Turkish Journal of Mathematics

Volume 41 | Number 4 Article 15

1-1-2017

p-Subordination chains and p-valence integral operators

ERHAN DENİZ

HALİT ORHAN

MURAT ÇAĞLAR

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 956 – 968
© TÜBİTAK
doi:10.3906/mat-1505-9

p-Subordination chains and p-valence integral operators

Erhan DENİZ[1,*], **Halit ORHAN**[2], **Murat ÇAĞLAR**[1]
[1]Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Turkey
[2]Department of Mathematics, Faculty of Science, Atatürk University, Erzurum, Turkey

Received: 04.05.2015 • **Accepted/Published Online:** 04.10.2016 • **Final Version:** 25.07.2017

Abstract: In the present investigation we obtain some sufficient conditions for the analyticity and the p-valence of an integral operator in the unit disk $\mathbb{D}$. Using these conditions we give some applications for a few different integral operators. The significant relationships and relevance to other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

Key words: Univalent functions, p-valent function, p-subordination chain, p-valence criterion

1. Introduction

Denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \quad (0 < r \leqslant 1)$ the disk of radius r and let $\mathbb{D} = \mathbb{D}_1$. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}$ that satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{P}$ denote the class of functions $\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,\ z \in \mathbb{D}$ that satisfy the condition $\Re \mathfrak{p}(z) > 0$. Let $\mathcal{A}_p$ denote the class of analytic functions in the open unit disk $\mathbb{D}$ that satisfy the normalizations $f^{(k)}(0) = 0$ for $k = 1, 2, ..., p-1$ $(p \in \mathbb{N} = \{1, 2, ...\})$ and $f^{(p)}(0) \neq 0$, and let $\mathcal{A}_p^*$ be the subclass of $\mathcal{A}_p$ consisting of functions of the form $f(z) = z^p + \sum_{n=1+p}^{\infty} a_n z^n$ in $\mathbb{D}$. These classes have been one of the most important subjects of research in geometric function theory for a long time (see [22]). For analytic functions f and g in $\mathbb{D}$, f is said to be subordinate to g, denoted by $f(z) \prec g(z)$, if there exists an analytic function w satisfying $w(0) = 0,\ |w(z)| < 1$, such that $f(z) = g(w(z))$ $(z \in \mathbb{D})$. In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

2. p-Normalized subordination chain and related theorem

Before proving our main theorem we need a brief summary of the method of p-subordination chains.

Definition 2.1 *(see Hallenbeck and Livingston [8]) Let $\mathcal{L}(z,t)$ be a function defined on $\mathbb{D} \times I$, where $I := [0, \infty)$. $\mathcal{L}(z,t)$ is called a p-subordination chain if $\mathcal{L}(z,t)$ satisfies the following conditions:*

1. *$\mathcal{L}(z,t)$ is analytic in $\mathbb{D}$ for all $t \in I$,*
2. *$\mathcal{L}^{(k)}(0,t) = 0,\ k = 1, 2, ..., p-1$, and $\mathcal{L}^{(p)}(0,t) \neq 0$,*
3. *$\mathcal{L}(z,t) \prec \mathcal{L}(z,s)$ for all $0 \le t \le s < \infty,\ z \in \mathbb{D}$.*

*Correspondence: edeniz36@gmail.com
2010 *AMS Mathematics Subject Classification:* 30C45, 30C55, 30C80.

A p-subordination chain is said to be normalized if $\mathcal{L}(0,t)=0$ and $\mathcal{L}^{(p)}(0,t)=p!e^{pt}$ for all $t\in I$.

In order to prove our main results we need the following lemma due to Hallenbeck and Livingston [8].

Lemma 2.1 *Let $\mathcal{L}(z,t)=a_p(t)z^p+a_{p+1}(t)z^{p+1}+...,\ a_p(t)\neq 0,$ be analytic in $\mathbb{D}_r$ for all $t\in I$. Suppose that:*

(i) $\mathcal{L}(z,t)$ is a locally absolutely continuous function in the interval I and locally uniformly with respect to $\mathbb{D}_r$.

(ii) $a_p(t)$ is a complex valued continuous function on I such that $|a_p(t)|\to\infty$ for $t\to\infty$ and

$$\left\{\frac{\mathcal{L}(z,t)}{a_p(t)}\right\}_{t\in I}$$

forms a normal family of functions in $\mathbb{D}_r$.

(iii) There exists an analytic function $h:\mathbb{D}\times I\to\mathbb{C}$ satisfying $\Re h(z,t)>0$ for all $z\in\mathbb{D},\ t\in I$ and

$$p\frac{\partial\mathcal{L}(z,t)}{\partial t}=z\frac{\partial\mathcal{L}(z,t)}{\partial z}h(z,t),\quad z\in\mathbb{D}_r,\ t\in I. \tag{2.1}$$

Then, for each $t\in I$, the function $\mathcal{L}(z,t)$ is the pth power of a univalent function in $\mathbb{D}$.

