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Generalized $*$ -Lie ideal of $*$ -prime ring

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Abstract: Let R be a $*$ -prime ring with characteristic not 2, $\sigma, \tau : R \rightarrow R$ be two automorphisms, U be a nonzero $*(\sigma, \tau)$ -Lie ideal of R such that τ commutes with $*$, and a, b be in R . (i) If $a \in S_*(R)$ and $[U, a] = 0$, then $a \in Z(R)$ or $U \subset Z(R)$. (ii) If $a \in S_*(R)$ and $[U, a]_{\sigma, \tau} \subset C_{\sigma, \tau}$, then $a \in Z(R)$ or $U \subset Z(R)$. (iii) If $U \not\subset Z(R)$ and $U \not\subset C_{\sigma, \tau}$, then there exists a nonzero $*$ -ideal M of R such that $[R, M]_{\sigma, \tau} \subset U$ but $[R, M]_{\sigma, \tau} \not\subset C_{\sigma, \tau}$. (iv) Let $U \not\subset Z(R)$ and $U \not\subset C_{\sigma, \tau}$. If $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.

Key words: $*$ -prime ring, $*(\sigma, \tau)$ -Lie ideal, (σ, τ) -derivation, derivation

1. Introduction

Let R be an associative ring with the center $Z(R)$. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. An involution $*$ of a ring R is an additive mapping satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R equipped with an involution $*$ is said to be $*$ -prime if $aRb = a^*Rb = 0$ or $aRb = aRb^* = 0$ implies that $a = 0$ or $b = 0$. R is said to be 2-torsion-free if whenever $2x = 0$ with $x \in R$ then $x = 0$. $S_*(R)$ will denote the set of symmetric and skew symmetric elements of R , i.e. $S_*(R) = \{x \in R \mid x^* = \pm x\}$. An ideal I of R is said to be a $*$ -ideal if I is invariant under $*$, i.e. $I^* = I$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. An additive mapping $h : R \rightarrow R$ is called a derivation if $h(xy) = h(x)y + xh(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \rightarrow R$ is given by $I_a(x) = [a, x]$ for $x \in R$ is a derivation, which is said to be an inner derivation determined by a . Let σ, τ be two mappings on R . Set $C_{\sigma, \tau} = \{c \in R \mid c\sigma(r) = \tau(r)c \text{ for all } r \in R\}$ and it is known as the (σ, τ) -center of R . In particular, $C_{1,1} = Z(R)$ is the center of R where $1 : R \rightarrow R$ is the identity map. As usual, the (σ, τ) -commutator $x\sigma(y) - \tau(y)x$ will be denoted by $[x, y]_{\sigma, \tau}$. An additive mapping $d : R \rightarrow R$ is called an (σ, τ) -derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $d_a : R \rightarrow R$ is given by $d_a(x) = [a, x]_{\sigma, \tau}$ for $x \in R$ is called a (σ, τ) -inner derivation determined by a . The definition of a (σ, τ) -Lie ideal was given in [4] as follows: let U be an additive subgroup of R . Then: (i) U is a (σ, τ) -right Lie ideal of R if $[U, R]_{\sigma, \tau} \subset U$; (ii) U is a (σ, τ) -left Lie ideal of R if $[R, U]_{\sigma, \tau} \subset U$; (iii) if U is both a (σ, τ) -right Lie ideal and a (σ, τ) -left Lie ideal of R then U is a (σ, τ) -Lie ideal of R . A (σ, τ) -Lie ideal of R is said to be a $*(\sigma, \tau)$ -Lie ideal if U is invariant under $*$, i.e. $U^* = U$. Every $*$ -Lie ideal of R is a $*(1, 1)$ -Lie ideal of R where $1 : R \rightarrow R$ is the identity map but every $*(\sigma, \tau)$ -Lie ideal of R is in general not a $*$ -Lie ideal of R .

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Example. As an example, set $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$. We define a map $*$: $R \rightarrow R$ as follows:
 $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^* = \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}$. Let $\sigma \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$, $\tau \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}$ be two endomorphisms of R . It is easy to check that $U = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{Z} \right\}$ is a $*$ - (σ, τ) -Lie ideal of R but not a $*$ -Lie ideal of R .

