Turkish Journal of Mathematics

Volume 41 | Number 5 | Article 4

1-1-2017

Some results on uniform statistical cluster points

TUĞBA YURDAKADİM

LEILA MILLER-VAN WIEREN

Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2017) 41: 1133 – 1139
© TÜBİTAK
doi:10.3906/mat-1607-21

Some results on uniform statistical cluster points

Tuğba YURDAKADİM[1,*], **Leila MILLER-VAN-WIEREN**[2]
[1]Department of Mathematics, Hitit University, Çorum, Turkey
[2]Faculty of Engineering and Natural Sciences, International University of Sarajevo, Sarajevo, Bosnia and Herzegovina

Received: 11.07.2016 • **Accepted/Published Online:** 25.10.2016 • **Final Version:** 28.09.2017

Abstract: In this paper, we present some results linking the uniform statistical limit superior and inferior, almost convergence and uniform statistical convergence of a sequence. We also study the relationship between the set of uniform statistical cluster points of a given sequence and its subsequences. The results concerning uniform statistical convergence and uniform statistical cluster points presented here are also closely related to earlier results regarding statistical convergence and statistical cluster points of a sequence.

Key words: Uniform statistical convergence, subsequences, uniform statistical cluster points

1. Introduction

The convergence of sequences has undergone numerous generalizations in order to provide deeper insights into summability theory. Convergence of sequences has different generalizations. One of the most important generalizations is uniform statistical convergence. This type of convergence has been introduced by Brown and Freedman [3] by using uniform density and has been studied by many authors in various directions [2, 14, 15, 19, 20]. This type of convergence is stronger than ordinary convergence and so it is quite effective, especially when the classical limit does not exist.

Buck [5] initiated the study of the relationship between the convergence of a given sequence and the summability of its subsequences. Later Agnew [1], Buck [6], Buck and Pollard [7], Miller and Orhan [18], and Zeager [22] studied this relation changing the concept of convergence. Moreover, Dawson [8] and Fridy [11] have studied analogous results by replacing subsequences with stretching and rearrangements, respectively. In [21], we studied some relationships between convergence and uniform statistical convergence of a given sequence and its subsequences. The related notions of statistical limit superior and inferior and statistical cluster points have been studied in recent papers including [12, 13] by Fridy and Orhan and [16] by Miller and Miller-Van Wieren.

In the present paper, we are concerned with the relationships between the uniform statistical limit superior and inferior, almost convergence and uniform statistical convergence of a sequence. We also show some results about the set of uniform statistical cluster points of a given sequence and its subsequences and stretchings, including the discussion of the Lebesgue measure of the set of subsequences that retain the same set of uniform statistical cluster points.

Now let us recall some known notions. Let $K \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. If $m, n \in \mathbb{N}$,

*Correspondence: tugbayurdakadim@hotmail.com
2010 *AMS Mathematics Subject Classification:* 40G99, 28A12.

by $K(m,n)$ we denote the cardinality of the set of numbers i in K such that $m \leq i \leq n$. Numbers

$$\underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}, \ \bar{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set K, respectively. If $\underline{d}(K) = \bar{d}(K)$ then it is said that $d(K) = \underline{d}(K) = \bar{d}(K)$ is the asymptotic density of K. The uniform density of $K \subseteq \mathbb{N}$ has been introduced in [3, 4] as follows:

$$\bar{u}(K) = \lim_{n\to\infty} \frac{\min\limits_{i\geq 0} K(i+1, i+n)}{n}, \bar{u}(K) = \lim_{n\to\infty} \frac{\max\limits_{i\geq 0} K(i+1, i+n)}{n}$$

are respectively called the lower and the upper uniform density of the set K (the existence of these bounds is also mentioned in [2]). If $\bar{u}(K) = \bar{u}(K)$, then $u(K) = \bar{u}(K)$ is called the uniform density of K. It is clear that for each $K \subseteq \mathbb{N}$ we have

$$\bar{u}(K) \leq \underline{d}(K) \leq \underline{d}(K) \leq \bar{u}(K).$$

The concept of statistical convergence has been introduced in [10] as follows: Let $x = \{x_n\}$ be a sequence of complex numbers. The sequence x is said to be statistically convergent to a complex number L provided that for every $\varepsilon > 0$ we have $d(K_\varepsilon) = 0$, where $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$. If $x = \{x_n\}$ converges statistically to L, then we write $st - \lim x = L$.

