k+1,1}^*M \rangle.$$

Hence, using the above isomorphism, we get

$$W_{k+1}/W'_{k+1} = \langle x_{1,1}W'_{k+1} \rangle \times \cdots \times \langle x_{k+1,1}W'_{k+1} \rangle,$$

which completes the induction. Now since $W = \bigcup_{k=1}^{\infty} W_k$ it follows easily that

$$W/W' = \prod_{k=1}^{\infty} \langle x_{k,1}W' \rangle$$

as a direct product. Suppose that $\langle x_{t,1}W' \rangle \cap \langle x_{k,1}W' : k \geq 1, k \neq t \rangle \neq 1$ for a $t \geq 1$. Then $\langle x_{t,1}W' \rangle \leq \langle x_{k,1}W' : k \geq 1, k \neq t \rangle$ since $|x_{t,1}| = p$. Hence, $x_{t,1}$ is a finite product of elements of certain cosets of the right side. Also, $W' = \bigcup_{k=1}^{\infty} W'_k$. Clearly then there exists an $n > t$ so that $x_{t,1} \in \langle x_{k,1}W'_n : 1 \leq k \leq n, k \neq t \rangle$, but since $W_n/W'_n = \langle x_{1,1}W'_n \rangle \times \cdots \times \langle x_{n,1}W'_n \rangle$, as was shown above, this is impossible. Therefore, the assumption is false and so W/W' is a direct product of the $\langle x_{k,1}W' \rangle$ as k ranges over the positive integers. \square

Remark. The commutator subgroup W' of W is perfect and transitive by [19, Theorem 1]. Also, W does not satisfy the normalizer condition by [1, Theorem 1.2(b)] since $G \leq W$ and G' is not an *MNFC*-group by Corollary 1.3. The reader may observe that W is exactly the same group that is constructed in [12, 18.2.2 Example], where it is shown also that this group does not satisfy the normalizer condition.

3. Proof of Theorem 1.1

We begin with a known result on the cyclic-block property for the convenience of the reader. (See also [5, Proposition 1.7].)

Lemma 3.1 3.1 Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$ satisfying the cyclic-block property, where Ω is infinite. Then the following hold:

(a) Let Δ be a finite block for G and let $\alpha \in \Delta$. Then for every $y \in G \setminus G_{\{\Delta\}}$, $\langle y^{s(y,\Delta)} \rangle(\alpha) = \Delta$.

(b) Let Δ be a finite block for G . Then

$$Ker_G(\Delta) = \{g \in G : |g| \leq |\Delta|\}.$$

Furthermore, $exp(G_{\{\Delta\}})$ is infinite.

(c) Any NFC -subgroup of G is transitive on Ω .

Proof (a) (See [4, Lemma 2.1].) Put $H = G_{\{\Delta\}}$ and let $y \in G \setminus H$. Put $t = s(y, \Delta)$. Then t is the smallest number such that $y^t \in H$. Also, $t = p^r$ for an $r \geq 1$. Next put $\langle y \rangle(\alpha) = \Gamma$ and $\langle y^t \rangle(\alpha) = \Lambda$. Then Γ and Λ are blocks for G by the cyclic-block property. Also, $\Delta \subset \Gamma$ and $\Lambda \subseteq \Delta$ by [4, Lemma 2.2] since $y \notin H$ but $y^t \in H$. Clearly $|\Gamma| = p^r |\Lambda|$. Assume if possible that there exists a $y^j(\alpha) \in \Delta \setminus \Lambda$. Then $j \nmid p^r$ and so $j < p^r$, but since $\alpha \in \Delta \cap y^{-j}(\Delta)$ and since Δ is a block, it follows that $y^j(\Delta) = \Delta$, which is a contradiction since $t = p^r$ is the smallest number with the property that $y^t(\Delta) = \Delta$. Therefore, the assumption is false and so $\langle y^t \rangle(\alpha) = \Delta$.