In [3], Bergen et al. proved the following results for a nonzero Lie ideal U such that $U \not\subset Z(R)$ of a prime ring R with characteristic not 2: (i) there exists a nonzero ideal M of R such that $[M, R] \subset U$ but $[M, R] \not\subset Z(R)$; (ii) if $a, b \in R$ such that $aUb = 0$, then $a = 0$ or $b = 0$. In [2], Aydın and Kandamar generalized these results for a nonzero (σ, τ) -Lie ideal that is not included in $Z(R)$ and $C_{\sigma, \tau}$ of a prime ring. Oukhtite and Salhi [7] generalized these results, which were proved in [3] for a nonzero $*$ -Lie ideal U such that $[U, U] \neq 0$ of a 2-torsion-free $*$ -prime ring R . In this paper our main goal will be to extend the above results to a nonzero $*$ - (σ, τ) -Lie ideal that is not included in $Z(R)$ and $C_{\sigma, \tau}$ of a $*$ -prime ring with characteristic not 2.

Throughout the present paper R will be a $*$ -prime ring, $Z(R)$ will be the center of R , $\sigma, \tau : R \rightarrow R$ will be two automorphisms, $C_{\sigma, \tau}$ will be the (σ, τ) -center of R , and $S_*(R)$ will be the set of symmetric and skew symmetric elements of R .

We will use the following basic commutator identities:

- $[x, yz] = y[x, z] + [x, y]z,$
- $[xy, z] = x[y, z] + [x, z]y,$
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$
- $[xy, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y,$
- $[x, yz]_{\sigma, \tau} = \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z),$
- $\left[[x, y]_{\sigma, \tau}, z \right]_{\sigma, \tau} = \left[[x, z]_{\sigma, \tau}, y \right]_{\sigma, \tau} + [x, [y, z]]_{\sigma, \tau}.$

2. Results

For the proof of our results we need the following lemmas.

Lemma 2.1. [1, Lemma 5] Let R be a ring and U be a nonzero (σ, τ) -left Lie ideal of R and $T = \{c \in R \mid [R, c]_{\sigma, \tau} \subset U\}$. Then the following hold:

- i) T is a subring of R .
- ii) If U is also a (σ, τ) -right Lie ideal of R then T is the largest Lie ideal of R such that $[R, T]_{\sigma, \tau} \subset U$ and $U \subset T$.

Lemma 2.2. [2, Lemma 4] Let R be a ring and U be a nonzero (σ, τ) -left Lie ideal of R . Then

$$R[T(U), \sigma(T(U))] \subset T(U) \text{ and } [T(U), \tau(T(U))]R \subset T(U).$$

Lemma 2.3. [5, Lemma 4] Let R be a σ -prime ring with characteristic not 2, d be a derivation of R satisfying $d\sigma = \pm\sigma d$, and I be a nonzero σ -ideal of R . If $d^2(I) = 0$ then $d = 0$.

Lemma 2.4. [6, Theorem 2.2] Let I be a nonzero σ -ideal of a σ -prime ring R . If a, b in R are such that $aIb = aI\sigma(b) = 0$ then $a = 0$ or $b = 0$.

Lemma 2.5. [8, 2.3. Lemma] Let I be a nonzero σ -ideal of a σ -prime ring R and $a \in R$. If $Ia = 0$ (or $aI = 0$) then $a = 0$.

Lemma 2.6. [8, 2.8. Theorem] Let R be a σ -prime ring with characteristic not 2, I be a nonzero σ -ideal of R , and d be a nonzero (α, β) -derivation of R such that β commutes with σ . If $d(I) \subset C_{\alpha, \beta}$ then R is commutative.

Lemma 2.7. [8, 2.9. Lemma] Let R be a σ -prime ring with characteristic not 2, I be a nonzero σ -ideal of R , d be a (α, β) -derivation of R such that β commutes with σ , and h be a derivation of R satisfying $h\sigma = \pm\sigma h$. If $dh(I) = 0$ and $h(I) \subset I$ then $d = 0$ or $h = 0$.

Lemma 2.8. Let U be a nonzero $*$ - (σ, τ) -left Lie ideal of R such that τ commutes with $*$. If $U \subset C_{\sigma, \tau}$ then $U \subset Z(R)$.