Pommerenke's theory of subordination chains (see [18, 19]) corresponds to $p=1$.

The univalence of complex functions is an important property, but unfortunately it is difficult and in many cases impossible to show directly that a certain complex function is univalent. For this reason, many authors obtained different types of sufficient conditions of univalence or not. Pommerenke [18, 19] and Becker [2] used the idea of normalized 1-subordination chains to obtain sufficient conditions for univalence. Two of the most important conditions of univalence are the well-known criteria of Becker [2] and Ahlfors [1], which were obtained by a clever use of the theory of 1-subordination chains and the generalized Loewner differential equation. Detailed information about 1-subordination chains can be found in Hotta's works (see [10] and [9]). Furthermore, Pascu [15] and Pescar [16] obtained some extensions of Becker and Ahlfors' univalence criteria for an integral operator, respectively, using 1-subordination chains.

For further results we refer to the recent papers [3–6, 9–12, 14, 20, 21] where, among other things, some interesting univalence criteria and quasiconformal extensions were established.

It is the purpose of this paper to use p-subordination chains to obtain conditions for an integral operator to be the pth power of a univalent function where $p=1,2,...$. In special cases our results contain the results obtained by some of the authors cited in the references. We also extend the aforementioned results of Hallenbeck and Livingston [8]. Our considerations are based on the theory of p-subordination chains.

3. p-Valence criteria

Making use of Lemma 2.1 we can prove now our main results.

Theorem 3.1 *Let α and c be complex numbers such that $\Re(\alpha)>0$, $|c|<p$ and $f\in\mathcal{A}_p^*$. If the inequality*

$$\left|c\,|z|^{2\alpha p}+\frac{\left(1-|z|^{2\alpha p}\right)}{\alpha}\left[1-p+\frac{zf''(z)}{f'(z)}\right]\right|\leqslant p \tag{3.1}$$

holds true for all $z \in \mathbb{D}$, *then the integral operator*

$$F_\alpha(z) = \left[\alpha \int_0^z u^{p(\alpha-1)} f'(u)du\right]^{1/\alpha} \tag{3.2}$$

is the p*th power of a univalent function in* $\mathbb{D}$, *where the principal branch is considered.*

Proof We will prove that there exists a real number $r \in (0,1]$ such that the function $\mathcal{L} : \mathbb{D}_r \times I \to \mathbb{C}$, defined formally by

$$\mathcal{L}(z,t) = \left(\alpha \int_0^{e^{-t}z} u^{p(\alpha-1)} f'(u)du + \frac{1}{p+c}\left(e^{2\alpha pt}-1\right)\left(e^{-t}z\right)^{(p(\alpha-1)+1)} f'(e^{-t}z)\right)^{1/\alpha} \tag{3.3}$$

is analytic in $\mathbb{D}_r$ for all $t \in I$.

Consider the function

$$\phi_1(z,t) = \alpha \int_0^{e^{-t}z} u^{p(\alpha-1)} f'(u)du = e^{-\alpha pt} z^{\alpha p} + ...,$$

and then we have

$$\phi_1(z,t) = \left(e^{-t}z\right)^{\alpha p} + \sum_{n=1}^{\infty} \frac{\alpha(n+p)}{\alpha p + n} a_{n+p} \left(e^{-t}z\right)^{\alpha p + n}.$$

Let the function $\phi_2(z,t)$ be such that

$$\phi_1(z,t) = z^{\alpha p}\phi_2(z,t).$$

It is easy to check that $\phi_2(z,t)$ is analytic in $\mathbb{D}$ for all $t \in I$ and

$$\phi_2(z,t) = \left(e^{-t}\right)^{\alpha p} + \sum_{n=1}^{\infty} \frac{\alpha(n+p)e^{-(\alpha p+n)t}}{\alpha p + n} a_{n+p} z^n.$$

Since the function $f(z)$ is analytic in $\mathbb{D}$, it follows that the function

$$\phi_3(z,t) = \left(e^{2\alpha pt}-1\right) e^{-t(p(\alpha-1)+1)} z^{1-p} f'(e^{-t}z)$$

is an analytic function in $\mathbb{D}$ for all $t \in I$. Then the function $\phi_4(z,t)$ given by

$$\phi_4(z,t) = \phi_2(z,t) + \frac{1}{p+c}\phi_3(z,t)$$

is also analytic in $\mathbb{D}$.

We have

$$\phi_4(0,t) = \phi_2(0,t) + \frac{1}{p+c}\phi_3(0,t) = e^{\alpha pt}\left[\frac{p+ce^{-2\alpha pt}}{p+c}\right].$$

The conditions $|c| < p$ and $\Re(\alpha) > 0$ yield $\phi_4(0,t) \neq 0$ for all $t \in I$. Therefore, there is a disk $\mathbb{D}_{r_1}$, $r_1 \in (0,1]$, in which $\phi_4(z,t) \neq 0$ for all $t \in I$. Then we can choose a uniform branch of $[\phi_4(z,t)]^{1/\alpha}$ analytic in $\mathbb{D}_{r_1}$, denoted by $\phi_5(z,t)$.