Next we introduce the concept of uniform statistical convergence, which is the primary topic of this paper. A generalized approach to convergence has been obtained by means of the notion of an ideal I of subsets of $\mathbb{N}$, i.e. I is an additive and hereditary class of sets. A sequence x is said to be I-convergent to L if for every ε the set $K_\varepsilon = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to I, and we write $I - limx = L$. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, then I_d-convergence coincides with statistical convergence. In the case $I = I_u = \{A \subseteq \mathbb{N} : u(A) = 0\}$ we obtain uniform statistical convergence to L or I_u convergence to L. Then we write $st_u - \lim x = L$.

In [18], Orhan and Miller have studied the concept of almost convergence of sequences and have obtained some results regarding subsequences. Namely a bounded sequence $x = \{x_n\}$ is almost convergent to L if

$$\lim_{n\to\infty} \frac{\sum_{i=m+1}^{m+n} x_i}{n} = L$$

uniformly in m (see [18]). Orhan and Miller [18] have shown that if x almost converges to L, then the set of $t \in (0,1]$ for which the associated subsequence (xt) almost converges to L must have measure 0 or 1 (both values may occur).

Definition 1 *γ is called a uniform statistical cluster point of $x = \{x_k\}$ if for every $\varepsilon > 0$ the set $\{k : |x_k - \gamma| < \varepsilon\}$ does not have uniform density* 0.

Let Γ_u denote the set of all uniform statistical cluster points of x and Γ_s denote the set of all statistical cluster points. Clearly $\Gamma_s \subseteq \Gamma_u$. It can happen that $\Gamma_s \subsetneqq \Gamma_u$, for example if

$$x = 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, ..., 0, 0, 0, 1, 1, 1, ...,$$

where segments of 0's of length 2^k, $k = 0, 1, 2, 3, ...$ and 1's of length $k+1$, $k = 0, 1, 2, 3, ...$ alternate. Then one can see that x is statistically convergent to 0 and so $\Gamma_s = \{0\}$ but $\Gamma_u = \{0, 1\}$.

In [12], Fridy and Orhan introduced the definitions of the statistical limit superior and statistical limit inferior of a sequence and proved some results concerning these notions. Here we proceed analogously for the case of uniform statistical convergence.

For a real sequence x, let A_x, B_x denote

$$B_x = \{b \in \mathbb{R} : \ u(\{k : x_k > b\}) \neq 0\},$$
$$A_x = \{a \in \mathbb{R} : \ u(\{k : x_k < a\}) \neq 0\}.$$

Definition 2 *For $x = \{x_k\}$ the uniform statistical limit superior and uniform statistical limit inferior are given by*

$$st_u - \limsup x := \begin{cases} \sup B_x & ; \ if \ B_x \neq \varnothing \\ -\infty & ; \ if \ B_x = \varnothing \end{cases},$$

$$st_u - \liminf x := \begin{cases} \inf A_x & ; \ if \ A_x \neq \varnothing \\ \infty & ; \ if \ A_x = \varnothing \end{cases}.$$

Theorem 1 *If $\beta = st_u - \limsup x$ is finite then for every positive ε*

$$u(\{k : x_k > \beta - \varepsilon\}) \neq 0 \ and \ u(\{k : x_k > \beta + \varepsilon\}) = 0. \tag{1.1}$$

Conversely if (1.1) holds for every positive ε then $\beta = st_u - \limsup x$.

Theorem 2 *If $\alpha = st_u - lim \inf x$ is finite then for every positive ε*

$$u(\{k : x_k < \alpha + \varepsilon\}) \neq 0 \ and \ u(\{k : x_k < \alpha - \varepsilon\}) = 0. \tag{1.2}$$

Conversely if (1.2) holds for every positive ε then $\alpha = st_u - \liminf x$.

Furthermore, it is easy to see that Γ_u is a closed set and if $\beta = st_u - \limsup x$ is finite then $\beta = \max \Gamma_u$ (and vice versa) and if $\alpha = st_u - \liminf x$ is finite then $\alpha = \min \Gamma_u$ (and vice versa).

In [9], the concepts of I-limit superior and inferior and I-core have been defined and studied in a different way.

Theorem 3 *x converges uniformly statistically to α if and only if $\alpha = st_u - \limsup x = st_u - \liminf x$.*

2. Results

Fridy and Orhan [12] proved the following theorem.

Theorem 4 *If the sequence x is bounded above and C_1 summable to $\beta = st - \limsup x$, then x is statistically convergent to β.*

Here we prove the following analogue.