Proof For any $u \in U$, $r \in R$, we have $[r\sigma(u), u]_{\sigma, \tau} = [r, u]_{\sigma, \tau}\sigma(u) \in U$. From the hypothesis, it holds that $[[r, u]_{\sigma, \tau}\sigma(u), s]_{\sigma, \tau} = 0$ for all $u \in U$, $r, s \in R$. Expanding this equation and using the hypothesis

$$\begin{aligned} 0 &= [[r, u]_{\sigma, \tau}\sigma(u), s]_{\sigma, \tau} = [r, u]_{\sigma, \tau}\sigma([u, s]) + [[r, u]_{\sigma, \tau}, s]_{\sigma, \tau}\sigma(u) \\ &= [r, u]_{\sigma, \tau}\sigma([u, s]), \end{aligned}$$

so it implies that

$$[r, u]_{\sigma, \tau}\sigma([u, s]) = 0, \forall r, s \in R, u \in U.$$

In this equation, taking $\tau(r)k$ instead of r where $k \in R$, it follows that

$$\tau([r, u])R\sigma([u, s]) = 0, \forall r, s \in R, u \in U. \tag{2.1}$$

Assume that $u \in U \cap S_*(R)$. In (2.1), replacing r by r^* and using that $\tau^* = *\tau$, we get

$$\tau^*([r, u])R\sigma([u, s]) = 0, \forall r, s \in R.$$

Thus,

$$\tau([r, u])R\sigma([u, s]) = \tau^*([r, u])R\sigma([u, s]) = 0, \forall r, s \in R$$

is obtained. By the $*$ -primeness of R , we have

$$u \in Z(R), \forall u \in U \cap S_*(R).$$

Thus, it holds that

$$U \cap S_*(R) \subset Z(R). \quad (2.2)$$

Now, for all $u \in U$, we know that $u - u^* \in U \cap S_*(R)$. From (2.2), it implies $u - u^* \in Z(R)$. This means that $[u, r] = [u^*, r]$ for all $r \in R$. In (2.1), taking r^* instead of r , and using that $\tau^* = *\tau$ and $[u, r] = [u^*, r]$ for all $r \in R$, we get

$$\tau^*([r, u])R\sigma([u, s]) = 0, \quad \forall r, s \in R, u \in U.$$

By the $*$ -primeness of R , we have

$$u \in Z(R), \quad \forall u \in U,$$

which implies that

$$U \subset Z(R).$$

Lemma 2.9. *Let U be a nonzero $*$ - (σ, τ) -left Lie ideal of R such that τ commutes with $*$ and $a \in R$. If $Ua = 0$, then $a = 0$ or $U \subset Z(R)$.*

Proof Since U is a $*$ - (σ, τ) -left Lie ideal of R , we know that $[r, u]_{\sigma, \tau} a = 0$ for all $r \in R, u \in U$. Replacing r by rs where $s \in R$ in the last equality, we get $[rs, u]_{\sigma, \tau} a = 0$ for all $r, s \in R, u \in U$. Expanding this equation,

$$0 = [rs, u]_{\sigma, \tau} a = r[s, u]_{\sigma, \tau} a + [r, \tau(u)]sa,$$

and using the hypothesis, we have

$$[r, \tau(u)]Ra = 0, \quad \forall r \in R, u \in U.$$

In the last equation, taking r^*, u^* instead of r, u respectively and using that τ commutes with $*$, we get

$$([r, \tau(u)])^* Ra = 0, \quad \forall r \in R, u \in U.$$

Thus,

$$[r, \tau(u)]Ra = ([r, \tau(u)])^* Ra = 0, \quad \forall r \in R, u \in U$$

is obtained. From the $*$ -primeness of R , it yields

$$a = 0 \text{ or } [R, \tau(U)] = 0.$$

Since τ is an automorphism, we arrive at $a = 0$ or $U \subset Z(R)$.

Lemma 2.10. *Let U be a nonzero $*$ - (σ, τ) -left Lie ideal of R such that τ commutes with $*$. If $a \in R$ and $[U, a] = 0$, then $[\sigma(U), a] = 0$.*

Proof Since U is a $*$ - (σ, τ) -left Lie ideal of R , we know that $[r\sigma(u), u]_{\sigma, \tau} = [r, u]_{\sigma, \tau}\sigma(u) \in U$ for all $u \in U, r \in R$. From the hypothesis, it holds that

$$[[r, u]_{\sigma, \tau}\sigma(u), a] = 0.$$

Expanding this equation,

$$0 = [r, u]_{\sigma, \tau}[\sigma(u), a] + [[r, u]_{\sigma, \tau}, a]\sigma(u),$$

and using the hypothesis,

$$[r, u]_{\sigma, \tau} [\sigma(u), a] = 0, \forall u \in U, r \in R$$

is obtained. In the last equation, replacing r by $\tau(s)r$ where $s \in R$, it implies

$$\tau([s, u]) R [\sigma(u), a] = 0, \forall u \in U, s \in R. \tag{2.3}$$

Suppose that $u \in U \cap S_*(R)$. Taking s^* instead of s in (2.3) and using that τ commutes with $*$, it follows that

$$\tau^*([s, u]) R [\sigma(u), a] = 0, \forall s \in R, u \in U \cap S_*(R).$$

Thus,

$$\tau([s, u]) R [\sigma(u), a] = \tau^*([s, u]) R [\sigma(u), a] = 0, \forall s \in R, u \in U \cap S_*(R).$$

