Turkish Journal of Mathematics

Volume 41 | Number 4 Article 8

1-1-2017

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 854 – 868
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doi:10.3906/mat-1511-82

Canonical involution on double jet bundles

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Received: 18.11.2015 • **Accepted/Published Online:** 26.09.2016 • **Final Version:** 25.07.2017

Abstract: In this study, we generalize double tangent bundles to double jet bundles. We present a secondary vector bundle structure on a 1-jet of a vector bundle. We show that the 1-jet of a vector bundle carries two vector bundle structures, namely primary and secondary structures. We also show that the manifold charts induced by primary and secondary structures belong to the same atlas. We prove that double jet bundles can be considered as a quotient of the second order jet bundle. We show that there exists a natural involution that interchanges between primary and secondary vector bundle structures on double jet bundles.

Key words: Double jet bundle, double vector bundle, second order jets, canonical involution, tangent bundle of higher order

1. Introduction

In general, there are two ways to define k-jets. The first definition is based on using the sections of a fibered manifold. In this definition, a k-jet is an equivalence class determined by an equivalence relation $\backsim_k$. Two sections of a fibered manifold are called k-related by the relation $\backsim_k$ if they have the same Taylor polynomial expansion at the point x truncated at order k. This definition usually leads to a geometric approach, which is applied to the study of systems of differential equations (we refer the reader to [1, 2, 4, 9–11] for more details).

The second definition of jet bundles is based on using the functions from N to M, where N and M are smooth manifolds. In this definition, a k-jet of a function f at x is an equivalence class defined by an equivalence relation $\backsim_k$. The equivalence of two functions is defined in the same way as the equivalence of sections. The collection of all k-jets is called a k-jet bundle. One particular case is when $N = \mathbb{R}$. In this case, the jet bundle is called a tangent bundle of higher order. This jet bundle possesses a certain kind of geometric structure, which is called an almost tangent structure of higher order. More generally, if $N = \mathbb{R}^p$, then the jet bundle is called the tangent bundle of p^k velocities. This concept was introduced by Ehresmann to develop classical field theory in an autonomous sense [6].

In this paper, we will consider the second definition: 1-jets with source at the origin of $\mathbb{R}^p$, and targeted in M, which gives a good approach in generalizing tangent bundles. The canonical involution in this work is a generalization of the canonical involution defined on tangent bundles. The canonical involution on tangent bundles is used to provide a practical description of a torsion of a connection. It was proven that symmetric connections are exactly the ones whose horizontal distributions are preserved by canonical involution

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2010 *AMS Mathematics Subject Classification:* Primary 58A20; Secondary 55R25, 58A05.

[8]. Moreover, there is a correspondence between symmetric connections and sprays on tangent bundles [8]. Since a jet bundle is a generalization of a tangent bundle, the canonical involution in this work can be used to find generalized symmetric connections and therefore generalized sprays on jet bundles.

The organization of this work is as follows: in Section 2, we provide some preliminary theorems that will be used in our proofs. In Section 3, we introduce a study of the 1-jet of an arbitrary vector bundle and present a secondary vector bundle structure on a 1-jet of the vector bundle. Due to two vector bundle structures, the 1-jet of a jet bundle is considered as a double vector bundle (DVB). In terms of the induced manifold structures on the total space of the DVB, we prove that both belong to the same atlas. We also prove that the 1-jet of a jet bundle (double jet bundle) can be considered as a quotient of a second order jet bundle. Moreover, we show that two vector bundle structures are isomorphic on a double jet bundle by defining a canonical involution that interchanges between such structures. In Section 4, we prove some identities and statements that we use throughout the paper.

2. Preliminaries

In this section we summarize some necessary preliminary materials that we need for a self-contained presentation of our paper.

2.1. First order jets: $J^1_p M$

Let $C^\infty(\mathbb{R}^p)$ be the algebra of C^∞-functions on the Euclidean space $\mathbb{R}^p$ with natural coordinates $(u_1, u_2, ..., u_p)$. Let $f, g \in C^\infty(\mathbb{R}^p)$; f is equivalent to g if $f(0) = g(0)$ and $\partial/\partial u_i(f) = \partial/\partial u_i(g)$ at $u = 0 \in \mathbb{R}^p$ for every $i = 1, 2, ..., p$; clearly, this is an equivalence relation.

Now let M be an m-dimensional manifold. Consider the set $C^\infty(\mathbb{R}^p; M)$ of all smooth maps $\phi : \mathbb{R}^p \to M$, and take elements $\phi, \xi \in C^\infty(\mathbb{R}^p; M)$. Then ϕ is equivalent to ξ if $f \circ \phi$ is equivalent to $f \circ \xi$ for every $f \in C^\infty(M)$. This is again an equivalence relation, denoted by $j^1(\phi)$ the equivalence class of $\phi \in C^\infty(\mathbb{R}^p; M)$ and called a 1-jet in M at $\phi(0)$. We denote $J^1_p M$ the set of all equivalence classes in $C^\infty(\mathbb{R}^p; M)$. If $(U, x_1, ...x_m)$ is a local chart in M, then $(J^1_p U, x_1, ..., x_m, x^1_\alpha, ...x^m_\alpha)$ is the local chart for $J^1_p M$, with $\alpha = 1, 2, ..., p$ by

$$x_i(j^1(\phi)) = x_i(\phi(0))$$
$$x^i_\alpha(j^1(\phi)) = \frac{\partial(x_i \circ \phi)}{\partial u_\alpha}|_0. \tag{1}$$

Let $j^1\phi$ be an arbitrary point, and let $\phi_\alpha : \mathbb{R} \to M$ be the differentiable curve given by $\phi_\alpha(u) = \phi(0, ..., u, ...0)$, with u at the α^{th} place; then, associated to $j^1\phi$ there is a unique $(p+1)$-tuple $[x; X_1, ..., X_p]$ given by

$$x = \phi(0) \qquad\qquad X_\alpha = \phi^\alpha_*(\frac{d}{du}|_0),$$

where d/du is the canonical vector field tangent to $\mathbb{R}$. From now on, we shall write $[x; X_1, ..., X_p]$ simply as $[x; X_\alpha]$ and shall identify $j^1\phi \equiv [x; X_\alpha]$ if there is no confusion.

Remark 1 *In later sections, we sometimes use the notation X_α for $\frac{\partial \Phi}{\partial u_\alpha}|_0$.*

Now we focus on some functorial properties of jet bundles.

Theorem 2 *[5]*

(i) If $h: M \to N$ is a differentiable function, then h induces a canonical differentiable map $h^1 : J_p^1 M \to J_p^1 N$ given by

$$h^1(j^1\phi) = j^1(h \circ \phi), \qquad \forall j^1\phi \in J_p^1 M,$$

and in terms of previous identification, we have $h^1([x; X_\alpha]) = [h(x); h_* X_\alpha]$.

(ii) If h is a diffeomorphism, then the induced map is also a diffeomorphism and, moreover, $(h^1)^{-1} = (h^{-1})^1$. For the manifolds M and N, $J_p^1(M \times N)$ is diffeomorphic to $J_p^1 M \times J_p^1 N$.

(iii) If M is a real vector space of dimension m, then $J_p^1 M$ inherits a vector space structure: for any $j^1 f, j^1 g \in J_p^1 M$ and $\lambda \in \mathbb{R}$, operations are given by

$$j^1 f + j^1 g = j^1(f+g), \qquad \lambda j^1 f = j^1(\lambda f),$$

where $f+g$ and λf are defined in the usual way. Vector space operations of $J_p^1 M$ as $[x; X_\alpha] + \lambda[y; Y_\alpha] = [x + \lambda y; X_\alpha + \lambda Y_\alpha]$.