Theorem 5 *If the sequence x is bounded and almost convergent to $\beta = st_u - \limsup x$, then x is uniformly statistically convergent to β.*

Proof Let us use the following notation:

$$K_{m+1}^{m+n} = K \cap \{m+1, m+2, ..., m+n\} \text{ for } K \subseteq \mathbb{N}.$$

Suppose x is not uniformly statistically convergent to β. Then $st_u - \liminf x < \beta$ and so there exists $\mu < \beta$ such that $u(\{k : x_k < \mu\}) \neq 0$. Let $K^{'} = \{k : x_k < \mu\}$. Since $u(K^{'}) \neq 0$ there exists $d > 0$ and infinitely many $n,\ n \to \infty$ such that for each n there exists $m = m(n)$ so that

$$\frac{\left|K_{m+1}^{'m+n}\right|}{n} = \frac{\left|\{k \in K^{'} : m+1 \leq k \leq m+n\}\right|}{n} \geq d.$$

Suppose ε is arbitrary fixed. Then $u(\{k : x_k > \beta + \varepsilon\}) = 0$. Define

$$K^{''} = \{k : \mu \leq x_k \leq \beta + \varepsilon\}\ , K^{'''} = \{k : x_k > \beta + \varepsilon\} \textit{ and let } B = \sup_k x_k < \infty.$$

Then for each previously mentioned $n, m = m(n)$ we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=m+1}^{m+n} x_k &= \frac{1}{n}\sum_{k\in K_{m+1}^{'m+n}} x_k + \frac{1}{n}\sum_{k\in K_{m+1}^{''m+n}} x_k + \frac{1}{n}\sum_{k\in K_{m+1}^{'''m+n}} x_k \\
&< \frac{\mu}{n}\left|K_{m+1}^{'m+n}\right| + \frac{\beta+\varepsilon}{n}\left|K_{m+1}^{''m+n}\right| + \frac{B}{n}\left|K_{m+1}^{'''m+n}\right| \\
&\leq \mu\frac{\left|K_{m+1}^{'m+n}\right|}{n} + (\beta+\varepsilon)(1 - \frac{\left|K_{m+1}^{'m+n}\right|}{n} - \frac{\left|K_{m+1}^{'''m+n}\right|}{n}) + \frac{B}{n}\left|K_{m+1}^{'''m+n}\right| \\
&\leq \mu\frac{\left|K_{m+1}^{'m+n}\right|}{n} + (\beta+\varepsilon)(1 - \frac{\left|K_{m+1}^{'m+n}\right|}{n}) + M\frac{\left|K_{m+1}^{'''m+n}\right|}{n},\ M \text{ some constant} \\
&\leq \mu\frac{\left|K_{m+1}^{'m+n}\right|}{n} + \beta(1 - \frac{\left|K_{m+1}^{'m+n}\right|}{n}) + \varepsilon(1-d) + M\frac{\left|K_{m+1}^{'''m+n}\right|}{n} \\
&\leq \beta + (\mu - \beta)\frac{\left|K_{m+1}^{'m+n}\right|}{n} + \varepsilon(1-d) + M\frac{\left|K_{m+1}^{'''m+n}\right|}{n} \\
&\leq \beta - (\beta - \mu)d + \varepsilon(1-d) + M\frac{\left|K_{m+1}^{'''m+n}\right|}{n}.
\end{aligned}$$

Since for any $\varepsilon > 0,\ M\frac{\left|K_{m+1}^{'''m+n}\right|}{n} \to 0$ as $n \to \infty$ and ε can be chosen arbitrarily small we get that for infinitely many $n,\ m = m(n)$

$$\frac{1}{n}\sum_{k=m+1}^{m+n} x_k \leq \beta - d(\beta - \mu) < \beta$$

and so x does not almost converge to β. □

Remark 1 *The assumption that x is bounded is necessary as of course almost convergent sequences must be bounded.*

Remark 2 *Clearly, the uniform statistical limit superior can be replaced by the uniform statistical limit inferior in the statement of the above theorem.*

In [16] the following theorem has been proved.

Theorem 6 *Suppose* $x = \{x_k\}$ *is bounded and has* L *as its set of limit points, and* $\Gamma \subseteq L$ *is closed and nonempty. Then there exists a subsequence* $\{x_{n_k}\}$ *of* x *such that* Γ *is the set of statistical cluster points of* $\{x_{n_k}\}$*.*

Using the same construction of subsequence as in that proof with $\Gamma \subseteq L$, Γ closed we get $\{x_{n_k}\}$ whose set of uniform statistical cluster points is Γ. Because the subsequence is constructed by taking "every other" repeatedly the asymptotic densities in the proof will be at the same time uniform densities. Thus we will have the following theorem.