(b) Put $M = Ker_G(\Delta)$. Then $M < H$ since $H \neq G$ due to the fact that Ω is infinite and G is transitive. Let $y \in G$ and put $|y| = t$. First suppose that $t \leq |\Delta|$. Then we claim that $y \in H$. This is trivial if $supp(y) \cap \Delta = \emptyset$ since then $y(\Delta) = \Delta$. Suppose that $y(\alpha) \neq \alpha$ for an $\alpha \in \Delta$. Put $\Gamma = \langle y \rangle(\alpha)$. Then Γ is a block for G by the hypothesis and $|\Gamma| \leq t \leq |\Delta|$. Also, since $\alpha \in \Gamma \cap \Delta$ applying [4, Lemma 2.2], we get $\Gamma \subseteq \Delta$, which implies that $y \in H$. Thus, $\{g \in G : |g| \leq |\Delta|\} \subseteq M$. Next suppose that $t > |\Delta|$. There exists a $\beta \in \Omega$ so that $t = |\langle y \rangle(\beta)|$. Also, there exists a $g \in G$ so that $g(\beta) = \alpha$. Since $\langle y \rangle(\beta) = \{\beta, y(\beta), \dots, y^{t-1}(\beta)\}$, it follows that $\langle gyg^{-1} \rangle(\alpha) = \{gy(\beta), \dots, gy^{t-1}(\beta), g(\beta)\}$. Now if $y \in M$ then also $gyg^{-1} \in M$, but since $\langle gyg^{-1} \rangle(\alpha)$ is a block containing α and has size greater than $|\Delta|$, this is a contradiction. Therefore, $M = \{g \in G : |g| \leq |\Delta|\}$. In particular it follows that any subgroup of finite exponent of G is contained in a kernel subgroup which is nilpotent of finite exponent. It is well-known that a transitive subgroup of $FSym(\Omega)$ has infinite exponent if Ω is infinite by [18, Lemma 3.1] or [10, Theorem 8.3A]). Let $\alpha \in \Omega$. We show that G_α contains a conjugate of every element of G . Let $g \in G$. There exists a $\beta \in \Omega$ so that $g(\beta) = \beta$ and so $g \in G_\beta$. Also, $\beta = x(\alpha)$ for an $x \in G$. Hence, $g \in G_{x(\alpha)} = xG_\alpha x^{-1}$ and so $g^x \in G_\alpha$, which completes the proof of (b).

(c) Let X be a proper NFC -subgroup of G . Then X cannot be contained in the set-wise stabilizer of a finite block for G since X is not an FC -group. However, if $exp(X) \leq |\Delta|$ for a finite block Δ , then $X \leq Ker(\Delta) \leq G_{\{\Delta\}}$ by (b), which is impossible. Therefore, $exp(X) = \infty$. Let $\alpha, \beta \in \Omega$ and let Δ be a finite block for G containing both of them. Then there exists a $g \in X \setminus X_{\{\Delta\}}$ so that $\langle g^p \rangle(\alpha) = \Delta$ by (a), which implies that $\beta = (g^p)^j(\alpha)$ for a $j \geq 1$, and so X is transitive. \square

Lemma 3.2 Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$ and let c, d be two cycles in G such that $supp(c), supp(d)$ are blocks for G and $supp(c) \subseteq supp(d)$. Let $|c| = p^a, |d| = p^b$ and put $t = s(d, supp(c))$.

Then $t = p^{b-a}$. If $d^{p^{b-a}}|_{\text{supp}(c)} = c^k$, then $(p, k) = 1$ and

$$d^{p^{b-a}} = c^k \times (c^k)^d \times \dots \times (c^k)^{d^{p^{b-a}-1}}.$$

Proof Put $\Delta = \text{supp}(c)$, $\Gamma = \text{supp}(d)$. Then $|\Delta| = p^a$ and $|\Gamma| = p^b$. Let $\alpha \in \Delta$. Now $\Delta \subseteq \Gamma$. Clearly $\Gamma = \Delta \cup d(\Delta) \cup \dots \cup d^{t-1}(\Delta)$ as a disjoint union since Δ is a block and d is a cycle. Hence, $p^b = tp^a$ and hence $t = p^{b-a}$.

Put $H = G_{\{\Delta\}}$. Then t is the smallest number with $d^t \in H$. Hence, $\langle d^{p^{b-a}} \rangle(\alpha) \subseteq \Delta$ and $|d^{p^{b-a}}| = |\langle d^{p^{b-a}} \rangle(\alpha)|$ since d is a cycle, which implies that $|\langle d^{p^{b-a}} \rangle(\alpha)| = p^a$. Now suppose that $d^{p^{b-a}}|_{\Delta} = c^k$. Then $p \nmid k$ since $|c|$ is a cycle of length p^a . Thus, $(p, k) = 1$ and c^k is a cycle.