2.2. Second order jets: $J_{2p}^2 M$

Now we consider the 2-jets by taking classes having equivalence up to all derivatives of second order. The natural atlas of this bundle can be obtained as follows:

Let $\phi, \phi' \in C^\infty(\mathbb{R}^{2p}, M)$. We say that ϕ is equivalent to ϕ' if

$$\begin{aligned} \phi(0,0) &= \phi'(0,0) \\ \frac{\partial \phi}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)} &= \frac{\partial \phi'}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)} \\ \frac{\partial^2 \phi}{\partial \bar{u}_{\bar{\alpha}} \partial \bar{u}_{\bar{\beta}}}|_{(0,0)} &= \frac{\partial^2 \phi'}{\partial \bar{u}_{\bar{\alpha}} \partial \bar{u}_{\bar{\beta}}}|_{(0,0)}. \qquad 1 \leq \bar{\alpha}, \bar{\beta} \leq 2p \end{aligned}$$

Let E be a finite dimensional real vector space, and then $J_{2p}^2 E$ is regarded as a finite dimensional real vector space. Let $L(\mathbb{R}^{2p}, E)$ denote the vector space of linear functions $A : \mathbb{R}^{2p} \to E$ and let $S_2(\mathbb{R}^{2p}; E)$ denote the vector space of all symmetric bilinear functions $B : \mathbb{R}^{2p} \times \mathbb{R}^{2p} \to E$. The function

$$j^2\phi \to \left(\phi(0,0), \frac{\partial \phi}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)}, \frac{\partial^2 \phi}{\partial \bar{u}_{\bar{\alpha}} \partial \bar{u}_{\bar{\beta}}}|_{(0,0)}\right), \qquad 1 \leq \bar{\alpha}, \bar{\beta} \leq 2p$$

is a canonical isomorphism, where

$$\bar{u}_{\bar{\alpha}} = \begin{cases} u_\alpha & \bar{\alpha} = 1, 2, ..., p, \\ w_\alpha & \bar{\alpha} = p+1, ..., 2p. \end{cases}$$

We note that we identify $\mathbb{R}^{2p}$ with $\mathbb{R}^p \times \mathbb{R}^p$. Let $\varphi : U \to E$ be a local chart of M that, without loss of generality, we assume maps onto a vector space E. Let $(x, A, B) \in J^2_{2p}(E)$; the corresponding 2-jet is the one represented by

$$\phi(\bar{u}) = x + A.\bar{u} + \frac{1}{2}\bar{u}^T.B.\bar{u}.$$

In terms of curve notations discussed for the first order jets, we may identify $j^2\phi \in J^2_{2p}$ with the triple

$$j^2\phi = [x; A_{\bar{\alpha}}; B_{\bar{\alpha}\bar{\beta}}], \qquad 1 \leq \bar{\alpha}, \bar{\beta} \leq 2p,$$

where

$$A_{\bar{\alpha}} = \frac{\partial \phi}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)}$$

and

$$B_{\bar{\alpha}\bar{\beta}} = \frac{\partial^2 \phi}{\partial \bar{u}_{\bar{\alpha}} \partial \bar{u}_{\bar{\beta}}}|_{(0,0)}.$$

3. Jet bundle to a vector bundle

Let $\pi_{\mathbb{E}} : \mathbb{E} \to M$ be a vector bundle with the local bundle trivialization

$$\psi : \pi_{\mathbb{E}}^{-1}(U) \to U \times E,$$

where U is an open subset of M. Then $J^1_p\mathbb{E}$ can be considered as a jet bundle on $\mathbb{E}$ (the total space of VB $\pi_{\mathbb{E}}$). The bundle trivialization on $J^1_p\mathbb{E}$ is

$$\tilde{\psi} : J^1_p\mathbb{E} \to \mathbb{E} \times L(\mathbb{R}^p, \mathbb{R}^{m+k})$$

by

$$\tilde{\psi}(j^1\Phi) = (\Phi(0), \frac{\partial \Phi}{\partial u_\alpha}|_0),$$

where $\Phi \in C^\infty(\mathbb{R}^p, \mathbb{E})$. Then $\Phi(0) = (\psi)^{-1}(x, y)$ and

$$\frac{\partial \Phi}{\partial u_\alpha}|_0 = (\frac{\partial(\bar{x}_i \circ \Phi)}{\partial u_\alpha}|_0, \frac{\partial(\bar{y}_j \circ \Phi)}{\partial u_\alpha}|_0),$$

where $\bar{x}_i = x_i \circ \pi_{\mathbb{E}}$, $\bar{y}_j = y_j \circ pr_2 \circ \psi$, and x_i, y_j are the local coordinate functions of M and E, respectively.

Using curve notation gives:

$$j^1\Phi \equiv [x, y; X_\alpha, Y_\alpha]$$

where

$$X_\alpha = \frac{\partial(\bar{x}_i \circ \Phi)}{\partial u_\alpha}|_0$$

and

$$Y_\alpha = \frac{\partial(\bar{y}_j \circ \Phi)}{\partial u_\alpha}|_0.$$

Moreover, for each smooth manifold M, $J^1_p M$ carries a vector bundle structure (see Lemma (12)). Since $\mathbb{E}$ is a smooth manifold, then $J^1_p\mathbb{E}$ carries a VB structure, where the intrinsic operations are given by the following:

For a local bundle chart $\tilde{\psi} : (\pi_\mathbb{E}\circ\tilde{\pi})^{-1} \to \pi_\mathbb{E}^{-1}(U)\times L(\mathbb{R}^p, \mathbb{R}^{m+k})$, the mapping $+_1$ and $\bullet_1$ on $(\tilde{\pi})^{-1}\{(x,y)\}$ are defined locally as

$$j^1\Phi +_1 j^1\Phi' = (\tilde{\psi})^{-1}((\psi)^{-1}(x,y), (pr_2 \circ \tilde{\psi})(j^1\Phi) + (pr_2 \circ \tilde{\psi})(j^1\Phi'))$$

and

$$\lambda \bullet_1 j^1\Phi = (\tilde{\psi})^{-1}((\psi)^{-1}(x,y), \lambda.(pr_2 \circ \tilde{\psi})(j^1\Phi)),$$

where $\tilde{\pi} : J^1_p\mathbb{E} \to \mathbb{E}$ is the projection map of the first jet bundle $(J^1_p\mathbb{E}, \tilde{\pi}, \mathbb{E}, L(\mathbb{R}^p, \mathbb{R}^{m+k}))$. Using curve notation, one can see that

$$j^1\Phi +_1 j^1\Phi' = [x, y; X_\alpha, Y_\alpha] +_1 [x, y; X'_\alpha, Y'_\alpha] = [x, y; X_\alpha + X'_\alpha, Y_\alpha + Y'_\alpha]$$

and

$$\lambda \bullet_1 [x, y; X_\alpha, Y_\alpha] = [x, y; \lambda.X_\alpha, \lambda.Y_\alpha],$$

where U is an open subset of M.

Remark 3 *Hereafter, we will refer to the above vector bundle structure as the primary structure.*

3.1. Secondary VB structure on $J^1_p\mathbb{E}$

We recall that there exists an induced canonical smooth function $\pi^1_\mathbb{E} : J^1_p\mathbb{E} \to J^1_pM$, where $\pi_\mathbb{E} : \mathbb{E} \to M$ a smooth bundle projection of $\mathbb{E}$. One can easily prove that $\pi^1_\mathbb{E}$ is a surjective map. Moreover, it can be seen from Theorem 2 that J^1_pE is a vector space (isomorphic to $L(\mathbb{R}^p, E)$). If we let the local trivialization of $\mathbb{E}$ be the map ψ, then $\psi^1 : J^1_p(\pi_\mathbb{E}^{-1}(U)) \to J^1_p(U \times E)$ is a diffeomorphism.