Theorem 7 *If* $x = \{x_k\}$ *is bounded and has* L *as its set of limit points, and* $\Gamma \subseteq L$ *is closed and nonempty. Then there exists a subsequence* $\{x_{n_k}\}$ *of* x *such that* Γ *is the set of uniform statistical cluster points of* $\{x_{n_k}\}$*.*

However, the theorem about "stretching" from that paper cannot be generalized. Consider

$$x = 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, ..., 0, 0, 0, 1, 1, 1, ...,$$

where segments of 0's of length 2^k, $k = 0, 1, 2, 3, ...$ and 1's of length $k+1$, $k = 0, 1, 2, 3, ...$ alternate. There is no stretching of x with $\{0\}$ (or $\{1\}$) as its set of uniform statistical cluster points.

If $x = \{x_n\}$ is a sequence of reals, for $t \in (0, 1]$ written in binary representation with infinitely many ones, let (xt) denote the usual generated subsequence of x.

The following theorem has been proved in [17].

Theorem 8 *Given a bounded sequence* x *if* $\Gamma_s(x)$ *is the set of its statistical cluster points, the set* $\{t \in (0, 1] : \Gamma_s(xt) = \Gamma_s(x)\}$ *has measure* 1*.*

Let $\Gamma_u(x)$ denote the set of uniform statistical cluster points of x; likewise we can prove:

Theorem 9 *Suppose* x *is a bounded sequence of reals. Then* $T = \{t \in (0, 1] : \Gamma_u(xt) = \Gamma_u(x)\}$ *has measure* 0 *or* 1 *(both can occur).*

Proof First T is a tail set and so T has measure 0 or 1 or it is unmeasurable. First we will check T is measurable. For $a \in [\liminf x, \limsup x]$, let

$$\Gamma_u(a) = \{t \in (0, 1] : \text{ a is uniform statistical cluster point of } (xt)\}.$$

Then

$$T = \Big(\bigcap_{a \in \Gamma_u(x)} \Gamma_u(a)\Big) \cap \Big(\bigcap_{a \in [\liminf x, \limsup x] \setminus \Gamma_u(x)} (0, 1] \setminus \Gamma_u(a)\Big)$$

since (xt) has the same uniform statistical cluster points as x. Now $\Gamma_u(x)$ is closed and separable and so there exists $\{a_n\}$, $a_n \in \Gamma_u(x)$ dense in $\Gamma_u(x)$.

Easily

$$\bigcap_{a\in\Gamma_u(x)}\Gamma_u(a)=\bigcap_n\Gamma_u(a_n).$$

For any a

$$\Gamma_u(a)=\bigcap_k\bigcup_j\bigcup_N\bigcap_{n\geq N}\bigcup_m\left\{t\in(0,1]:\frac{\left|\{m+1\leq i\leq m+n:|(xt)_i-a|<\frac{1}{k}\}\right|}{n}>\frac{1}{j}\right\}$$

is measurable and so

$$\bigcap_{a\in\Gamma_u(x)}\Gamma_u(a)=\bigcap_n\Gamma_u(a_n) \tag{2.1}$$

is measurable. Looking at the set $[\liminf x,\limsup x]\setminus\Gamma_u(x)$, since $\Gamma_u(x)$ is closed, it can be represented as an union of countably many intervals $\{I_k\}$ that are mutually disjoint and open (or half open/closed). Further each $I_k=\bigcup_{i=1}^{\infty}J_{k_i}$ where J_{k_i} are closed intervals and $J_{k_1}\subseteq J_{k_2}\subseteq ....\subseteq J_{k_i}\subseteq ....\subseteq I_k$. It is not hard to show that for $t\in(0,1]$:

$$x\notin\Gamma_u(a),\forall a\in[\liminf x,\limsup x]\setminus\Gamma_u(x)\text{ if and only if }u(\{j:(xt)_j\in J_{k_i}\})=0,\ \forall k,\ \forall i.$$

Therefore

$$\bigcap_{a\in[\liminf x,\limsup x]\setminus\Gamma_u(x)}(0,1]\setminus\Gamma_u(a)=\bigcap_k\bigcap_i\{t\in(0,1]:u(\{j:(xt)_j\in J_{k_i}\})=0\}. \tag{2.2}$$

Now

$$\{t\in(0,1]:u(\{j:(xt)_j\in J_{k_i}\})=0\}=$$
$$\bigcap_l\bigcup_N\bigcap_{n\geq N}\bigcap_m\left\{t\in(0,1]:\frac{|\{m+1\leq j\leq m+n:(xt)_j\in J_{k_i}\}|}{n}<\frac{1}{l}\right\}$$

and so it is measurable and thus the set in (2.2) is measurable. From (2.1) and (2.2) finally we get that T is measurable.