Now $d^{p^{b-a}}|_{d^i(\Delta)} = d^i c^k d^{-i}$ for every $1 \leq i \leq p^{b-a}$ and $\Gamma = \Delta \cup d(\Delta) \cup \dots \cup d^{p^{b-a}-1}(\Delta)$. Obviously then

$$d^{p^{b-a}} = d^{p^{b-a}}|_{\Gamma} = c^k \times (c^k)^d \times \dots \times (c^k)^{p^{b-a}-1}.$$

□

Lemma 3.3 *Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property. Suppose also that X satisfies $(*)$. Then X contains an ascending subset $Y = \{y_i : i \geq 1\}$ of G such that the following holds. Each y_i can be expressed as a direct product of cycles as*

$$y_i = c_{i,1} \times \dots \times c_{i,r(i)}$$

so that $\text{supp}(y_i) \subset \text{supp}(c_{i+1,1})$, $m(y_i) \leq m(y_{i+1})$, and the following hold. Let $1 \leq j \leq r(i)$ and $k \geq i$. Put $|c_{i,j}| = p^a$ and $|c_{k,1}| = p^b$. Then

$$[c_{k,1}^{p^{b-a}}|_{\text{supp}(c_{i,j})}, c_{i,j}] = 1.$$

Proof Choose a $y_1 \neq 1$ in X so that $m(y_1) \leq m(x)$ for every $x \in X$ and let

$$y_1 = c_{1,1} \times \dots \times c_{1,r(1)}$$

be the cycle decomposition of y_1 . Let Γ_1 be the smallest block containing $\text{supp}(y_1)$. Next choose a y_2 in $X \setminus G_{\{\Gamma_1\}}$ so that $m(y_2) \leq m(x)$ for every $x \in X \setminus G_{\{\Gamma_1\}}$. Now $\langle y_2 \rangle(\alpha)$ is a block by the cyclic-block property and $\Gamma_1 \subset \text{supp}(\langle y_2 \rangle(\alpha))$ by [4, Lemma 2.2] since $y_2 \notin G_{\{\Gamma_1\}}$. Also, $m(y_1) \leq m(y_2)$. Put $c_{2,1} = (\alpha, \dots, y_2^{t_2-1}(\alpha))$, where t_2 is the smallest number such that $y_2^{t_2}(\alpha) = \alpha$. Thus, $\Gamma_1 \subset \text{supp}(c_{2,1})$. Continuing in this way we obtain an infinite subset $Y = \{y_i : i \geq 1\}$ of X such that $m(y_i) \leq m(y_{i+1})$ and $\text{supp}(y_i) \subset \text{supp}(c_{i+1,1})$ for every $i \geq 1$, where

$$y_i = c_{i,1} \times \dots \times c_{i,r(i)}$$

is the cycle decomposition of y_i . Let $1 \leq i < k$ and let $1 \leq j \leq r(i)$. Then $\text{supp}(c_{i,j}) \subset \text{supp}(c_{k,1})$. Also, Y satisfies $(*)$ since Y is a subset of X . Therefore,

$$[c_{k,1}^{s(c_{k,1}, \text{supp}(c_{i,j}))}|_{\text{supp}(c_{i,j})}, c_{i,j}] = 1.$$

Let $|c_{i,j}| = p^a$ and $|c_{k,1}| = p^b$. Then since $s(c_{k,1}, \text{supp}(c_{i,j})) = p^{b-a}$ by Lemma 3.2, substituting this value above the desired equality is obtained. Furthermore, Y is ascending since $\text{supp}(c_{i,1})$ is a block and $\text{supp}(c_{i,1}) \subset \text{supp}(c_{i+1,1})$ for every $i \geq 1$. \square

Lemma 3.4 *Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property and $(*)$. Let $Y = \{y_i : i \geq 1\}$ be the subset of X obtained in Lemma 3.3. Thus, for each $i \geq 1$, the cycle decomposition of y_i can be written as*

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

such that $\text{supp}(y_i) \subset \text{supp}(c_{i+1,1})$ and $m_1(y_i) \leq m_1(y_{i+1})$ for every $i \geq 1$. Moreover, if we put $|c_{i,j}| = p^{a(i,j)}$, for every $i \geq 1$ and $1 \leq j \leq r(i)$, then for $1 \leq i \leq k$ the equality

$$c_{k,1}^{p^{a(k,1)-a(i,j)}}|_{\text{supp}(c_{i,j})} = c_{i,j}^{q(i,j)} \tag{1}$$

holds for a $q(i,j) \geq 1$ with $(p, q(i,j)) = 1$ by Lemma 3.3.