Now we consider the trivialization domains:

Lemma 4 *Let $\pi_\mathbb{E}^{-1}(U)$ be a local trivialization domain of the bundle $\mathbb{E}$. Then*

$$J^1_p(\pi_\mathbb{E}^{-1}(U)) = (\pi^1_\mathbb{E})^{-1}(J^1_pU) = (\pi_\mathbb{E} \circ \tilde{\pi})^{-1}(U). \tag{2}$$

Proof To prove equation (2), we will prove the following:

(i) $J^1_p(\pi_\mathbb{E}^{-1}(U)) \subset (\pi^1_\mathbb{E})^{-1}(J^1_pU)$,

(ii) $(\pi^1_\mathbb{E})^{-1}(J^1_pU) \subset (\pi_\mathbb{E} \circ \tilde{\pi})^{-1}(U)$, and

(iii) $(\pi_{\mathbb{E}} \circ \tilde{\pi})^{-1}(U) \subset J^1_p(\pi_{\mathbb{E}}^{-1}(U))$.

Suppose that

$$j^1\phi \in J^1_p(\pi_{\mathbb{E}}^{-1}(U)).$$

By its own definition, $\phi(0) \in \pi_{\mathbb{E}}^{-1}(U)$, and then

$$\begin{aligned} \pi_{\mathbb{E}}(\phi(0)) \in U \quad &\Rightarrow \quad j^1(\pi_{\mathbb{E}} \circ \Phi) \in J^1_p U \\ &\Rightarrow \quad \pi^1_{\mathbb{E}}(j^1\Phi) \in J^1_p U \\ &\Rightarrow \quad j^1\phi \in (\pi^1_{\mathbb{E}})^{-1}(J^1_p U). \end{aligned}$$

Then $J^1_p((\pi^1_{\mathbb{E}})^{-1}(U)) \subset (\pi^1_{\mathbb{E}})^{-1}(J^1_p U)$. The first statement is proven.

Suppose that $j^1\Phi \in (\pi^1_{\mathbb{E}})^{-1}(J^1_p U)$. Then

$$\begin{aligned} \pi^1_{\mathbb{E}}(j^1\Phi) \in J^1_p U \quad &\Rightarrow \quad j^1(\pi_{\mathbb{E}} \circ \Phi) \in J^1_p U \\ &\Rightarrow \quad (\pi_{\mathbb{E}} \circ \Phi)(0) \in U \\ &\Rightarrow \quad \Phi(0) \in (\pi_{\mathbb{E}})^{-1}(U) \\ &\Rightarrow \quad \tilde{\pi}^{-1}(\Phi(0)) \in \tilde{\pi}^{-1}(\pi_{\mathbb{E}}^{-1}(U)) \\ &\Rightarrow \quad j^1\Phi \in (\pi_{\mathbb{E}} \circ \tilde{\pi})^{-1}(U). \end{aligned}$$

Then $(\pi^1_{\mathbb{E}})^{-1}(J^1_p U) \subset (\pi_{\mathbb{E}} \circ \tilde{\pi})^{-1}(U)$. The second statement is proven.

On the other hand, suppose that $j^1\Phi \in (\pi_{\mathbb{E}} \circ \tilde{\pi})^{-1}(U)$. Then

$$\begin{aligned} (\pi_{\mathbb{E}} \circ \tilde{\pi})(j^1\Phi) \in U \quad &\Rightarrow \quad (\pi_{\mathbb{E}}(\tilde{\pi}(j^1\Phi))) \in U \\ &\Rightarrow \quad \pi_{\mathbb{E}}(\Phi(0)) \in U \\ &\Rightarrow \quad \Phi(0) \in \pi_{\mathbb{E}}^{-1}(U) \\ &\Rightarrow \quad j^1\Phi \in J^1_p(\pi_{\mathbb{E}}^{-1}(U)), \end{aligned}$$

which shows that $(\pi_{\mathbb{E}} \circ \tilde{\pi})^{-1}(U) \subset J^1_p(\pi_{\mathbb{E}}^{-1}(U))$. This completes the proof. □

Proposition 5 *Let $\pi_{\mathbb{E}} : \mathbb{E} \to M$ be a vector bundle. Then $\pi^1_{\mathbb{E}} : J^1_p\mathbb{E} \to J^1_p M$ is a vector bundle so that the manifold $J^1_p\mathbb{E}$ has two vector bundle structures, namely its primary vector bundle structure as the first jet bundle of manifold $\mathbb{E}$ and a secondary structure with $J^1_p M$ as the base manifold. Moreover, the induced charts on $J^1_p\mathbb{E}$ from the primary and secondary structures belong to the same atlas.*

Proof First, we start the proof by showing that ψ^1 is the bundle trivialization of the secondary jet bundle $J^1_p\mathbb{E}$:

Since $Pr_1 \circ \psi^1 = \pi^1_{\mathbb{E}}$, then $\pi^1_{\mathbb{E}} : J^1_p\mathbb{E} \to J^1_pM$ is a smooth fiber bundle. (Here $Pr_1 : J^1_pU \times J^1_pE \to J^1_pU$ represents the first projection. We note that $J^1_p(U \times E)$ and $J^1_pU \times J^1_pE$ are diffeomorphic by setting $j^1f \cong (j^1(f_1), j^1(f_2))$, where $f = (f_1, f_2) : \mathbb{R}^p \to U \times E$.)

Now we identify $j^1\Phi \in J^1_p\mathbb{E}$ as quadruple $[x, X_\alpha; y, Y_\alpha]$ such that

$$X_\alpha = \frac{\partial(\pi_{\mathbb{E}} \circ \Phi)}{\partial u_\alpha}|_0 \tag{3}$$

and

$$Y_\alpha = \frac{\partial(pr_2 \circ \psi \circ \Phi)}{\partial u_\alpha}|_0. \tag{4}$$

Showing that the fiber map is linear proves that $\pi^1_{\mathbb{E}}$ is a vector bundle over J^1_pM.

Let $j^1\Phi, j^1\Phi' \in (\pi^1_{\mathbb{E}})^{-1}\{j^1\theta\}$, for each $j^1\theta = [x; X_\alpha] \in J^1_pM$. Then $j^1\Phi = [x, X_\alpha; y, Y_\alpha]$ for $\alpha = 1, 2, ..., p$, and $j^1\Phi' = [x, X_\alpha; y', Y'_\alpha]$. The secondary vector bundle operations on this fiber are defined by:

$$\begin{aligned} j^1\Phi +_2 j^1\Phi' &= (\psi^1)^{-1}(j^1\theta, (pr_2 \circ \psi^1)(j^1\Phi) + (j^1\theta, pr_2 \circ \psi^1)(j^1\Phi')) \\ &= [x, X_\alpha; y + y', Y_\alpha + Y'_\alpha] \end{aligned}$$

and

$$\begin{aligned} \lambda \bullet_2 j^1\Phi &= (\psi^1)^{-1}(j^1\theta, \lambda.(pr_2 \circ \psi^1)(j^1\Phi)) \\ &= [x, X_\alpha; \lambda y, \lambda Y_\alpha], \end{aligned}$$

where $j^1\Phi, j^1\Phi' \in (\pi^1_{\mathbb{E}})^{-1}\{j^1\theta\}$, and $\lambda \in \mathbb{R}$. Thus,

$$\begin{aligned} \psi^1_{[x,X_\alpha]}([x, X_\alpha; y, Y_\alpha] +_2 \lambda \bullet_2 [x, X_\alpha; y', Y'_\alpha] &= \psi^1([x, X_\alpha; y + \lambda y', Y_\alpha + \lambda Y'_\alpha]) \\ &= [y + \lambda y', Y_\alpha + \lambda Y'_\alpha] \\ &= [y, Y_\alpha] +_{\mathbb{E}} \lambda \bullet_{\mathbb{E}} [y', Y'_\alpha], \end{aligned}$$

which is equal to

$$\psi^1_{[x,X_\alpha]}([x, X_\alpha; y, Y_\alpha]) +_2 \lambda \bullet_{\mathbb{E}} \psi^1_{[x,X_\alpha]}([x, X_\alpha; y', Y'_\alpha]).$$

This shows that $\psi^1_{[x,X_\alpha]}$ is a linear function. Therefore, $J^1_p\mathbb{E}$ is a vector bundle with its secondary structure.