It follows from (3.3) that

$$\mathcal{L}(z,t) = z^p \phi_5(z,t) = a_p(t) z^p + a_{p+1}(t) z^{p+1} + ...$$

and thus the function $\mathcal{L}(z,t)$ is analytic in $\mathbb{D}_{r_1}$.

We have

$$a_p(t) = e^{pt} \left[\frac{p + ce^{-2\alpha pt}}{p+c} \right]^{1/\alpha}.$$

From $|c| < p$ and $\Re(\alpha) > 0$, we obtain

$$\lim_{t \to \infty} |a_p(t)| = \infty.$$

Moreover, $a_p(t) \neq 0$ for all $t \in I$.

From the analyticity of $\mathcal{L}(z,t)$ in $\mathbb{D}_{r_1}$, it follows that there exists a number r_2, $0 < r_2 < r_1$ where $\mathcal{L}(z,t)/a_p(t)$ is analytic in disk $\mathbb{D}_{r_2}$ and a constant $K = K(r_2)$ such that

$$\left| \frac{\mathcal{L}(z,t)}{a_p(t)} \right| < K, \quad \forall z \in \mathbb{D}_{r_2},\ t \in I.$$

Then, by Montel's theorem, $\left\{ \frac{\mathcal{L}(z,t)}{a_p(t)} \right\}_{t \in I}$ is a normal family in $\mathbb{D}_{r_2}$. From the analyticity of $\frac{\partial \mathcal{L}(z,t)}{\partial t}$, we obtain that for all fixed numbers $T > 0$ and r_3, $0 < r_3 < r_2$, there exists a constant $K_1 > 0$ (that depends on T and r_3) such that

$$\left| \frac{\partial \mathcal{L}(z,t)}{\partial t} \right| < K_1, \quad \forall z \in \mathbb{D}_{r_3},\ t \in [0,T].$$

Therefore, the function $\mathcal{L}(z,t)$ is locally absolutely continuous in I, locally uniform with respect to $\mathbb{D}_{r_3}$.

Let $h : \mathbb{D} \times I \to \mathbb{C}$ be the function defined by

$$h(z,t) = p \frac{\partial \mathcal{L}(z,t)}{\partial t} \Big/ z \frac{\partial \mathcal{L}(z,t)}{\partial z}.$$

If the function

$$w(z,t) = \frac{h(z,t) - 1}{h(z,t) + 1} = \frac{p\frac{\partial \mathcal{L}(z,t)}{\partial t} - \frac{z\partial \mathcal{L}(z,t)}{\partial z}}{p\frac{\partial \mathcal{L}(z,t)}{\partial t} + \frac{z\partial \mathcal{L}(z,t)}{\partial z}} \tag{3.4}$$

is analytic in $\mathbb{D} \times I$ and $|w(z,t)| < 1$, for all $z \in \mathbb{D}$ and $t \in I$, then $h(z,t)$ is an analytic function with positive real part in $\mathbb{D}$, for all $t \in I$.

From equality (3.4), we have

$$w(z,t) = \frac{(p+1)\Psi(z,t) - 2p^2}{(p-1)\Psi(z,t) - 2p^2}, \tag{3.5}$$

where

$$\Psi(z,t) = ce^{-2\alpha pt} + \frac{\left(1-e^{-2\alpha pt}\right)}{\alpha}\left[1-p+\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}\right] + p \tag{3.6}$$

for $z \in \mathbb{D}$ and $t \in I$.

The inequality $|w(z,t)| < 1$ for all $z \in \mathbb{D}$ and $t \in I$, where $w(z,t)$ is defined by (3.5), is equivalent to

$$|\Psi(z,t) - p| < p, \quad \forall z \in \mathbb{D},\ t \in I. \tag{3.7}$$

From the hypothesis of the theorem and (3.6), we have

$$|\Psi(z,0) - p| = |c| < p, \quad \text{for all } z \in \mathbb{D} \tag{3.8}$$

and

$$|\Psi(0,t) - p| = \left|ce^{-2\alpha pt}\right| = |c|\, e^{-2pt\Re(\alpha)} < p, \quad \text{for all } t \in I. \tag{3.9}$$

Let $t > 0$ and let $z \in \mathbb{D}\backslash\{0\}$. Since $|e^{-t}z| \leqslant e^{-t} < 1$ for all $z \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : \ |z| \leqslant 1\}$ we find that $\Psi(z,t) - p$ is an analytic function in $\overline{\mathbb{D}}$. Using the maximum modulus principle it follows that for all $z \in \mathbb{D}\backslash\{0\}$ and each $t > 0$ arbitrarily fixed there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|\Psi(z,t) - p| < \lim_{|z|=1} |\Psi(z,t) - p| = \left|\Psi(e^{i\theta},t) - p\right|. \tag{3.10}$$

Denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$, and from (3.6), we have

$$\left|\Psi(e^{i\theta},t) - p\right| = \left|c\,|u|^{2\alpha p} + \frac{\left(1-|u|^{2\alpha p}\right)}{\alpha}\left[1-p+\frac{uf''(u)}{f'(u)}\right]\right|.$$

Since $u \in \mathbb{D}$, the inequality (3.1) implies that

$$\left|\Psi(e^{i\theta},t) - p\right| \leqslant p,$$

and from (3.8), (3.9), and (3.10), we conclude that

$$\left|\Psi(e^{i\theta},t) - p\right| < p$$

for all $z \in \mathbb{D}$ and $t \in I$. Therefore, $|w(z,t)| < 1$ for all $z \in \mathbb{D}$ and $t \in I$.

Since all the conditions of Lemma 2.1 are satisfied, we obtain that the function $\mathcal{L}(z,t)$ is the pth power of a univalent function whole unit disk $\mathbb{D}$, for all $t \in I$. For $t = 0$ we have $\mathcal{L}(z,0) = F_\alpha(z)$, for $z \in \mathbb{D}$ and therefore the function $F_\alpha(z)$ is the pth power of a univalent function in $\mathbb{D}$. □

For $p = 1$, condition (3.1) is a well-known sufficient condition of univalence given by Pescar [16].

Condition (3.1) of Theorem 3.1 can be replaced with a simpler one.

Theorem 3.2 *Let $f \in \mathcal{A}_p^*$ and let α be a complex number such that $\Re(\alpha) > 0$. Supposing that*

$$\left|1-p+\frac{zf''(z)}{f'(z)}\right| \leqslant p\Re(\alpha) \tag{3.11}$$

is true for all $z \in \mathbb{D}$, then the integral operator $F_\alpha(z)$ defined by (3.2) is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Proof It is known (see [15]) that for all $z \in \mathbb{D} \setminus \{0\}$ and $\Re(\alpha) > 0$,

$$\left| \frac{1 - |z|^{2\alpha p}}{\alpha} \right| \leq \frac{1 - |z|^{2p\Re(\alpha)}}{\Re(\alpha)}. \tag{3.12}$$

Making use of (3.11), we obtain

$$\begin{aligned} & \left| c\,|z|^{2\alpha p} + \frac{\left(1 - |z|^{2\alpha p}\right)}{\alpha} \left[1 - p + \frac{zf''(z)}{f'(z)}\right] \right| \\ \leq \;& |c|\,|z|^{2p\Re(\alpha)} + \frac{1 - |z|^{2p\Re(\alpha)}}{\Re(\alpha)} \left| 1 - p + \frac{zf''(z)}{f'(z)} \right| \\ \leq \;& p|z|^{2p\Re(\alpha)} + \frac{1 - |z|^{2p\Re(\alpha)}}{\Re(\alpha)} p\Re(\alpha) = p. \end{aligned}$$

Since the conditions of Theorem 3.1 are satisfied, it follows that the function $F_\alpha(z)$ defined by (3.2) is the pth power of a univalent function in $\mathbb{D}$. □

We now give some results that follow from Theorem 3.1. If we set $c = 0$, then by Theorem 3.1 we obtain the following:

Corollary 3.3 *Let $f \in \mathcal{A}_p^*$ and let α be a complex number such that $\Re(\alpha) > 0$. Supposing that*

$$\left| \frac{\left(1 - |z|^{2\alpha p}\right)}{\alpha} \left[1 - p + \frac{zf''(z)}{f'(z)}\right] \right| \leqslant p$$

is true for all $z \in \mathbb{D}$, then the integral operator $F_\alpha(z)$ defined by (3.2) is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Becker's univalence criterion can also be obtained from Corollary 3.3 for $\alpha = p = 1$. Using the inequality (3.12) in Corollary 3.3, we obtain the following result:

Corollary 3.4 *Let $f \in \mathcal{A}_p^*$ and let α be a complex number such that $\Re(\alpha) > 0$. Supposing that*

$$\frac{1 - |z|^{2p\Re(\alpha)}}{\Re(\alpha)} \left| 1 - p + \frac{zf''(z)}{f'(z)} \right| \leqslant p$$

is true for all $z \in \mathbb{D}$, then the integral operator $F_\alpha(z)$ defined by (3.2) is the p power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Example 3.1 *Let α be complex number such that $\Re(\alpha) > 1 - \frac{1}{p}$. Then the integral operator*

$$E_\alpha(z) = \left[\alpha p \int_0^z u^{p\alpha - 1} e^{u(p-1)} du \right]^{1/\alpha} \tag{3.13}$$

is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Proof In the integral operator (3.2) we get $f'(z) = p(ze^z)^{p-1}$. Then we have

$$\frac{zf''(z)}{f'(z)} = (p-1)(1+z).$$

From Corollary 3.4 we see that E_α given by (3.13) *is the* pth power of a univalent function in $\mathbb{D}$. □

For $p = 1$, Corollary 3.4 in turn implies the well-known univalence citerion of Pascu [15].