Since R is a $*$ -prime ring and τ is an automorphism, we obtain $u \in Z(R)$ or $[\sigma(u), a] = 0$ for all $u \in U \cap S_*(R)$. That is,

$$[\sigma(u), a] = 0, \forall u \in U \cap S_*(R). \tag{2.4}$$

Now, suppose that $u \in U$. In this case, we know $u - u^* \in U \cap S_*(R)$. It follows that $[\sigma(u - u^*), a] = 0$ from (2.4). Thus, we have

$$[\sigma(u), a] = [\sigma(u^*), a], \forall u \in U. \tag{2.5}$$

In (2.3), replacing s, u by s^*, u^* , respectively, and using that τ commutes with $*$ and (2.5), we get

$$\tau^*([s, u]) R [\sigma(u), a] = 0, \forall u \in U, s \in R.$$

Thus,

$$\tau([s, u]) R [\sigma(u), a] = \tau^*([s, u]) R [\sigma(u), a] = 0, \forall u \in U, s \in R.$$

Since R is a $*$ -prime ring and τ is an automorphism, it implies that $u \in Z(R)$ or $[\sigma(u), a] = 0$ for all $u \in U$. This means

$$[\sigma(U), a] = 0.$$

Theorem 2.11. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U, a] = 0$, then $a \in Z(R)$ or $U \subset Z(R)$.*

Proof Let

$$T(U) = \{c \in R \mid [R, c]_{\sigma, \tau} \subset U\}.$$

Since U is a (σ, τ) -Lie ideal of R , from Lemma 2.1, $T(U)$ is a Lie ideal of R such that $U \subset T(U)$. Hence, it follows that

$$[R, U] \subset [R, T(U)] \subset T(U).$$

From the definition of $T(U)$, we have

$$[R, [R, U]]_{\sigma, \tau} \subset [R, T(U)]_{\sigma, \tau} \subset U.$$

From the hypothesis, we have $[[R, [R, U]]_{\sigma, \tau}, a] = 0$. For any $r, s \in R$ and $u \in U$, it holds that

$$[[r, [s, u]]_{\sigma, \tau}, a] = 0. \tag{2.6}$$

Replacing r by ra in (2.6) and expanding by using (2.6),

$$\begin{aligned} 0 &= [ra, [s, u]]_{\sigma, \tau}, a = [r [a, \sigma ([s, u])], a] + [r, [s, u]]_{\sigma, \tau} a, a \\ &= r [[a, \sigma ([s, u])], a] + [r, a] [a, \sigma ([s, u])] + [r, [s, u]]_{\sigma, \tau} [a, a] \\ &\quad + [r, [s, u]]_{\sigma, \tau}, a a \\ &= r [[a, \sigma ([s, u])], a] + [r, a] [a, \sigma ([s, u])] \end{aligned}$$

is obtained. It follows that

$$r [[a, \sigma ([s, u])], a] + [r, a] [a, \sigma ([s, u])] = 0, \forall r, s \in R, u \in U. \tag{2.7}$$

In (2.7), taking rm instead of r where $m \in R$ and expanding by using (2.7),

$$\begin{aligned} 0 &= rm [[a, \sigma ([s, u])], a] + [rm, a] [a, \sigma ([s, u])] \\ &= rm [[a, \sigma ([s, u])], a] + r [m, a] [a, \sigma ([s, u])] + [r, a] m [a, \sigma ([s, u])] \\ &= r (m [[a, \sigma ([s, u])], a] + [m, a] [a, \sigma ([s, u])]) + [r, a] m [a, \sigma ([s, u])] \\ &= [r, a] m [a, \sigma ([s, u])] \end{aligned}$$

is obtained. This implies that

$$[r, a] R [a, \sigma ([s, u])] = 0, \forall r, s \in R, u \in U.$$

Replacing r by r^* and using $a \in S_*(R)$ in the last equation, we have

$$[r, a]^* R [a, \sigma ([s, u])] = 0, \forall r, s \in R, u \in U.$$

It holds that

$$[r, a] R [a, \sigma ([s, u])] = [r, a]^* R [a, \sigma ([s, u])] = 0, \forall r, s \in R, u \in U.$$