By now, we have shown that $J^1_p\mathbb{E}$ carries two vector bundle structures (namely primary and secondary). One can easily see that these two structures define coordinate charts on the total space $J^1_p\mathbb{E}$. Now we will show that these two charts belong to the same atlas. To do this, we will show that the identity map of $J^1_p\mathbb{E}$ is a diffeomorphism between two VB structures based on J^1_pM and $\mathbb{E}$.

Let $\mathbb{E}$ have the local coordinate maps $\bar{x}_i, \bar{y}_j$ with $1 \le i \le m, 1 \le j \le k$. Considering the primary structure on $J^1_p\mathbb{E}$, for all $j^1\phi \in J^1_p\mathbb{E}$, there exist triples (x, y, ρ) with

$\Phi(0)=\psi^{-1}(x,y)$ and $\rho:\mathbb{R}^p\to\mathbb{R}^{m+k}$, which is a linear function that corresponds to the matrix $A=\left[\frac{\partial\Phi_{\bar{a}}}{\partial u_\alpha}|_0\right]$. Here the term $\Phi_{\bar{a}}$ is defined by

$$\Phi_{\bar{a}}=\begin{cases}(\bar{x}_i\circ\Phi) & \bar{a}=1,2,...,m,\\ (\bar{y}_j\circ\Phi) & \bar{a}=m+1,....,m+k.\end{cases}$$

Therefore, the matrix A consists of two submatrices: they are

$$\left[\frac{\partial(\bar{x}_i\circ\Phi)}{\partial u_\alpha}|_0\right] \qquad \textit{and} \qquad \left[\frac{\partial(\bar{y}_j\circ\Phi)}{\partial u_\alpha}|_0\right]. \tag{5}$$

On the other hand, considering the secondary structure on $J^1_p\mathbb{E}$, given any $j^1\Phi\in J^1_p\mathbb{E}$, there exists quadruple $(x,f;y,g)$, where $\Phi(0)=\psi^{-1}(x,y)$, $f=\left[\frac{\partial(\bar{x}_i\circ\Phi)}{\partial u_\alpha}|_0\right]$, and $g=\left[\frac{\partial(\bar{y}_j\circ\Phi)}{\partial u_\alpha}|_0\right]$.

It is clear that the matrix representations of f and g are defined the same as in Equation 5. Let Ω be the identity map of $J^1_p\mathbb{E}$. Let $\Psi_{\mathbb{E}}$, Ψ_M, and Ψ_E be the local trivializations of jet bundles $J^1_p\mathbb{E}, J^1_pM$, and J^1_pE, respectively. Suppose further that

$\varphi:U\subset M\to\mathbb{R}^m$ is a coordinate chart. Then the local form of Ω (which we denote by $\hat{\Omega}$) is given by the following commutative diagram:

$$\begin{array}{ccc}(\pi^1_{\mathbb{E}})^{-1}(J^1_pU) & \xrightarrow{\quad\Omega\quad} & (\pi_{\mathbb{E}}\circ\tilde{\pi})^{-1}(U)\\ \downarrow\varphi_2 & & \downarrow\varphi_1\\ \varphi(U)\times L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k) & \xrightarrow{\quad\hat{\Omega}\quad} & \varphi(U)\times E\times L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^k)\end{array}$$

where

$$\varphi_1=(\varphi\times id_{E\times L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^k)})\circ(\psi\times\xi)\circ\tilde{\psi},$$

and

$$\varphi_2=(\varphi\times id_{L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k)})\circ(\Psi_M\times\Psi_E)\circ\psi^1,$$

and

$$\begin{array}{rcl}\xi:L(\mathbb{R}^p,\mathbb{R}^{m+k}) & \to & L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^k)\\ f_g & \to & \xi(f_g)=(f,g),\end{array}$$

where $f_g(u)=(f(u),g(u))$.

From Lemma 16, the local form $\hat{\Omega}$ is $(x,f,y,g)\to(x,y,f,g)$. It is clear that the local form is a surjective map. Since $\hat{\Omega}=pr_1\times pr_3\times pr_2\times pr_4$, then the local form is differentiable with its own inverse. Therefore, $\hat{\Omega}$ is a diffeomorphism, which implies that two structure charts belong to the same atlas. □

Remark 6 *The secondary structure defined on tangent bundles can be found in [8].*

3.2. $J_p^1(J_p^1 M)$ as a quotient manifold

Let us consider the case $\mathbb{E} = J_p^1 M$ for the smooth manifold M. By Lemma 12, $J_p^1 M$ is a vector bundle. By Proposition 5, $J_p^1(J_p^1 M)$ has two vector bundle structures both based on $J_p^1 M$. To define the double jet manifold, we begin with the idea of smooth functions on $J_p^1 M$.

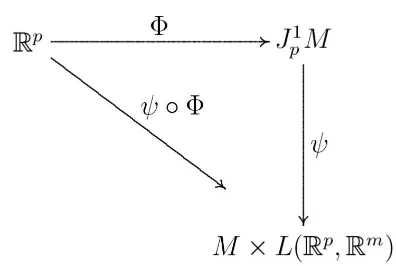
Let $\phi : \mathbb{R}^p \times \mathbb{R}^p \to M$ be a smooth function. We define

$$\begin{aligned} \Phi \quad &: \mathbb{R}^p \quad \to J_p^1 M \\ u \quad &\quad\;\; \to \Phi(u) = J_p^1(\phi_u), \end{aligned} \tag{6}$$

where ϕ_u is defined by

$$\begin{aligned} \phi_u \quad &: \mathbb{R}^p \quad \to M \\ v \quad &\quad\;\; \to \phi_u(v) = \phi(u, v). \end{aligned}$$

It can easily be seen that ϕ_u is a smooth function.



Here

$$\begin{aligned} (\psi \circ \Phi)(u) &= (\phi_u(0), \frac{\partial \phi_u}{\partial w_\alpha}|_{w=0}), \qquad\qquad u \in \mathbb{R}^p \\ &= (\phi(u,0), \frac{\partial \phi_u}{\partial w_\alpha}|_{w=0}) \end{aligned}$$

is a smooth function. Therefore, $\Phi \in C^\infty(\mathbb{R}^p, J_p^1 M)$.

Proposition 7 *Let Φ and ϕ be the functions defined by Equation 6. Then $J_p^1(J_p^1 M)$ is a quotient of $J_{2p}^2 M$, with the quotient map Λ defined by the following:*

$$\begin{aligned} \Lambda : \quad J_{2p}^2(M) \to &\quad J_p^1(J_p^1 M) \\ J^2\phi \to &\quad J^1\Phi. \end{aligned}$$

Proof Let ψ be a local trivialization of the vector bundle $J_p^1 M$; then considering local charts for $J_{2p}^2 M$, and

$J^1_p(J^1_p M)$, we have the following commutative diagram:

$$\begin{array}{ccc} J^2_{2p}M & \xrightarrow{\quad\Lambda\quad} & J^1_p(J^1_p M) \\ \Big\downarrow \psi^2 & & \Big\downarrow \bar{\psi}\circ\psi^1 = \widetilde{\psi} \\ M\times L(\mathbb{R}^{2p},\mathbb{R}^m)\times S_2(\mathbb{R}^{2p},\mathbb{R}^m) & \xrightarrow{\quad\hat{\Lambda}\quad} & M\times L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^{mp}) \end{array} \tag{7}$$

where $\bar{\psi}$ denotes the local trivialization of $J^1_p(M\times L(\mathbb{R}^p,\mathbb{R}^m))$, and ψ^2 stands for the local trivialization of $J^2_{2p}M$. We note that, due to Proposition 5, it is also possible to use the induced chart of the primary structure.