Now let $j, k, t \geq 1$ be integers with $j \leq k, t$ and suppose that $|y_j| \leq \min\{m(y_k), m(y_t)\}$. Let $m, n \geq 1$. Then

$$c_{j,1}^{y_k^m y_t^n} = c_{j,1}^{y_r^s}$$

for an $r \in \{k, t\}$ and $s \geq 1$.

Proof Put $c_i = c_{i,1}$ and let $|\text{supp}(c_i)| = p^{a(i)}$ for $i = j, k, t$. Then c_i is a factor of the cycle decomposition of y_i for $i = j, k, t$ and $\text{supp}(y_u) \subset \text{supp}(c_v)$ for every $1 \leq u < v$. We may suppose that $j < k, t$.

Case 1 $j < k < t$. Now

$$c_j^{y_k^m y_t^n} = c_j^{c_k^m y_t^n} \tag{2}$$

since $\text{supp}(c_j) \subseteq \text{supp}(c_k)$. On the other hand,

$$c_t^{p^{a(t)-a(k)}} = c_k^{q(k,1)} \times \cdots \times (c_k^{q(k,1)}) c_t^{p^{a(t)-a(k)}-1} = c_k^{q(k,1)} \times v_k$$

by (1) and Lemma 3.2, where $\text{supp}(v_k) \cap \text{supp}(c_k) = \emptyset$. Also, $bq(k,1) \equiv 1 \pmod{p^{a(k)}}$ for an integer b since $(q(k,1), p) = 1$ by Lemma 3.2. Using this above gives

$$c_t^{bp^{a(t)-a(k)}} = c_k \times \cdots \times c_k^{bc_t^{p^{a(t)-a(k)}-1}} = c_k \times v_k^b.$$

Hence, $c_k^m = c_t^{mbp^{a(t)-a(k)}} v_k^{-bm}$. Substituting this in (2) gives

$$c_j^m y_t^n = c_j^{v_k^{-bm} c_t^{mbp^{b-a}}} y_t^n = c_j^{mbp^{a(t)-a(k)}} y_t^n = c_j^{mbp^{a(t)-a(k)}+n} = c_j^{y_t^{mbp^{b-a}+n}}$$

since $\text{supp}(c_j) \subset \text{supp}(c_k)$.

Case 2 $k > t > j$. We may suppose that $\text{supp}(c_j^{c_k^m}) \cap \text{supp}(y_t^n) \neq \emptyset$; otherwise, $c_j^{c_k^m} y_t^n = c_j^{c_k^m}$ and we are done. Then there exists a cycle $c_{t,r}$ in the cycle decomposition of y_t so that $\text{supp}(c_j^{c_k^m}) \subseteq \text{supp}(c_{t,r})$ by

the cyclic-block property since $|y_j| < m(y_t)$ by the hypothesis. For simplicity, put $u_t = c_{t,r}$ and $q(t) = q(t, r)$. Clearly now $c_j^m y_t^n = c_j^m u_t^n$. Let $|u_t| = p^z$. Then

$$c_k^{p^{a(k)-z}} = u_t^{q(t)} \times \dots \times (u_t^{q(t)})^{c_k^{p^{a(k)-z}-1}} = u_t^{q(t)} \times v_t$$

where $(q(t), p) = 1$ by (1). Then, as in Case 1, there exists an integer b so that

$$c_k^{bp^{a(k)-z}} = u_t \times v_t^b$$

where $\text{supp}(u_t) \cap \text{supp}(v_t) = \emptyset$. Hence $u_t^n = v_t^{-bn} c_k^{nbp^{a(k)-z}}$. Substituting this above gives

$$\begin{aligned} c_j^m u_t^n &= c_j^m v_t^{-bn} c_k^{nbp^{a(k)-z}} = c_j^m c_k^{nbp^{a(k)-z}} \\ &= c_j^{m+nbp^{a(k)-z}} = c_j^{y_k^{m+nbp^{a(k)-z}}} \end{aligned}$$

since $\text{supp}(c_j^m) \subseteq \text{supp}(u_t)$ and $\text{supp}(u_t) \cap \text{supp}(v_t) = \emptyset$, which completes the proof of the lemma. □