Theorem 3.5 *Let α and c be complex numbers such that $\Re(\alpha) > 0$, $|c| < p$ and $g \in \mathcal{A}$. Supposing that*

$$\left| c\,|z|^{2\alpha p} + \frac{\left(1-|z|^{2\alpha p}\right)}{\alpha}\left[(1-\alpha p)\left(1-\frac{zg'(z)}{g(z)}\right)+\frac{zg''(z)}{g'(z)}\right]\right| \leqslant p$$

is true for all $z \in \mathbb{D}$, then the function g is univalent in $\mathbb{D}$.

Proof Let $F_\alpha(z) = [g(z)]^p$. Thus, we obtain

$$f'(z) = pg'(z)(g(z))^{\alpha p-1} z^{p(1-\alpha)}.$$

It is easy to see that F_α satisfies the assumption of Theorem 3.1 if it satisfies the assumption of this theorem. Thus, g is a univalent function in $\mathbb{D}$ because F_α in view of Theorem 3.1 is the pth power of a univalent function. □

Reasoning along the same lines as in the proof of the Theorem 3.1 for the p-subordination chain

$$\mathcal{L}(z,t) = \left(\alpha \int_0^{e^{-t}z} u^{p(\alpha-1)} f'(u)du + \frac{\alpha}{p+c}\left(e^{2pt}-1\right)\left(e^{-t}z\right)^{(p(\alpha-1)+1)} f'(e^{-t}z)\right)^{1/\alpha}, \tag{3.14}$$

we obtain the following theorem. We omit the details.

Theorem 3.6 *Let α and c be complex numbers such that $|\alpha - 1| < 1$, $|c| < p$ and $f \in \mathcal{A}_p^*$. If the inequality*

$$\left| c\,|z|^{2p} + \left(1-|z|^{2p}\right)\left[p(\alpha-2)+1+\frac{zf''(z)}{f'(z)}\right]\right| \leqslant p \tag{3.15}$$

holds true for all $z \in \mathbb{D}$, then the integral operator $F_\alpha(z)$ defined by (3.2) is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

4. Applications

The problem of the univalence of integral operators in $\mathbb{D}$ was discussed by many authors. For example, Pfaltzgraff [17] proved that for $f \in \mathcal{S}$ the integral operator

$$G_\beta(z) = \int_0^z \left(f'(u)\right)^\beta du$$

is in the class $\mathcal{S}$ if $|\beta| \leqslant \frac{1}{4}$. He showed that the bound $\frac{1}{4}$ is sharp.

On the other hand, Kim and Merkes [13] showed that for $f \in \mathcal{S}$ the integral operator

$$G_\gamma(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\gamma du$$

is in the class $\mathcal{S}$ if $|\gamma| \leqslant \frac{1}{4}$.

The following lemma is of fundamental importance in our investigation.

Lemma 4.1 *(Wesolowski [23]). For each function $f \in \mathcal{S}$ and a fixed z, $z \in \mathbb{D}$, the inequality*

$$\left|\frac{z}{f(z)} - 1 + |z|^2\right| \leq 2\,(1 + |z|)$$

holds.

Proof By using a rotation of the form $f_\lambda(z) = \overline{\lambda} f(\lambda z)$, $|\lambda| = 1$, if needed, we see that it is enough to prove the inequality

$$\left|\frac{r}{f(r)} - 1 + r^2\right| \leq 2(1 + r), \qquad |z| = r.$$

Grunsky [7, p. 323] proved that the domain of variability in $\frac{z}{f(z)}$ is the closed disk

$$\left|\ln\frac{z}{f(z)} - \ln(1 - r^2)\right| \leq \ln\frac{1+r}{1-r}, \quad |z| = r, \quad z \in \mathbb{D}.$$

Hence, arguing as in [7, pp. 323-326] and denoting $\frac{1+r}{1-r} = a$, for any θ, $\theta \in [0, 2\pi]$ we have

$$\begin{aligned}
\left|\frac{r}{f(r)} - 1 + r^2\right| &= \left|(1 - r^2)a^{e^{i\theta}} - 1 + r^2\right| \\
&= (1 - r^2)\sqrt{a^{2\cos\theta} - 2a^{\cos\theta}\cos(\sin\theta\ln a) + 1} \\
&\leq (1 - r^2)\left(\frac{1+r}{1-r}\right)^{\cos\theta} + 1 - r^2 \leq 2(1 + r).
\end{aligned}$$