For any $A\in L(\mathbb{R}^{2p},\mathbb{R}^m)$, we can identify A in terms of the submatrices of form $m\times p$ with $A=[A_\alpha A_{\dot\alpha}]$ where $A_\alpha, A_{\dot\alpha}\in L(\mathbb{R}^p,\mathbb{R}^m)$. Similarly, for any $B\in S_2(\mathbb{R}^{2p},\mathbb{R}^m)$, we can identify B as the submatrices

$$B_{\bar\alpha\bar\beta}=\begin{pmatrix} B_{\alpha\beta} & B_{\dot\alpha\beta} \\ B_{\alpha\dot\beta} & B_{\dot\alpha\dot\beta}\end{pmatrix}.$$

Suppose that $(x,A,B)\in M\times L(\mathbb{R}^{2p},\mathbb{R}^m)$. Let $(\psi^2)^{-1}(x,A,B)=j^2\phi$ with the following properties:

$$x=\phi(0,0),\quad A=\frac{\partial\phi}{\partial\bar{u}_\alpha}|_{(0,0)},\quad [B_{\alpha\beta}]=\frac{\partial^2\phi}{\partial\bar{u}_\alpha\bar{u}_\beta}|_{(0,0)}.$$

Therefore, the submatrices A_α, $A_{\dot\alpha}$, $B_{\alpha\beta}$, $B_{\dot\alpha\beta}$, $B_{\alpha\dot\beta}$, and $B_{\dot\alpha\dot\beta}$ are

$$A_\alpha=\frac{\partial\phi}{\partial u_\alpha}|_{(0,0)}\qquad A_{\dot\alpha}=\frac{\partial\phi}{\partial w_\alpha}|_{(0,0)}$$

and

$$[B_{\alpha\beta}]=\frac{\partial^2\phi}{\partial u_\alpha u_\beta}|_{(0,0)}\qquad [B_{\dot\alpha\beta}]=\frac{\partial^2\phi}{\partial w_\alpha u_\beta}|_{(0,0)}\qquad [B_{\alpha\dot\beta}]=\frac{\partial^2\phi}{\partial u_\alpha w_\beta}|_{(0,0)}\qquad [B_{\dot\alpha\dot\beta}]=\frac{\partial^2\phi}{\partial w_\alpha w_\beta}|_{(0,0)}.$$

Then,

$$\begin{aligned}\hat\Lambda(x,A,B) &= ((\bar\psi\circ\psi^1)\circ\Lambda\circ(\psi^2)^{-1})(x,A,B)\\ &= ((\bar\psi\circ\psi^1)\circ\Lambda)(j^2(\phi))\\ &= (\bar\psi\circ\psi^1)(j^1\Phi)\\ &= \bar\psi(j^1(\psi\circ\Phi))\\ &= ((\psi\circ\Phi)(0),\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}|_0).\end{aligned}\tag{8}$$

From the last line of Equation 8, one can see that

$$(\psi\circ\Phi)(0)=\psi(j^1(\phi_0))=(\phi(0,0),\frac{\partial\phi(0,w)}{\partial w_\alpha}|_{w=0})\in M\times L(\mathbb{R}^p,\mathbb{R}^m),$$

and

$$\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}|_0 = (\frac{\partial\phi(u,0)}{\partial u_\alpha}|_{u=0}, \frac{\partial^2\phi}{\partial u_\alpha \partial w_\beta}|_{(0,0)}) \in L(\mathbb{R}^p,\mathbb{R}^m)\times L(\mathbb{R}^p,\mathbb{R}^{mp}).$$

Continuing to Equation (8), we have

$$\begin{aligned}\hat{\Lambda}(x,A,B) &= \left((\phi(0,0), \frac{\partial\phi(0,w)}{\partial w_\alpha}|_{w=0}), (\frac{\partial\phi(u,0)}{\partial u_\alpha}|_{u=0}, \frac{\partial^2\phi}{\partial u_\alpha\partial w_\beta}|_{u=w=0})\right)\\ &= \left(\phi(0,0), \frac{\partial\phi(u,w)}{\partial w_\alpha}|_{(u,w)=(0,0)}, \frac{\partial\phi(u,0)}{\partial u_\alpha}|_{u=0}, \frac{\partial^2\phi}{\partial u_\alpha\partial w_\beta}|_{(u,w)=(0,0)}\right)\\ &= (x, A_{\dot\alpha}, A_\alpha, B_{\alpha\dot\beta}).\end{aligned}$$

The map $(A,B)\to(x,A_{\dot\alpha},A_\alpha,B_{\alpha\dot\beta})$ is a surjective linear map. It follows immediately that $\hat{\Lambda}$ is a submersion, and thus Λ is. □

Remark 8 *Using curve notation on each* $j^2\phi$ *and* $j^1\Phi$ *in* $J^2_{2p}M$ *and* $J^1_p(J^1_pM)$ *respectively as follows:*

$$j^2\phi = [x;[A_\alpha A_{\dot\alpha}];B_{\bar\alpha\bar\beta}],$$

and considering the primary structure on $J^1_p(J^1_pM)$*, we have*

$$j^1\Phi = [(x,X_\alpha);(Y_\alpha,C_{\alpha\beta})],$$

where $(x,X_\alpha)=\Phi(0)$*, and* $(Y_\alpha,C_{\alpha\beta}) = \frac{\partial\Phi}{\partial u_\alpha}|_0$.

Considering the secondary structure on $J^1_p(J^1_pM)$*, we have*

$$j^1\Phi = [(x,X'_\alpha);(Y'_\alpha,C'_{\alpha\beta})].$$

It follows from equations (3) and (4) that X'_α*,* Y'_α*, and* $C'_{\alpha\beta}$ *are defined by the following:*

$$X'_\alpha = \frac{\partial(\pi\circ\Phi)}{\partial u_\alpha}, \qquad Y'_\alpha = (pr_2\circ\psi\circ\Phi)(0) = \frac{\partial\phi_0}{\partial u_\alpha}|_0,$$

and

$$C'_{\alpha\beta} = \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0 = \frac{\partial^2\Phi}{\partial u_\alpha\partial w_\beta}|_0.$$

The local form of Λ *follows immediately such that the curve notation (discussed in Section 2) is as follows:*

$$\Lambda([x;[A_\alpha A_{\dot\alpha}];B_{\bar\alpha\bar\beta}]) = [x,A_{\dot\alpha};A_\alpha,B_{\alpha\dot\beta}].$$

Now we shall define an involution on second order jet bundles that will help us to define the involution of $J^1_p(J^1_pM)$.

Theorem 9 *There exists a canonical involution* $\hat{\ell} : J^2_{2p}M \to J^2_{2p}M$ *that descends via* Λ *to* $J^1_p(J^1_pM)$ *giving the involution* $\ell : J^1_p(J^1_pM) \to J^1_p(J^1_pM)$.

To prove this theorem, we need to show that $\hat{\ell}$ is a diffeomorphism, and the induced map ℓ is well defined.

Proof Let $\mathfrak{f} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ denote the flip map $(u,w) \to (w,u)$ (the map that flips first and second p-tuples). Define

$$\tilde{\ell} : J^2_{2p}M \to J^2_{2p}M$$

by

$$\tilde{\ell}(j^2\phi) = j^2(\phi \circ \mathfrak{f}).$$

The local form of $\tilde{\ell}$ is the involution $(x, A, B) \to (x, \bar{A}, \bar{B})$ such that

$$(x, \bar{A}, \bar{B}) = (x, [A_{\dot{\alpha}} A_{\alpha}], \begin{pmatrix} B_{\dot{\alpha}\dot{\beta}} & B_{\alpha\dot{\beta}} \\ B_{\dot{\alpha}\beta} & B_{\alpha\beta} \end{pmatrix}). \tag{9}$$

(For detailed proof of Equation (9), see Appendix, Section 4, Corollary 15.)