Lemma 3.5 *Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$. Let X be an ascending subset of G satisfying the cyclic-block property and $(*)$. Let $Y = \{y_i : i \geq 1\}$ be the subset of X obtained in Lemma 3.3 and suppose that $|y_i| < m(y_{i+1})$ for every $i \geq 1$. Let $j \geq 1$ and put $Y_j^* = \{y_i : i \geq j\}$. Let $y = y_{k_1}^{m_1} \dots y_{k_r}^{m_r} \in Y_j^*$, where $k_i \geq j$, $r \geq 1$, $m_i \geq 1$, and $k_u \neq k_{u+1}$. Then $c_{j,1}^y = c_{j,1}^{y_u^s}$ for a $u \in \{k_1, \dots, k_r\}$ and $s \geq 1$.*

Proof We may use induction on $r \geq 1$. For $r = 1$ the assertion is obvious. Suppose that $r > 1$ and the assertion holds for numbers less than r . Note that $|y_j| < m(y_{k_i})$ for $i = 1, \dots, r$ by the hypothesis. Hence, applying Lemma 3.4 we obtain a $k \in \{k_1, k_2\}$ and an $m \geq 1$ so that $c_{j,1}^{y_{k_1}^{m_1} y_{k_2}^{m_2} \dots y_{k_r}^{m_r}} = c_{j,1}^{y_k^m y_{k_3}^{m_3} \dots y_{k_r}^{m_r}}$. Then the induction hypothesis applies to the right side of the preceding equality. Therefore, there exist a $u \in \{k, k_3, \dots, k_r\}$ and an $s \geq 1$ so that $c_{j,1}^{y_k^m y_{k_3}^{m_3} \dots y_{k_r}^{m_r}} = c_{j,1}^{y_u^s}$. Then since $c_{j,1}^y = c_{j,1}^{y_u^s}$ the induction and the proof of the lemma are complete. □

Lemma 3.6 *Let the hypothesis and the notation be as in Lemma 3.5. Let $j \geq 1$. Then $[c_{j,1}^y, c_{j,1}] = 1$ for every $y \in Y_j^*$.*

Proof Put $c_j = c_{j,1}$. Let $y \in Y_j^*$. We have $c_j^y = c_j^{y_k^s}$ for a $k \geq j$ and an $s \geq 1$ by Lemma 3.5. Let $\text{supp}(c_j) = \Gamma_j$ and put $H = G_{\{\Gamma_j\}}$. If $y_k^s \notin H$, then $y_k^s(\Gamma_j) \cap \Gamma_j = \emptyset$ and since $\text{supp}(c_j^{y_k^s}) = y_k^{-s}(\Gamma_j)$ it follows that $[c_j^{y_k^s}, c_j] = 1$, and the assertion holds in this case since $c_j^y = c_j^{y_k^s}$.

Next suppose that $y_k^s \in H$. Let $c_{k,1} = c_k$, $\text{supp}(c_k) = \Gamma_k$, $|\Gamma_j| = p^{a(j)}$, and $|\Gamma_k| = p^{a(k)}$. Then $p^{a(k)-a(j)}|_s$ by Lemma 3.2 and hence $s = p^{a(k)-a(j)}t$ for a $t \geq 1$. Also, $\text{supp}(y_j) \subseteq \Gamma_k$ and $c_k^{p^{a(k)-a(j)}} = c_j^{q(j)} \times v_k$ for a $v_k \in FSym(\Omega)$ and $q(j) \geq 1$ by (1) in Lemma 3.4. Hence, $c_k^s = c_j^{tq(j)} v_k^t$, but also $y_k = c_k \times z_k$ for a