By the $*$ -primeness of R , we get

$$a \in Z(R) \text{ or } [a, \sigma ([s, u])] = 0, \forall s \in R, u \in U.$$

That is,

$$[\sigma ([s, u]), a] = 0, \forall s \in R, u \in U.$$

Since σ is an automorphism, it implies

$$[[R, \sigma (u)], a] = 0, \forall u \in U.$$

Thus, we have $[[r, \sigma (u)], a] = 0$ for all $r \in R, u \in U$. Using the identity $[[x, y], z] = [[x, z], y] + [x, [y, z]]$ for all $x, y, z \in R$,

$$0 = [[r, \sigma (u)], a] = [[r, a], \sigma (u)] + [r, [\sigma (u), a]]$$

is obtained. From Lemma 2.10, we know that $[\sigma (U), a] = 0$. It holds that

$$[[r, a], \sigma (u)] = 0, \forall r \in R, u \in U. \tag{2.8}$$

From (2.8), it implies

$$(I_{\sigma(u)}I_a)(R) = 0, \forall u \in U.$$

Since $a \in S_*(R)$, we know that $I_a * = \pm * I_a$. Hence, according to Lemma 2.7, we have

$$a \in Z(R) \text{ or } \sigma(u) \in Z(R), \forall u \in U,$$

which implies that

$$a \in Z(R) \text{ or } U \subset Z(R).$$

Corollary 2.12. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $U \subset S_*(R)$ and $[U, U] = 0$, then $U \subset Z(R)$.*

Theorem 2.13. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U, a]_{\sigma, \tau} = 0$, then $a \in Z(R)$ or $U \subset Z(R)$.*

Proof Since U is a $*$ - (σ, τ) -Lie ideal of R , we have $[r\sigma(u), u]_{\sigma, \tau} = [r, u]_{\sigma, \tau} \sigma(u) \in U$ for all $r \in R, u \in U$.

From the hypothesis, we have $\left[[r, u]_{\sigma, \tau} \sigma(u), a \right]_{\sigma, \tau} = 0$ for all $r \in R, u \in U$. Expanding this equation by using the hypothesis,

$$\begin{aligned} 0 &= \left[[r, u]_{\sigma, \tau} \sigma(u), a \right]_{\sigma, \tau} = \left[[r, u]_{\sigma, \tau}, a \right]_{\sigma, \tau} \sigma(u) + [r, u]_{\sigma, \tau} \sigma([u, a]) \\ &= [r, u]_{\sigma, \tau} \sigma([u, a]) \end{aligned}$$

is obtained. Thus, we have

$$[r, u]_{\sigma, \tau} \sigma([u, a]) = 0, \forall u \in U, r \in R.$$

In the last equality, replacing r by $\tau(s)r$ where $s \in R$, it implies that

$$\tau([s, u]) R \sigma([u, a]) = 0, \forall u \in U, s \in R. \tag{2.9}$$

Suppose that $u \in U \cap S_*(R)$. In (2.9), replacing s by s^* and using that τ commutes with $*$, we have

$$\tau^*([s, u]) R \sigma([u, a]) = 0, \forall s \in R, u \in U \cap S_*(R).$$

It follows that

$$\tau([s, u]) R \sigma([u, a]) = \tau^*([s, u]) R \sigma([u, a]) = 0, \forall s \in R, u \in U \cap S_*(R).$$

Since R is a $*$ -prime ring and σ, τ are automorphisms, we get $u \in Z(R)$ or $[u, a] = 0$ for all $u \in U \cap S_*(R)$. Hence, we get

$$[u, a] = 0, \forall u \in U \cap S_*(R).$$

Now, suppose that $u \in U$. In this case, we know that $u - u^* \in U \cap S_*(R)$. It follows that $[u - u^*, a] = 0$ from the above equation. Thus, we have

$$[u, a] = [u^*, a], \forall u \in U. \tag{2.10}$$

In (2.9), replacing s, u by s^*, u^* , respectively, and using that τ commutes with $*$ and (2.10), we get

$$\tau^*([s, u]) R \sigma([u, a]) = 0, \forall u \in U, s \in R.$$

Thus,

$$\tau^*([s, u])R\sigma([u, a]) = \tau([s, u])R\sigma([u, a]) = 0, \forall u \in U, s \in R$$

is obtained. Since R is a $*$ -prime ring and σ, τ are automorphisms, it implies that

$$u \in Z(R) \text{ or } [u, a] = 0, \forall u \in U.$$

That is,

$$[U, a] = 0.$$

Therefore, according to Theorem 2.11,

$$a \in Z(R) \text{ or } U \subset Z(R)$$

is obtained.