The linear function corresponding to matrix $\bar{A}$ is

$$\bar{A}(U,W) = A_{\dot{\alpha}}U + A_{\alpha}W = [A_{\alpha}A_{\dot{\alpha}}].(W,U) = A(f(U,W)).$$

Therefore, $\bar{A} = A \circ f$. The symmetric bilinear form corresponding to matrix $\bar{B}$ is

$$\begin{aligned} \bar{B}((U,W),(\bar{U},\bar{W})) &= (\bar{U},\bar{W})^T.\bar{B}.(U,W) \\ &= B_{\dot{\alpha}\dot{\beta}}U\bar{U} + B_{\alpha\dot{\beta}}W\bar{U} + B_{\dot{\alpha}\beta}U\bar{W} + B_{\alpha\beta}W\bar{W} \\ &= \mathfrak{f}(\bar{U},\bar{W})^T.B.\mathfrak{f}(U,W) \\ &= B(\mathfrak{f}(U,W),\mathfrak{f}(\bar{U},\bar{W})). \end{aligned}$$

Thus, $\bar{B} = B \circ (\mathfrak{f},\mathfrak{f})$. Then the local form of $\tilde{\ell}$ is the function

$$(x, A, B) \to (x, A \circ \mathfrak{f}, B \circ (\mathfrak{f},\mathfrak{f})),$$

which is a diffeomorphism. To finish the proof, we need to show that the above defined ℓ is well defined.

Let $j^2\phi, j^2\phi' \in J^2_{2p}M$, and suppose that $\Lambda(j^2\phi) = \Lambda(j^2\phi')$. Then,

$$\begin{aligned} & \widetilde{\psi}(\Lambda(j^2\phi)) = \widetilde{\psi}(\Lambda(j^2\phi')) \\ \Rightarrow \quad & (\widetilde{\psi} \circ \Lambda)(j^2\phi) = (\widetilde{\psi} \circ \Lambda)(j^2\phi') \end{aligned}$$

By Diagram (7), we have

$$\begin{aligned}
(\hat{\Lambda}\circ\psi^2)(j^2\phi) &= (\hat{\Lambda}\circ\psi^2)(j^2\phi') \\
\Rightarrow (\phi(0,0), \frac{\partial\phi}{\partial w_\alpha}, \frac{\partial\phi}{\partial u_\alpha}, \frac{\partial^2\phi}{\partial u_\alpha \partial w_\beta}) &= (\phi'(0,0), \frac{\partial\phi'}{\partial w_\alpha}, \frac{\partial\phi'}{\partial u_\alpha}, \frac{\partial^2\phi'}{\partial u_\alpha \partial w_\beta}) \\
\Rightarrow (\phi(0,0), A_{\dot{\alpha}}, A_\alpha, B_{\alpha\dot{\beta}}) &= (\phi'(0,0), A'_{\dot{\alpha}}, A'_\alpha, B'_{\alpha\dot{\beta}}).
\end{aligned}$$

With the last equation, we can conclude that $B_{\alpha\dot{\beta}} = B'_{\alpha\dot{\beta}}$. Since the matrices B and B' are symmetric, then

$$B_{\alpha\dot{\beta}} = [B_{\dot{\alpha}\beta}]^T,$$

which implies

$$B_{\dot{\alpha}\beta} = B'_{\dot{\alpha}\beta}.$$

Now we focus on the function $\tilde{\ell}$. Using commutative Diagram (7), we have

$$\tilde{\psi}\circ\Lambda = \hat{\Lambda}\circ\psi^2.$$

Then

$$\begin{aligned}
(\tilde{\psi}\circ\Lambda\circ\tilde{\ell})(j^2\phi) &= (\tilde{\psi}\circ\Lambda)(j^2(\phi\circ\mathfrak{f})) \\
&= (\hat{\Lambda}\circ\psi^2)(j^2(\phi\circ\mathfrak{f})) \\
&= \hat{\Lambda}(\phi(0,0), \frac{\partial(\phi\circ\mathfrak{f})}{\partial\bar{u}_{\bar{\alpha}}}, \frac{\partial^2(\phi\circ\mathfrak{f})}{\partial\bar{u}_{\bar{\alpha}}\partial\bar{u}_{\bar{\beta}}}) \\
&= \hat{\Lambda}(x, [A_{\dot{\alpha}}, A_\alpha], \begin{pmatrix} B_{\dot{\alpha}\dot{\beta}} & B_{\alpha\dot{\beta}} \\ B_{\dot{\alpha}\beta} & B_{\alpha\beta} \end{pmatrix}) \\
&= (x, A_\alpha, A_{\dot{\alpha}}, B_{\dot{\alpha}\beta}) \\
&= (x', A'_\alpha, A'_{\dot{\alpha}}, B'_{\dot{\alpha}\beta}) \\
&= (\tilde{\psi}\circ\Lambda\circ\tilde{\ell})(j^2\phi').
\end{aligned}$$

On the other hand, because $\tilde{\psi}$ is injective, it follows immediately that

$$(\Lambda\circ\tilde{\ell})(j^2\phi) = (\Lambda\circ\tilde{\ell})(j^2\phi'). \tag{10}$$

Equation (10) implies that ℓ is a function on J^1_pM. Since Λ is a surjective submersion, and $\tilde{\ell}$ is a diffeomorphism, then ℓ is a differentiable function (see [3] and [7], 16.7.7,ii). Since $\tilde{\ell}$ is an involution, i.e. $\tilde{\ell}$ has its own inverse, then ℓ is also an involution, and the function ℓ is a diffeomorphism. □

In the following proposition, we define the aforementioned involution ℓ.

Proposition 10 *The involution $\ell : J^1_p(J^1_pM) \to J^1_p(J^1_pM)$ is defined by the following:*

$$\ell([y, Y_\alpha; X_\alpha, C_{\alpha\beta}]) = [y, X_\alpha; Y_\alpha, [C_{\alpha\beta}]^T].$$

Proof Suppose that $v=[y,Y_\alpha;X_\alpha,C_{\alpha\beta}]\in J^1_p(J^1_pM)$. Due to Λ being a surjective function, there exists $j^2\phi=[x,A_{\bar\alpha},B_{\bar\alpha\bar\beta}]\in J^2_{2p}M$ such that

$$A_{\bar\alpha}=[A_\alpha A_{\dot\alpha}],$$

$$B_{\bar\alpha\bar\beta}=\begin{pmatrix}B_{\alpha\beta} & B_{\dot\alpha\beta}\\ B_{\alpha\dot\beta} & B_{\dot\alpha\dot\beta}\end{pmatrix},$$

and

$$[y,Y_\alpha;X_\alpha,C_{\alpha\beta}]=\Lambda(j^2\phi)=\Lambda([x,A_{\bar\alpha},B_{\bar\alpha\bar\beta}])=[x,A_{\dot\alpha},A_\alpha,B_{\alpha\dot\beta}].$$

Then,

$$x=y,\qquad Y_\alpha=A_{\dot\alpha},\qquad X_\alpha=A_\alpha,\qquad C_{\alpha\beta}=B_{\alpha\dot\beta}.$$

On the other hand, since the matrix $B_{\bar\alpha\bar\beta}$ is symmetric, then $[B_{\alpha\dot\beta}]^T=B_{\dot\alpha\beta}$. Then

$$\begin{aligned}\ell([y,Y_\alpha;X_\alpha,C_{\alpha\beta}])=(\ell\circ\Lambda)(j^2\phi)) &= (\Lambda\circ\tilde\ell)(j^2\phi)\\ &= [x,A_\alpha;A_{\dot\alpha},B_{\dot\alpha\beta}].\\ &= [y,X_\alpha;Y_\alpha,[B_{\alpha\dot\beta}]^T]\\ &= [y,X_\alpha;Y_\alpha,[C_{\alpha\beta}]^T],\end{aligned}$$

which finishes the proof. □

In the following proposition, we shall show that the primary and secondary vector bundle structures on $J^1_p(J^1_pM)$ are isomorphic.