□

Theorem 4.1 *Let $f \in \mathcal{S}$. If α and β are any complex numbers such that $|\alpha - 1| < 1$ and*

$$|\beta| \leqslant \frac{p(1 - |\alpha - 1|)}{6p - 2},$$

then the integral operator

$$G_{\alpha,\beta}(z) = \left[\alpha p \int_0^z u^{\alpha p - 1}\left(f'(u)\right)^\beta du\right]^{1/\alpha} \tag{4.1}$$

is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Proof We begin by setting

$$F(z)=\int_0^z pu^{p-1}(f'(u))^{\beta}du \tag{4.2}$$

so that, obviously,

$$F'(z)=pz^{p-1}(f'(z))^{\beta}, \tag{4.3}$$

and from (4.3), we obtain

$$\frac{zF''(z)}{F'(z)}=p-1+\beta\left(\frac{zf''(z)}{f'(z)}\right). \tag{4.4}$$

It is well known that for any arbitrary point $z_0\in\mathbb{D}$, the function $f\in\mathcal{S}$ can be written as

$$f(z)=\frac{k\left(\frac{z+z_0}{1+z\bar{z}_0}\right)-k(z_0)}{k'(z_0)(1-|z_0|^2)},\quad z\in\mathbb{D}, \tag{4.5}$$

where k is a function in the class $\mathcal{S}$.

Therefore, we get that for all such z_0,

$$\frac{-z_0f''(-z_0)}{f'(-z_0)}=\frac{2|z_0|^2-2a_2z_0}{1-|z_0|^2} \tag{4.6}$$

where $a_2=a_2(z_0)$ is the second coefficient in the Taylor series expansion of the function k. The classical Bieberbach theorem states that $|a_2(z_0)|\leq 2$ for every $z_0\in\mathbb{D}$.

From (4.4) and (4.6), putting $z_0=-z$, we have

$$\frac{zF''(z)}{F'(z)}=p-1+\beta\frac{2|z|^2+2a_2(-z)z}{1-|z|^2},$$

where $|a_2|=|a_2(-z)|\leq 2$.

Putting $c=p(\alpha-1)-2\beta$ and F instead of f in (3.15) and using the above equality, we have

$$\begin{aligned}
&\left|(p(\alpha-1)-2\beta)|z|^{2p}+\left(1-|z|^{2p}\right)\left[p(\alpha-1)+\beta\frac{2|z|^2+2a_2z}{1-|z|^2}\right]\right|\\
=\;&\left|-2\beta|z|^{2p}+p(\alpha-1)+2\beta\left(1-|z|^{2p}\right)\left[\frac{|z|^2+a_2z}{1-|z|^2}\right]\right|\\
\leq\;& p|\alpha-1|+2|\beta|\left|a_2z\left(1+|z|^2+...+|z|^{2(p-1)}\right)+|z|^2\left(1+|z|^2+...+|z|^{2(p-2)}\right)\right|\\
\leq\;& p|\alpha-1|+2|\beta|(3p-1).
\end{aligned}$$

Finally, in view of the assumption $|\beta|\leqslant\frac{p(1-|\alpha-1|)}{6p-2}$ and Theorem 3.6, we conclude that the function $G_{\alpha,\beta}$ defined by (4.1) is the pth power of a univalent function in $\mathbb{D}$. This completes the proof. □

For $p=\alpha=1$ in Theorem 4.1 we obtain the following result of Pfaltzgraff [17].

Corollary 4.2 *Let* $f \in \mathcal{S}$. *If* $\beta \in \mathbb{C}$ *satisfies* $|\beta| \leqslant 1/4$, *then the integral operator*

$$G_\beta(z) = \int_0^z (f'(u))^\beta \, du \tag{4.7}$$

is univalent in $\mathbb{D}$, *where the principal branch is considered.*

Theorem 4.3 *Let* $f \in \mathcal{S}$. *If* α *and* γ *are any complex numbers such that* $|\alpha - 1| < 1$ *and*

$$|\gamma| \leqslant \frac{1 - |\alpha - 1|}{4},$$

then the integral operator

$$G_{\alpha,\gamma}(z) = \left[\alpha p \int_0^z u^{\alpha p - 1} \left(\frac{f(u)}{u} \right)^\gamma du \right]^{1/\alpha} \tag{4.8}$$

is the p*th power of a univalent function in* $\mathbb{D}$, *where the principal branch is considered.*

Proof We begin by setting

$$F(z) = \int_0^z p u^{p-1} \left(\frac{f(u)}{u} \right)^\gamma dt \tag{4.9}$$

so that, obviously,

$$F'(z) = p z^{p-1} \left(\frac{f(z)}{z} \right)^\gamma, \tag{4.10}$$

and from (4.10), we obtain

$$\frac{zF''(z)}{F'(z)} = p - 1 + \gamma \left(\frac{zf'(z)}{f(z)} - 1 \right). \tag{4.11}$$

For the class of univalent functions $\mathcal{S}$ we use the well-known Koebe transformation defined by (4.5) and we have

$$\frac{-z_0 f'(-z_0)}{f(-z_0)} = \frac{z_0}{k(z_0)(1 - |z_0|^2)}, \quad k \in \mathcal{S}. \tag{4.12}$$