$z_k \in F\text{Sym}(\Omega)$. Combining these values we get $y_k^s = c_j^{tq(j)}(v_k^t z_k^s)$, where $\text{supp}(v_k z_k) \cap \Gamma_k = \emptyset$. Using this last equality we get

$$c_j^{y_k^s} = c_j^{c_j^{tq(j)}(v_k^t z_k^s)} = c_j^{v_k^t z_k^s} = c_j$$

and hence

$$[c_j^{y_k^s}, c_j] = [c_j, c_j] = 1,$$

which was to be shown. □

Lemma 3.7 *Let G be a totally imprimitive p -subgroup of $F\text{Sym}(\Omega)$, where Ω is infinite. Let $y \in G$ and let $j > 1$ so that $\alpha \in \text{supp}(y) \subset \langle c_{j,1}^{p^2} \rangle(\alpha)$ and $|c_{j,1}| = p^t$ for a $t \geq 4$. Then $[c_{j,1}^y, c_{j,1}] \neq 1$.*

Proof Put $c = c_{j,1}$. Then $\text{supp}(y) \subseteq \{\alpha, c^{p^2}(\alpha), \dots, (c^{p^2})^{p^{t-2}-1}(\alpha)\}$ by the hypothesis. This means that if y moves an element of Ω , then it must be of the form $(c^{p^2})^k(\alpha)$ for a $0 \leq k \leq p^{t-2} - 1$.

Assume if possible that $c^y c = c c^y$. Then

$$y c y^{-1} c(\alpha) = c y c y^{-1}(\alpha). \quad (1)$$

Now

$$y c y^{-1} c(\alpha) = y c c(\alpha) = y(c^2(\alpha))$$

since y cannot move $c(\alpha)$ and

$$c y c y^{-1}(\alpha) = c y c(c^{kp^2}(\alpha)) = c y(c^{kp^2+1}(\alpha)) = c^{kp^2+2}(\alpha)$$

since y cannot move $c^{kp^2+1}(\alpha)$, where $y^{-1}(\alpha) = (c^{p^2})^k(\alpha)$ and $1 \leq k \leq p^{t-2} - 1$ since $y(\alpha) \neq \alpha$. Thus the equality (1) takes the form

$$y(c^2(\alpha)) = c^{kp^2+2}(\alpha).$$

Now if $p > 2$, then $y(c^2(\alpha)) = c^2(\alpha)$ since $c^2(\alpha)$ is not of the form $(c^{p^2})^k(\alpha)$. Indeed, if $c^2(\alpha) = (c^{p^2})^k(\alpha)$, then $c^{kp^2-2}(\alpha) = \alpha$, which implies that $p^t | kp^2 - 2$ since $|c| = p^t$, which is impossible. Therefore, $c^2(\alpha) = c^{kp^2+2}(\alpha)$ and hence $\alpha = c^{kp^2}(\alpha)$, which is a contradiction since $1 \leq k \leq p^{t-2} - 1$, c is a cycle, $|c| = p^t$, and $t \geq 4$. Next suppose that $p = 2$. Again since y can move only elements of the form $(c^{p^2})^k(\alpha) = c^{4k}(\alpha)$ and since $c^2(\alpha)$ is not of this form, we get $y(c^2(\alpha)) = c^2(\alpha)$ and hence $c^2(\alpha) = c^{4k+2}(\alpha)$. Hence, $c^{4k}(\alpha) = \alpha$, which is another contradiction since $|c| = 2^t$, $t \geq 4$, and $1 \leq k \leq 2^{t-2} - 1$. □

Lemma 3.8 *Let G be a totally imprimitive p -subgroup of $F\text{Sym}(\Omega)$, where Ω is infinite. Let X be the ascending subset of G satisfying the cyclic-block property such that for every $x \in X$ there exists a $y \in X$ such that $m(x) < m(y)$. Then there exists an ascending subset $Z = \{z_i : i \geq 1\}$ of G so that $m(z_i) < m(z_{i+1})$ for every $i \geq 1$.*