Now T can have measure 1 since if x is a convergent sequence, then $T=[0,1)$ and so it has measure 1. Moreover, let x be given by:

$$\underbrace{0,1,0,1,\ldots,0,1}_{n_1};\underbrace{0,0,1,0,0,1,\ldots,0,0,1}_{n_2};\underbrace{0,0,0,1,0,0,0,1,0,0,0,1\ldots,0,0,0,1}_{n_3};\ldots$$

with $n_1,n_2,n_3,\ldots$ as in part b) of the proof of Theorem 2.3. in [18]. Then x is almost convergent and hence from Lemma 1 of [21] uniformly statistically convergent to 0 and from the proof of Theorem 1 of [21], the set

$$\{t\in(0,1]:(xt)\text{ converges uniformly statistically to }0\}$$

has measure 0. However, that set is the same as the set T, and so for this sequence $m(T)=0$. Therefore $m(T)=0$ or 1, and both occur. □

References

[1] Agnew RP. Summability of subsequences. Bull Amer Math Soc 1944; 50: 596-598.

[2] Balaz V, Šalát T. Uniform density u and corresponding I_u-convergence. Math Commun 2006; 11: 1-7.

[3] Brown TC, Freedman AR. Arithmetic progressions in Lacunary sets. Rocky Mountain J Math 1987; 17: 587-596.

[4] Brown TC, Freedman AR. The uniform density of sets of integers and Fermat's last theorem. C R Math Ref Acad Sci Canad 1990; 12: 1-6.

[5] Buck RC. A note on subsequences. Bull Amer Math Soc 1943; 49: 898-899.

[6] Buck RC. An addentum to "a note on subsequences". Proc Amer Math Soc 1956; 7: 1074-1075.

[7] Buck RC, Pollard H. Convergence and summability properties of subsequences. Bull Amer Math Soc 1943; 49: 924-931.

[8] Dawson DF. Summability of subsequences and strechings of sequences. Pacific J Math 1973; 44: 455-460.

[9] Demirci K. I-limit superior and inferior. Math Commun 2001; 6: 165-172.

[10] Fast H. Sur la convergence statistique. Colloq Math 1951; 2: 241-244.

[11] Fridy JA. Summability of rearrangements of sequences. Math Z 1975; 143: 187-192.

[12] Fridy JA, Orhan C. Statistical limit superior and limit inferior. Proc Amer Math Soc 1997; 125: 3625-3631.

[13] Fridy JA, Orhan C. Statistical core theorems. J Math Anal Appl 1997; 208: 520-527.

[14] Antonini RG, Grekos G. Weighted uniform densities. Journal de Théorie des Nombres de Bordeaux 2007; 19: 191-204.

[15] Kostyrko P, Šalát T. I-convergence. Real Analysis Exchange 2000/2001; 26: 669-686.

[16] Miller HI, Miller-Wan Wieren L. Some statistical cluster point theorems. Hacet J Math Stat 2015; 44: 1405-1409.

[17] Miller HI, Miller-Wan Wieren L. Statistical cluster point and statistical limit point sets of subsequences of a given sequence, submitted for publication.

[18] Miller HI, Orhan C. On almost convergence and statistically convergent subsequences. Acta Math Hungar 2001; 93: 135-151.

[19] Pehlivan S. Strongly almost convergent sequences defined by a modulus and uniformly statistical convergence. Soochow J Math 1994; 20: 205-211.

[20] Tas E, Yurdakadim T. Characterization of uniform statistical convergence for double sequences. Miskolc Math Notes 2012; 13: 543-553.

[21] Yurdakadim T, Miller-Wan Wieren L. Subsequential results on uniform statistical convergence. Sarajevo J Math 2016; 12: 1-9.

[22] Zeager J. Buck-type theorems for statistical convergence. Radovi Math 1999; 9: 59-69.