From (4.11) and (4.12), putting $z_0 = -z$, we have

$$\frac{zF''(z)}{F'(z)} = p - 1 + \gamma \left(\frac{z}{-k(-z)(1 - |z|^2)} - 1 \right).$$

Putting $c = p(\alpha - 1)$ in (3.15) and using the above equality and Lemma 4.1, we have

$$
\begin{aligned}
&\left| p(\alpha-1)|z|^{2p} + \left(1-|z|^{2p}\right)\left[p(\alpha-1)+\gamma\left(\frac{z}{-k(-z)(1-|z|^2)}-1\right)\right]\right| \\
= &\left| p(\alpha-1)+\gamma\frac{\left(1-|z|^{2p}\right)}{1-|z|^2}\left[\frac{z}{-k(-z)}-1+|z|^2\right]\right| \\
\leq \ & p\,|\alpha-1| + 2\,|\gamma|\,(1+|z|)\left(1+|z|^2+...+|z|^{2(p-1)}\right) \\
\leq \ & p\,|\alpha-1| + 4p\,|\gamma|\,.
\end{aligned}
$$

In view of the assumption $|\gamma| \leqslant \frac{1-|\alpha-1|}{4}$ and Theorem 3.6, we obtain the assertion of the theorem. □

For $p = \alpha = 1$ in Theorem 4.3, we obtain the following result of Kim and Merkes [13].

Corollary 4.4 *Let $f \in \mathcal{S}$. If $\gamma \in \mathbb{C}$ satisfies $|\gamma| \leqslant 1/4$ then the integral operator*

$$
G_\gamma(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\gamma du \tag{4.13}
$$

is univalent in $\mathbb{D}$, where the principal branch is considered.

Another application is as follows.

Theorem 4.5 *Let $f \in \mathcal{A}_p^*$ be the pth power of a univalent function in $\mathbb{D}$. If α and μ are any complex numbers such that $|\alpha - 1| < 1$ and*

$$
|\mu| \leqslant \frac{p(1-|\alpha-1|)}{4p^2+2p-2},
$$

then the integral operator

$$
H_{\alpha,\mu}(z) = \left[\alpha p \int_0^z u^{\alpha p-1}\left(\frac{f'(u)}{pu^{p-1}}\right)^\mu du\right]^{1/\alpha} \tag{4.14}
$$

is the pth power of a univalent function in $\mathbb{D}$, where the principal branch is considered.

Proof We begin by setting

$$
F(z) = \int_0^z pu^{p-1}\left(\frac{f'(u)}{pu^{p-1}}\right)^\mu dt
$$

so that, obviously,

$$
F'(z) = pz^{p-1}\left(\frac{f'(z)}{pz^{p-1}}\right)^\mu, \tag{4.15}
$$

and from (4.15), we obtain

$$
\frac{zF''(z)}{F'(z)} = (p-1)(1-\mu) + \mu\frac{zf''(z)}{f'(z)}. \tag{4.16}
$$

Let $f(z) = (h(z))^p$ where $h \in \mathcal{S}$. Thus, we have

$$\frac{zf''(z)}{f'(z)} = (p-1)\frac{zh'(z)}{h(z)} + \frac{zh''(z)}{h'(z)}. \tag{4.17}$$

Now, from (4.16) and (4.17), we rewrite

$$\frac{zF''(z)}{F'(z)} = (p-1)(1-\mu) + \mu\left((p-1)\frac{zh'(z)}{h(z)} + \frac{zh''(z)}{h'(z)}\right). \tag{4.18}$$

By using the identities (4.6) and (4.12) for h instead of f, putting $c = p(\alpha-1) - 2\mu$ in (3.15) and Lemma 4.1, we find that

$$\begin{aligned}
&\Bigg|(p(\alpha-1)-2\mu)|z|^{2p} + \left(1-|z|^{2p}\right)\Big[p(\alpha-1) \\
&+\mu(p-1)\left(\frac{z}{-k(-z)(1-|z|^2)}-1\right) + \mu\frac{2|z|^2+2a_2z}{1-|z|^2}\Big]\Bigg| \\
= &\left|p(\alpha-1)-2\mu|z|^{2p} + \frac{\left(1-|z|^{2p}\right)}{(1-|z|^2)}\left[\mu(p-1)\left(\frac{z}{-k(-z)}-1+|z|^2\right)+\mu(2|z|^2+2a_2z)\right]\right| \\
= &\Bigg|p(\alpha-1)+2\mu\left(-|z|^{2p}+\frac{1-|z|^{2p}}{1-|z|^2}|z|^2\right)+2\mu a_2 z\frac{1-|z|^{2p}}{1-|z|^2} \\
&+\mu(p-1)\frac{1-|z|^{2p}}{1-|z|^2}\left(\frac{z}{-k(-z)}-1+|z|^2\right)\Bigg| \\
\leq\ & p|\alpha-1| + 2|\mu|\left|-|z|^{2p}+|z|^2\frac{1-|z|^{2p}}{1-|z|^2}\right| + 2|\mu|\frac{1-|z|^{2p}}{1-|z|^2}\left(|a_2||z|+(p-1)(1+|z|)\right) \\
=\ & p|\alpha-1| + 2|\mu||z|^2\left(1+|z|^2+\ldots+|z|^{2(p-2)}\right) \\
&+2|\mu|\left(|a_2||z|+(p-1)(1+|z|)\right)\left[1+|z|^2+\ldots+|z|^{2(p-1)}\right] \\
\leq\ & p|\alpha-1| + |\mu|\left[4p^2+2p-2\right].
\end{aligned}$$