Corollary 2.14. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $U \subset S_*(R)$ and $[U, U]_{\sigma, \tau} = 0$, then $U \subset Z(R)$.*

Theorem 2.15. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $a \in S_*(R)$ and $[U, a]_{\sigma, \tau} \subset C_{\sigma, \tau}$, then $a \in Z(R)$ or $U \subset Z(R)$.*

Proof From the hypothesis, we know that $[u, a]_{\sigma, \tau}, r]_{\sigma, \tau} = 0$ for all $u \in U, r \in R$. Using the identity

$$[[x, y]_{\sigma, \tau}, z]_{\sigma, \tau} = [[x, z]_{\sigma, \tau}, y]_{\sigma, \tau} + [x, [y, z]_{\sigma, \tau}]_{\sigma, \tau} \text{ for all } x, y, z \in R, \text{ for any } r \in R \text{ and } u \in U$$

$$0 = [[u, a]_{\sigma, \tau}, r]_{\sigma, \tau} = [[u, r]_{\sigma, \tau}, a]_{\sigma, \tau} + [u, [a, r]]_{\sigma, \tau}$$

is obtained. Since $[u, r]_{\sigma, \tau}, a]_{\sigma, \tau} \in C_{\sigma, \tau}$, from the last equation, we have

$$[u, [a, r]]_{\sigma, \tau} \in C_{\sigma, \tau}, \forall u \in U, r \in R.$$

In this case, we get $[u, [a, r]]_{\sigma, \tau}, s]_{\sigma, \tau} = 0$ for all $s \in R$. Replacing r by ra in this equality, we have

$$[u, [a, ra]]_{\sigma, \tau}, s]_{\sigma, \tau} = 0 \text{ for all } r, s \in R \text{ and } u \in U. \text{ Expanding this equation,}$$

$$\begin{aligned} 0 &= [[u, [a, ra]]_{\sigma, \tau}, s]_{\sigma, \tau} \\ &= [\tau([a, r])[u, a]_{\sigma, \tau}, s]_{\sigma, \tau} + [[u, [a, r]]_{\sigma, \tau} \sigma(a), s]_{\sigma, \tau} \\ &= \tau([a, r]) [[u, a]_{\sigma, \tau}, s]_{\sigma, \tau} + \tau([a, r], s) [u, a]_{\sigma, \tau} + \\ &\quad [[u, [a, r]]_{\sigma, \tau}, s]_{\sigma, \tau} \sigma(a) + [u, [a, r]]_{\sigma, \tau} \sigma([a, s]) \end{aligned}$$

is obtained. Using that $[u, a]_{\sigma, \tau}, [u, [a, r]]_{\sigma, \tau} \in C_{\sigma, \tau}$, it holds that

$$\tau([a, r], s) [u, a]_{\sigma, \tau} + [u, [a, r]]_{\sigma, \tau} \sigma([a, s]) = 0, \forall u \in U, r, s \in R.$$

In the last equation, taking a instead of s , we have

$$\tau ([[a, r], a]) [u, a]_{\sigma, \tau} = 0, \forall u \in U, r \in R.$$

Since $[u, a]_{\sigma, \tau} \in C_{\sigma, \tau}$, it implies

$$\tau ([[a, r], a]) R [u, a]_{\sigma, \tau} = 0, \forall u \in U, r \in R.$$

In the above equality, replacing r by r^* and using that $a \in S_*(R)$ and τ commutes with $*$, we get

$$(\tau ([[a, r], a]))^* R [u, a]_{\sigma, \tau} = 0, \forall u \in U, r \in R.$$

Thus,

$$\tau ([[a, r], a]) R [u, a]_{\sigma, \tau} = (\tau ([[a, r], a]))^* R [u, a]_{\sigma, \tau} = 0, \forall u \in U, r \in R.$$

Since R is a $*$ -prime ring and τ is an automorphism,

$$[[r, a], a] = 0 \text{ or } [u, a]_{\sigma, \tau} = 0, \forall u \in U, r \in R \tag{2.11}$$

is obtained. From (2.11), it follows that

$$I_a^2(R) = 0 \text{ or } [U, a]_{\sigma, \tau} = 0.$$

Since $a \in S_*(R)$, we know that $I_a^* = \pm * I_a$. According to Lemma 2.3 and Theorem 2.13,

$$a \in Z(R) \text{ or } U \subset Z(R)$$

is obtained.