Proposition 11 *For any smooth manifold M, the function*

$$\ell: J^1_p(J^1_pM)\to J^1_p(J^1_pM),$$

which is defined in Theorem 9, is a J^1_pM-isomorphism of vector bundles in the following way:

$$\begin{array}{ccc} J^1_p(J^1_pM) & \xrightarrow{\ \ell\ } & J^1_p(J^1_pM)\\ \Big\downarrow{\scriptstyle\tilde\pi} & & \Big\downarrow{\scriptstyle\pi^1_M}\\ J^1_pM & \xrightarrow{\ =\ } & J^1_pM \end{array} \tag{11}$$

Proof To prove the theorem, we only need to show that ℓ preserves fibers.

Suppose that $j^1\Phi \in \tilde{\pi}^{-1}([x, X_\alpha])$. Then from Remark 8, $j^1\Phi = [x, X_\alpha; Y_\alpha, C_{\alpha\beta}]$. Using Proposition 10, we have

$$\begin{aligned}(\pi^1 \circ \ell)(j^1\Phi) &= (\pi^1 \circ \ell)([x, X_\alpha; Y_\alpha, C_{\alpha\beta}]) \\ &= \pi^1([x, Y_\alpha; X_\alpha, [C_{\alpha\beta}]^T) \\ &= [x, X_\alpha] \\ &= \tilde{\pi}(j^1\Phi),\end{aligned}$$

which proves that ℓ preserves fibers. Thus, ℓ is a bundle isomorphism. □

Acknowledgments

The author would like to thank the Idaho State University Department of Mathematics and in particular Prof Dr Robert Fisher Jr for being such nice hosts during her visit in Fall 2015. She also acknowledges library support from ISU.

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A. Appendix

Below, we present some identities and statements that are used in this paper.

Lemma 12 *$(J^1_pM, \pi, M, L(\mathbb{R}^p, \mathbb{R}^m))$ is a vector bundle with the bundle projection $\pi_M : J^1_pM \to M$, which is defined by $\pi_M(j^1\phi) = \phi(0)$ for all $j^1\phi \in J^1_pM$, the above manifold structure on J^1_pM, and the intrinsic operations on $\pi_M^{-1}\{x\}$ defined by*

$$[x; X_\alpha] + [x; X'_\alpha] = [x; X_\alpha + X'_\alpha] \qquad \lambda \bullet [x; X_\alpha] = [x; \lambda X_\alpha] \tag{A.1}$$

for all $x \in M$.

Proof In [5], it is proven that J^1_pM is a bundle with the local trivialization

$$\Psi_M : J^1_pU \to U \times L(\mathbb{R}^p, \mathbb{R}^m),$$

where $J^1_pU = \{j^1\phi : \phi(0) \in U\}$. Showing that $(\Psi_M)_x : \pi_M^{-1}\{x\} \to L(\mathbb{R}^p, \mathbb{R}^m))$ is an isomorphism implies that $(J^1_pM, \pi, M, L(\mathbb{R}^p, \mathbb{R}^m)$ is a vector bundle. From Equation 1, we have

$$(\Psi_M)_x(j^1\phi) = \frac{\partial(x_i \circ \phi)}{\partial u_\alpha}|_0.$$

If we denote $j^1\phi = [x; X_\alpha]$ where $X_\alpha = \phi^\alpha_*(d/du)$, then

$$(\Psi_M)_x(j^1\phi) = \left(X_\alpha[x_i]\right).$$

Therefore, we have

$$\begin{aligned}(\Psi_M)_x([x; X_\alpha] + [x; X'_\alpha]) = (X_\alpha + X'_\alpha)[x_i] &= (X_\alpha[x_i] + X'_\alpha)[x_i] \\ &= (\Psi_M)_x([x; X_\alpha]) + (\Psi_M)_x([x; X'_\alpha]).\end{aligned}$$

On the other hand, we have

$$(\Psi_M)_x(\lambda \bullet [x; X_\alpha]) = (\lambda X_\alpha)[x_i] = \lambda(X_\alpha)[x_i] = \lambda(\Psi_M)_x([x; X_\alpha]),$$

which shows that $(\Psi_M)_x$ is a linear function. □

Lemma 13 *The function Λ defined in Lemma 9 is well defined.*

Proof Suppose that $j^2\phi = j^2\phi'$, where $j^2\phi, j^2\phi' \in J^2_{2p}(M)$. To prove the lemma, we should show that $j^1\Phi = j^1\Phi'$.

The equality of 2-jets implies the following:

$$\phi(0,0) = \phi'(0,0), \tag{A.2}$$

$$\frac{\partial \phi}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)} = \frac{\partial \phi'}{\partial \bar{u}_{\bar{\alpha}}}|_{(0,0)}, \tag{A.3}$$

$$\frac{\partial^2 \phi}{\partial \bar{u}_\alpha \bar{u}_\beta}|_{(0,0)} = \frac{\partial^2 \phi'}{\partial \bar{u}_\alpha \bar{u}_\beta}|_{(0,0)}. \tag{A.4}$$

Equation A.2 shows that

$$\phi_0(0) = \phi'_0(0). \tag{A.5}$$

Equation (A.3) leads to

$$\frac{\partial \phi}{\partial u_\alpha}|_{(0,0)} = \frac{\partial \phi'}{\partial u_\alpha}|_{(0,0)} \qquad \textit{and} \qquad \frac{\partial \phi}{\partial w_\alpha}|_{(0,0)} = \frac{\partial \phi'}{\partial w_\alpha}|_{(0,0)},$$

which implies

$$\Rightarrow \frac{\partial \phi(u,0)}{\partial u_\alpha}|_{u=0} = \frac{\partial \phi'(u,0)}{\partial u_\alpha}|_{u=0} \tag{A.6}$$

and

$$\Rightarrow \frac{\partial \phi(0,w)}{\partial w_\alpha}|_{w=0} = \frac{\partial \phi'(0,w)}{\partial w_\alpha}|_{w=0}. \tag{A.7}$$

Equation (A.7) leads to

$$\frac{\partial \phi_0}{\partial w_\alpha} = \frac{\partial \phi'_0}{\partial w_\alpha}. \tag{A.8}$$

Combining Equations (A.5) and (A.8), we have

$$\Phi(0) = j^1(\phi_0) = j^1(\phi'_0) = \Phi'(0).$$

On the other hand, by setting $\bar{u}_{\bar{\alpha}} = u_\alpha$ and $\bar{u}_{\bar{\beta}} = w_\beta$ in Equation A.4, we have

$$\frac{\partial^2 \phi}{\partial u_\alpha w_\beta}|_{(0,0)} = \frac{\partial^2 \phi'}{\partial u_\alpha w_\beta}|_{(0,0)}. \tag{A.9}$$

For all $u \in \mathbb{R}^p$, $j^1(\phi_u)$ can be associated to the ordered pair $[\phi_u(0), \frac{\partial \phi_u}{\partial w_\beta}]$. Then, combining Equations A.6 and A.9, we have

$$\begin{aligned}
\frac{\partial \Phi}{\partial u_\alpha}|_{u=0} &= (\frac{\partial \phi_u(0)}{\partial u_\alpha}, \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\beta}|_{(0,0)}) \\
&= (\frac{\partial \phi(u,0)}{\partial u_\alpha}, \frac{\partial^2 \phi}{\partial u_\alpha \partial w_\beta}|_{(0,0)}) \\
&= (\frac{\partial \phi'_u(0)}{\partial u_\alpha}, \frac{\partial^2 \phi'}{\partial u_\alpha \partial w_\beta}|_{(0,0)}) \\
&= \frac{\partial \Phi'}{\partial u_\alpha}|_{u=0},
\end{aligned} \tag{A.10}$$

which shows that $j^1\Phi = j^1\Phi'$. Thus, Λ is well defined. $\square$

Lemma 14 *Let $\phi \in C^\infty(\mathbb{R}^{2p}, M)$, and let u_α, w_α denote the first p coordinate functions and the second p coordinate functions respectively on $\mathbb{R}^{2p}$. Let $\mathfrak{f}$ be the flip map defined in Lemma 9. Then there exist the following equations:*