In view of the assumption $|\mu| \leqslant \frac{p(1-|\alpha-1|)}{4p^2+2p-2}$ and Theorem 3.6, the proof is completed. □

For $\alpha = 1$ in Theorem 4.5 we obtain the following result of Hallenbeck and Livingston [8].

Corollary 4.6 *Let* $f \in \mathcal{A}_p^*$ *be the* p*th power of a univalent function in* $\mathbb{D}$*. If* μ *is any complex number such that*

$$|\mu| \leqslant \frac{p}{4p^2+2p-2}, \tag{4.19}$$

then the integral operator

$$H_\mu(z) = p\int_0^z u^{p-1}\left(\frac{f'(u)}{pu^{p-1}}\right)^\mu du \tag{4.20}$$

is the p*th power of a univalent function in* $\mathbb{D}$*, where the principal branch is considered.*

Acknowledgment

The research of the first author was supported by the Commission for the Scientific Research Projects of Kafkas University, project number 2012-FEF-30. The authors would like to thank the referees for their careful reading of the paper and for their helpful comments to improve it.

References

[1] Ahlfors LV. Sufficient conditions for quasiconformal extension. Ann Math Studies 1974; 79: 23-29.

[2] Becker J. Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. J Reine Angew Math 1972; 255: 23-43 (in German).

[3] Çağlar M, Orhan H. Sufficient conditions for univalence and quasiconformal extensions. Indian J Pure Appl Math 2015; 46: 41-50.

[4] Deniz E, Orhan H. Some notes on extensions of basic univalence criteria. J Korean Math Soc 2011; 48: 179-189.

[5] Deniz E, Orhan H. Loewner chains and univalence criteria related with Ruscheweyh and Sălăgean derivatives. J Appl Anal Comput 2015; 3: 465-478.

[6] Deniz E, Răducanu D, Orhan H. On an improvement of an univalence criterion. Math Balkanica (N.S.) 2010; 24: 33-39.

[7] Duren PL. Univalent Functions. Berlin, Germany: Springer Verlag, 1983.

[8] Hallenbeck DJ, Livingston AE. Subordination chains and p-valent functions. Not Am Math Soc 1976: A-731.

[9] Hotta I. Explicit quasiconformal extensions and Loewner chains. P Jpn Acad A-Math 2009; 85: 108-111.

[10] Hotta I. Loewner chains with complex leading coefficient. Monatsh Math 2011; 163: 315-325.

[11] Kanas S, Lecko A. Univalence criteria connected with arithmetic and geometric means. In: Proceedings of the Second Workshop on Transform Methods and Special Functions (Varna, 1996). Singapore: Science Culture Technology Publishing Company, 1998, pp. 201-209.

[12] Kanas S, Srivastava HM. Some criteria for univalence related to Ruscheweyh and Sălăgean derivatives. Complex Variables Theory Appl 1997; 38: 263-275.

[13] Kim YJ, Merkes EP. On an integral of power of a spirallike functions. Kyungpook Math J 1972; 12: 249-253.

[14] Ovesea H. A generalization of Ruscheweyh's univalence criterion. J Math Anal Appl 2001; 258: 102-109.

[15] Pascu NN. On a univalence criterion. In: Itinerant Seminar on Functional Equations Approximation and Convexity, Cluj-Napoca, pp. 153-154.

[16] Pescar V. A new generalization of Ahlfors's and Becker's criterion of univalence. Bull Malaysian Math Soc 1996; 19: 53-54.

[17] Pfaltzgraff JA. Univalence of the integral of $(f'(z))^{\lambda}$. B Lond Math Soc 1975; 7: 254-256.

[18] Pommerenke C. Über die Subordination analytischer Funktionen. J Reine Angew Math 1965; 218: 159-173 (in German).

[19] Pommerenke C. Univalent Functions. Gottingen, the Netherlands: Vandenhoeck Ruprecht, 1975.

[20] Răducanu D, Orhan H, Deniz E. On some sufficient conditions for univalence. Analele Stiint Univ 2010; 18: 217-222.

[21] Ruscheweyh S. An extension of Becker's univalence condition. Math Ann 1976; 220: 285-290.

[22] Srivastava HM, Owa S, editors. Current Topics in Analytic Function Theory. Singapore: World Scientifc Publishing Company, 1992.

[23] Wesolowski A. On the univalence of certain integrals meromorphic functions. Glas Mat 1990; 25: 43-47.