Proof By the hypothesis we can obtain easily an infinite subset $X^* = \{x_i : i \geq 1\}$ of X so that $m(x_i) < m(x_{i+1})$ for every $i \geq 1$. We may suppose that $x_1 \neq 1$. Let d_i be a cycle of x_i of the smallest length, that is, of length $m(x_i)$ for every $i \geq 1$. Then $|d_i| < |d_{i+1}|$ for every $i \geq 1$. Choose an $\alpha \in \text{supp}(d_1)$. By the transitivity of G for every $i \geq 1$ there exists an $a_i \in G$ so that $\alpha \in \text{supp}(d_i^{a_i})$. Then $\text{supp}(d_i^{a_i}) \subset \text{supp}(d_{i+1}^{a_{i+1}})$ by [4, Lemma 2.2 since $\alpha \in \text{supp}(d_i^{a_i}) \cap \text{supp}(d_{i+1}^{a_{i+1}})$ for every $i \geq 1$. Put $z_i = x_i^{a_i}$ for every $i \geq 1$ and define $Z = \{z_i : i \geq 1\}$. Since $\text{supp}(d_i^{a_i}) \subset \text{supp}(d_{i+1}^{a_{i+1}})$ and since each $\text{supp}(d_i^{a_i})$ is a block for G it follows that $\bigcup_{i=1}^{\infty} \text{supp}(d_i^{a_i}) = \Omega$ due to the fact that every proper block is finite by the transitivity of G on Ω . Hence, it follows that Z cannot be contained in the set stabilizer of a finite subset of G and also $\text{exp}(Z)$ is infinite. Therefore, Z is an ascending subset of G . \square

Proof of Theorem 1.1 Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$, where Ω is infinite. Let X be an ascending subset of G satisfying the cyclic-block property so that conditions (a) and (b) are satisfied. Then applying Lemma 3.8 we obtain an ascending subset $Z = \{z_i : i \geq 1\}$ of G so that $m(z_i) < m(z_{i+1})$ for every $i \geq 1$. Next we can choose an infinite subset $U = \{u_i : i \geq 1\}$ of Z so that $|u_i| < m(u_{i+1})$ for every $i \geq 1$ since the numbers $m(z_i)$ are increasing without bound. We now substitute U in place of X in Lemma 3.3. This gives an ascending subset $Y = \{y_i : i \geq 1\}$ of G so that the following hold. The cycle decomposition of each y_i can be expressed as

$$y_i = c_{i,1} \times \cdots \times c_{i,r(i)}$$

so that $\text{supp}(y_i) \subset \text{supp}(c_{i+1,1})$ and if $1 \leq j \leq r(i)$, $k \geq i$, $|c_{i,j}| = p^a$, $|c_{k,1}| = p^b$, then

$$c_{k,1}^{p^{b-a}}|_{\text{supp}(c_{i,j})} = c_{i,j}^{q(i,j)}$$

for a $q(i, j) \geq 1$. Furthermore, for each $i \geq 1$, the inequality $|y_i| < m(y_{i+1})$ is satisfied by definition of U . Thus, Lemmas 3.4, 3.5, and 3.6 can be applied to Y .

Next we may suppose that $y_1 \neq 1$. Choose an $\alpha \in \text{supp}(y_1)$. Let Δ be the smallest block such that $\text{supp}(y_1) \subseteq \Delta$ and let $|\Delta| \leq p^t$, for a $t \geq 4$. There exists a $j > 1$ so that $|c_{j,1}| \geq p^{2t}$ and $\Delta \subseteq \text{supp}(c_{j,1})$. Put $c_j = c_{j,1}$. Then $c_j = (\alpha, c_j(\alpha), \dots, c_j^{|c_j|-1}(\alpha))$. Now $c_j^{p^2}$ is a product of p^2 cycles each of length $\geq p^{2t-2} = p^{2(t-1)} \geq p^t$ since $t \geq 4$. Then it is easy to see that $\Delta \subseteq \langle c_j^{p^2} \rangle(\alpha)$ by the cyclic-block property since $\langle c_j^{p^2} \rangle(\alpha)$ is a block and $\alpha \in \Delta \cap \langle c_j^{p^2} \rangle(\alpha)$.

Put $Y_j^* = \langle y_i : i \geq j \rangle$. Then the application of Lemmas 3.4, 3.5, and 3.6 gives $[c_{j,1}^y, c_{j,1}] = 1$ for every $y \in Y^*$, but application of Lemma 3.7 gives $[c_j^{y_1}, c_j] \neq 1$, which implies that $y_1 \notin Y^*$ and so $Y^* \neq G$. However, since $\{y_i : i \geq j\}$ is ascending by definition of Y , the subgroup Y^* cannot be an FC -subgroup of G . Therefore, G cannot be an $MNFC$ -group and so the proof of the theorem is complete. \square