Corollary 2.16. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $U \subset S_*(R)$ and $[U, U]_{\sigma, \tau} \subset C_{\sigma, \tau}$, then $U \subset Z(R)$.*

Theorem 2.17. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$. If $U \not\subset Z(R)$ and $U \not\subset C_{\sigma, \tau}$, then there exists a nonzero $*$ -ideal M of R such that $[R, M]_{\sigma, \tau} \subset U$ but $[R, M]_{\sigma, \tau} \not\subset C_{\sigma, \tau}$.*

Proof It follows from Lemma 2.1 that $T(U) = \{c \in R \mid [R, c]_{\sigma, \tau} \subset U\}$ is a subring and Lie ideal of R . Moreover, $U \subset T(U)$. On the other hand, it follows from Lemma 2.2 that $[T(U), \tau(T(U))] R \subset T(U)$. Since $U \subset T(U)$, it implies that

$$[U, \tau(U)] R \subset [T(U), \tau(T(U))] R \subset T(U).$$

That is,

$$[U, \tau(U)] R \subset T(U).$$

Since $T(U)$ is a Lie ideal of R , it holds that $[R, [U, \tau(U)] R] \subset T(U)$. Thus, for any $s, r \in R, u, v \in U$

$$[s, [u, \tau(v)] r] = [u, \tau(v)] [s, r] + [s, [u, \tau(v)]] r \in T(U).$$

Using that $[U, \tau(U)]R \subset T(U)$, we have $[s, [u, \tau(v)]]r \in T(U)$ for all $u, v \in U, r, s \in R$. Expanding this equation,

$$[s, [u, \tau(v)]]r = s[u, \tau(v)]r - [u, \tau(v)]sr \in T(U)$$

and using that $[U, \tau(U)]R \subset T(U)$,

$$s[u, \tau(v)]r \in T(U), \forall u, v \in U, r, s \in R$$

is obtained. It holds that

$$R[U, \tau(U)]R \subset T(U). \tag{2.12}$$

Set $M = R[U, \tau(U)]R$. Thus, M is an ideal of R . Now, assume that $M = 0$. It follows that $R[U, \tau(U)]R = 0$. Since R is a $*$ -ideal of R , according to Lemma 2.5, it holds that

$$[U, \tau(U)] = 0.$$

On the other hand, we know that $\tau(U) \cap S_*(R) \subset S_*(R)$ and $\tau(U) \cap S_*(R) \subset \tau(U)$. It follows that $[U, \tau(U) \cap S_*(R)] \subset [U, \tau(U)] = 0$. Therefore, it implies

$$[U, \tau(U) \cap S_*(R)] = 0.$$

That is, we have $\tau(U) \cap S_*(R) \subset S_*(R)$ and $[U, \tau(U) \cap S_*(R)] = 0$. Since $U \not\subset Z(R)$, it holds from Theorem 2.11 that

$$\tau(U) \cap S_*(R) \subset Z(R).$$

Now, for any $u \in U$, we get $\tau(u - u^*), \tau(u + u^*) \in \tau(U) \cap S_*(R) \subset Z(R)$. It yields $2\tau(u) \in Z(R)$. Since the characteristic is not 2, we obtain $u \in Z(R)$ for all $u \in U$. This is a contradiction, which implies $M \neq 0$. Moreover, we have

$$M^* = (R[U, \tau(U)]R)^* = R[U, \tau(U)]R = M.$$

Thus, M is a nonzero $*$ -ideal of R . It also follows from (2.12) that $0 \neq M \subset T(U)$. From the definition of $T(U)$, we get

$$[R, M]_{\sigma, \tau} \subset [R, T(U)]_{\sigma, \tau} \subset U,$$

which implies that

$$[R, M]_{\sigma, \tau} \subset U.$$

Suppose that $[R, M]_{\sigma, \tau} \subset C_{\sigma, \tau}$. From the assumption, $d_r(M) \subset C_{\sigma, \tau}$ for all $r \in R$. According to Lemma 2.6, R is commutative. This is a contradiction, so we have

$$[R, M]_{\sigma, \tau} \not\subset C_{\sigma, \tau}.$$

Theorem 2.18. *Let R be a $*$ -prime ring with characteristic not 2, and let U be a nonzero $*$ - (σ, τ) -Lie ideal of R such that τ commutes with $*$ and $a, b \in R$. If $U \not\subset Z(R)$ and $U \not\subset C_{\sigma, \tau}$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Proof Since $U \not\subset Z(R)$ and $U \not\subset C_{\sigma,\tau}$, by Theorem 2.17, there exists a nonzero $*$ -ideal M of R such that $[R, M]_{\sigma,\tau} \subset U$ but $[R, M]_{\sigma,\tau} \not\subset C_{\sigma,\tau}$. Since $[R, M]_{\sigma,\tau} \subset U$, using the hypothesis, we have $a[R, M]_{\sigma,\tau} b = a^*[R, M]_{\sigma,\tau} b = 0$. For any $m \in M$ and $u \in U$, it implies $a[a^*u, m]_{\sigma,\tau} b = 0$. Expanding this equation,