$$\frac{\partial(\phi\circ\mathfrak{f})}{\partial u_\alpha}|_0 = \frac{\partial\phi}{\partial w_\alpha}|_0 \qquad \frac{\partial(\phi\circ\mathfrak{f})}{\partial w_\alpha}|_0 = \frac{\partial\phi}{\partial u_\alpha}|_0 \tag{A.11}$$

$$\frac{\partial^2(\phi\circ\mathfrak{f})}{\partial u_\beta\partial u_\alpha}|_0 = \frac{\partial^2\phi}{\partial w_\beta\partial w_\alpha}|_0 \tag{A.12}$$

$$\frac{\partial^2(\phi\circ\mathfrak{f})}{\partial w_\beta\partial u_\alpha}|_0 = \frac{\partial^2\phi}{\partial u_\beta\partial w_\alpha}|_0 \tag{A.13}$$

$$\frac{\partial^2(\phi\circ\mathfrak{f})}{\partial w_\beta w_\alpha}|_0 = \frac{\partial\phi}{\partial u_\beta u_\alpha}|_0 \tag{A.14}$$

Proof Since $\mathfrak{f}(u,w) = (w,u)$, then

$$u_\alpha\circ\mathfrak{f} = w_\alpha, \qquad w_\alpha\circ\mathfrak{f} = u_\alpha. \tag{A.15}$$

Due to the Chern rule, we have

$$\frac{\partial(\phi\circ\mathfrak{f})}{\partial u_{\bar\alpha}} = (\frac{\partial\phi}{\partial u_{\bar\beta}}\circ\mathfrak{f}).\frac{\partial(u_{\bar\beta}\circ\mathfrak{f})}{\partial u_{\bar\alpha}}. \tag{A.16}$$

Combining A.15 and A.16 at point 0, we obtain Equation A.11.

On the other hand, taking the partial derivatives of the two sides of Equation A.16 at point 0 and using the Chern rule gives Equations A.12, A.13, and A.14. $\square$

Corollary 15 *Entries of the matrices $\bar{A}$ and $\bar{B}$ in Lemma 9 can be given as follows:*

$$\bar{A}_\alpha = A_{\dot\alpha} \qquad \bar{A}_{\dot\alpha} = A_\alpha$$

$$\bar{B}_{\alpha\beta} = B_{\dot\alpha\dot\beta} \qquad \bar{B}_{\alpha\dot\beta} = B_{\dot\alpha\beta} \qquad \bar{B}_{\dot\alpha\beta} = B_{\alpha\dot\beta} \qquad \bar{B}_{\dot\alpha\dot\beta} = B_{\alpha\beta}$$

Proof Using the definition of matrix $\bar{B}$ and Equations (A.12), (A.13), and (A.14), we have

$$[\bar{B}_{\alpha\beta}] = \frac{\partial^2(\phi\circ\mathfrak{f})}{\partial u_\alpha\partial u_\beta}|_0 = \frac{\partial^2\phi}{\partial w_\alpha\partial w_\alpha}|_0 = B_{\dot\alpha\dot\beta} \qquad [\bar{B}_{\dot\alpha\beta}] = \frac{\partial^2(\phi\circ\mathfrak{f})}{\partial w_\alpha\partial u_\beta}|_0 = \frac{\partial^2\phi}{\partial u_\alpha\partial w_\alpha}|_0 = B_{\alpha\dot\beta}$$

and

$$[\bar{B}_{\alpha\dot\beta}] = \frac{\partial^2(\phi\circ\mathfrak{f})}{\partial u_\alpha\partial w_\beta}|_0 = \frac{\partial^2\phi}{\partial w_\alpha\partial u_\beta}|_0 = B_{\alpha\dot\beta} \qquad [\bar{B}_{\dot\alpha\dot\beta}] = \frac{\partial^2(\phi\circ\mathfrak{f})}{\partial w_\alpha\partial w_\beta}|_0 = \frac{\partial^2\phi}{\partial u_\alpha\partial u_\beta}|_0 = B_{\alpha\beta}.$$

$\square$

Lemma 16 *The local form of the function* Ω *is*

$$\hat{\Omega}(u,f,y,g) = (u,y,f,g).$$

Proof For all $(u,f,y,g) \in \varphi_2((\pi_{\mathbb{E}})^{-1}(J^1_pU))$, then there exists a unique $j^1\Phi \in (\pi_{\mathbb{E}})^{-1}(J^1_pU)$ such that $\varphi_2(j^1\Phi) = (u,f,y,g)$. Thus,

$$\begin{aligned}
& (u,f,y,g) \\
= \; & ((\varphi \times id_{L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k)}) \circ (\Psi_M \times \Psi_E) \circ \psi^1)(j^1\Phi) \\
= \; & ((\varphi \times id_{L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k)}) \circ (\Psi_M \times \Psi_E))(j^1(\psi\circ\Phi)) \\
= \; & ((\varphi \times id_{L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k)}) \circ (\Psi_M \times \Psi_E))(j^1(\pi_{\mathbb{E}}\circ\Phi), j^1(pr_2\circ\psi\circ\Phi)) \\
= \; & (\varphi \times id_{L(\mathbb{R}^p,\mathbb{R}^m)\times E\times L(\mathbb{R}^p,\mathbb{R}^k)})((\pi_{\mathbb{E}}\circ\Phi)(0), (\frac{\partial(\pi_{\mathbb{E}}\circ\Phi)}{\partial u_\alpha}|_0, (pr_2\circ\psi\circ\Phi)(0), \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0)) \\
= \; & ((\varphi\circ\pi_{\mathbb{E}}\circ\Phi)(0), \frac{\partial(\pi_{\mathbb{E}}\circ\Phi)}{\partial u_\alpha}|_0, (pr_2\circ\psi\circ\Phi)(0), \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0).
\end{aligned}$$

Therefore, there exist the following equations:

$$u = (\varphi\circ\pi_{\mathbb{E}}\circ\Phi)(0), \qquad f = \frac{\partial(\pi_{\mathbb{E}}\circ\Phi)}{\partial u_\alpha}|_0, \qquad y = (pr_2\circ\psi\circ\Phi)(0), \qquad g = \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0.$$

On the other hand, $\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}|_0 : \mathbb{R}^p \to \mathbb{R}^{m+k}$. The derivative can be written of form

$$\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}|_0 = [\frac{\partial(pr_1\circ\psi\circ\Phi)}{\partial u_\alpha}|_0 \qquad \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0].$$

Thus,

$$\xi(\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}|_0) = (\frac{\partial(pr_1\circ\psi\circ\Phi)}{\partial u_\alpha}|_0, \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0).$$

Using this equation, we have

$$\begin{aligned}
& ((\varphi\times id)\circ(\psi\times\xi)\circ\tilde{\psi})(j^1\Phi) = ((\varphi\times id)\circ(\psi\times\xi))(\Phi(0), \frac{\partial(\psi\circ\Phi)}{\partial u_\alpha}) \\
= \; & (\varphi\times id)((pr_1\circ\psi\circ\Phi)(0), (pr_2\circ\psi\circ\Phi)(0), \xi(\frac{\partial(\psi\circ\Phi)}{\partial u_\alpha})) \\
= \; & ((\varphi\circ\pi_{\mathbb{E}}\circ\Phi)(0), (pr_2\circ\psi\circ\Phi)(0), (\frac{\partial(pr_1\circ\psi\circ\Phi)}{\partial u_\alpha}|_0, \frac{\partial(pr_2\circ\psi\circ\Phi)}{\partial u_\alpha}|_0) \\
= \; & (u,y,f,g).
\end{aligned}$$

This proves that $\hat{\Omega}(u,f,y,g) = (u,y,f,g)$. □