Proof of Corollary 1.2 Let G be a totally imprimitive p -subgroup of $FSym(\Omega)$, where Ω is infinite. Let X be an ascending subset of homogeneous elements of G satisfying the cyclic-block property so that X satisfies the (*) condition. Then condition (a) of Theorem 1.1 is satisfied. Therefore, we need only show that condition (b) of Theorem 1.1 is satisfied. Since X is ascending by the hypothesis, $\text{exp}(X)$ is infinite and $\langle X \rangle$ is a non- FC -subgroup of G . Also, since G is locally finite, it follows that for every $x \in X$ there exists a $y \in X$ so that

$|x| < |y|$. Now the homogeneity of the elements of X shows that (b) is satisfied by X . Therefore, G cannot be *MNFC* by Theorem 1.1. \square

Proof of Corollary 1.3 Let G be the p -subgroup of $FSym(\mathbb{N}^*)$ described in Section 2. Then G satisfies the cyclic-block property by [4, Theorem 1.1]. We have $G = \langle g_k : k \geq 1 \rangle$, where $g_k = u_k \times v_k$, $u_k = (a_1, \dots, a_{p^k})$, $v_k = u_k \times \dots \times u_k^{x_{k+1,1}^{p-1}}$, $supp(u_k) = \Delta_k$, and $supp(v_k) = \Delta_{k+1} \setminus \Delta_k$. Hence, it follows that each g_k is homogeneous; that is, $|g_k| = m(g_k) = p^k$ for every $k \geq 1$. Furthermore,

$$g_{k+1}^p|_{\Delta_k} = g_k$$

since $u_{k+1}^p = g_k$ as was shown in Section 2. Thus, G satisfies the hypothesis of Corollary 1.2 and therefore G cannot be an *MNFC*-group.

Next we show that G' cannot be *MNFC*. For each $s \geq 2$ let $Y_s = \{g_k^{-1}g_k^{g_s} : 1 \leq k < s\}$ and put $Y = \bigcup_{s \geq 2} Y_s$. Then Y is an ascending subset of homogeneous elements of G' . To see this let $1 \leq k < s$. Then $g_k^{-1}g_k^{g_s} = g_k^{-1}g_k^{u_s}$ since $supp(g_k) = \Delta_{k+1} = supp(u_{k+1}) \subseteq supp(u_s)$. Also $u_{k+1}^p = g_k$ (see Section 2). Hence $g_k^{-1}g_k^{g_{k+1}} = g_k^{-1}g_k^{u_{k+1}} = 1$. So suppose that $s > k + 1$. Then $u_s(\Delta_{k+1}) \cap \Delta_{k+1} = \emptyset$. Also, $supp(g_k^{u_s}) = u_s^{-1}(supp(g_k)) = u_s^{-1}(\Delta_{k+1})$. Clearly it follows from this that $g_k^{-1}g_k^{g_s} = g_k^{-1} \times g_k^{g_s}$ and so $g_k^{-1}g_k^{g_s}$ is homogeneous since g_k is homogeneous. Furthermore, $g_k \notin G_{\{\Delta_{k-1}\}}$ since $g_k = u_k \times v_k$, $supp(u_k) = \Delta_k$ and $\Delta_{k-1} \subset \Delta_k$. Now suppose that $s > k + 1$. Then also $g_k^{-1}g_k^{u_s} \notin G_{\{\Delta_{k-1}\}}$ since $\Delta_{k-1} \subset supp(g_k)$ and $g_k^{u_s} \in G_{\Delta_{k-1}}$ due to the fact that $supp(g_k) \cap supp(g_k^{u_s}) = \emptyset$. Therefore, Y is an ascending subset of homogeneous elements of G' . In particular, (b) of Theorem 1.1 is satisfied.

Finally, let $1 \leq k + 1 < s$. Then

$$(g_{k+1}^{-1}g_{k+1}^{g_s})^p|_{\Delta_k} = g_{k+1}^{-p}|_{\Delta_k} = u_{k+1}^{-p}|_{\Delta_k} = g_k^{-1}|_{\Delta_k} = g_k^{-1} \times g_k^{g_s}|_{\Delta_k}$$

and so (a) of Theorem 1.1 is satisfied. Therefore, G' cannot be *MNFC* by Theorem 1.1. (A different proof of this result is given in [5, Theorem 1.6].) \square

Acknowledgment

The author is very grateful to the editor and the referee(s) for accepting this work for publication in Turkish Journal of Mathematics.

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