$$0 = a[a^*u, m]_{\sigma,\tau} b = aa^*[u, m]_{\sigma,\tau} b + a[a^*, \tau(m)] ub,$$

and using the hypothesis,

$$a[a^*, \tau(m)] ub = 0$$

is obtained. Expanding this by using the hypothesis, it yields

$$\begin{aligned} 0 &= aa^*\tau(m) ub - a\tau(m) a^*ub \\ &= aa^*\tau(m) ub. \end{aligned}$$

That is,

$$aa^*\tau(M) ub = 0, \forall u \in U.$$

Since $(aa^*)^* = aa^*$, it also holds that

$$aa^*\tau(M) ub = (aa^*)^*\tau(M) ub = 0, \forall u \in U.$$

Since M is a nonzero $*$ -ideal of R and τ commutes with $*$, $\tau(M)$ is a nonzero $*$ -ideal of R . By Lemma 2.4 and Lemma 2.9, it follows that

$$aa^* = 0 \text{ or } b = 0.$$

Assume that $b \neq 0$. In this case $aa^* = 0$. Since $[R, M]_{\sigma,\tau} \subset U$, we get $a[a^*n, m]_{\sigma,\tau} b = 0$ for all $n, m \in M$. Expanding this equation by using the assumption,

$$\begin{aligned} 0 &= a[a^*n, m]_{\sigma,\tau} b = aa^*n\sigma(m) b - a\tau(m) a^*nb \\ &= a\tau(m) a^*nb \end{aligned}$$

is obtained, so it holds that

$$a\tau(M) a^*Mb = 0.$$

Since $(a\tau(M) a^*)^* = a\tau(M) a^*$, we have

$$a\tau(M) a^*Mb = (a\tau(M) a^*)^* Mb = 0.$$

According to Lemma 2.4, since $b \neq 0$, it follows that

$$a\tau(M) a^* = 0. \tag{2.13}$$

On the other hand, since $[R, M]_{\sigma,\tau} \subset U$, we have $a[au, m]_{\sigma,\tau} b = 0$ for all $u \in U$, $m \in M$. Expanding this equation by using hypothesis, we have

$$\begin{aligned} 0 &= a[au, m]_{\sigma,\tau} b = aa[u, m]_{\sigma,\tau} b + a[a, \tau(m)] ub \\ &= a[a, \tau(m)] ub \\ &= a^2\tau(m) ub - a\tau(m) aub \\ &= a^2\tau(m) ub. \end{aligned}$$

That is,

$$a^2\tau(M)ub = 0, \forall u \in U.$$

Similarly, $a^*[a^*u, m]_{\sigma, \tau}b = 0$ for all $u \in U, m \in M$. Expanding this equation by using the hypothesis, we get

$$(a^2)^*\tau(M)ub = 0, \forall u \in U.$$

This implies that

$$a^2\tau(M)ub = (a^2)^*\tau(M)ub = 0, \forall u \in U.$$

Since $b \neq 0$, by Lemma 2.4 and Lemma 2.9, we get

$$a^2 = 0.$$

Moreover, for any $n, m \in M$, we have $a[an, m]_{\sigma, \tau}b = 0$ from the hypothesis. Expanding this equation by using that $a^2 = 0$,

$$\begin{aligned} 0 &= a[an, m]_{\sigma, \tau}b = a^2n\sigma(m)b - a\tau(m)anb \\ &= a\tau(m)anb \end{aligned}$$

is obtained. Thus, we have

$$a\tau(M)aMb = 0.$$

Similarly, we know that $a^*[a^*n, m]_{\sigma, \tau}b = 0$ for all $n, m \in M$. Using that $a^2 = 0$, we get

$$a^*\tau(M)a^*Mb = 0.$$

Since $(a\tau(M)a)^* = a^*\tau(M)a^*$, it holds that

$$(a\tau(M)a)^*Mb = 0.$$

Thus,

$$a\tau(M)aMb = (a\tau(M)a)^*Mb = 0$$

is obtained. By Lemma 2.4, we have

$$a\tau(M)a = 0. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$a\tau(M)a^* = a\tau(M)a = 0.$$

According to Lemma 2.4, it is implied that $a = 0$.

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