Turkish Journal of Mathematics

Volume 41 | Number 6 Article 18

1-1-2017

On Hölder continuity of approximate solution maps to vector equilibrium problems

LAM QUOC ANH

KIEN TRUNG NGUYEN

TRAN NGOC TAM

Recommended Citation

ANH, LAM QUOC; NGUYEN, KIEN TRUNG; and TAM, TRAN NGOC (2017) "On Hölder continuity of approximate solution maps to vector equilibrium problems," *Turkish Journal of Mathematics*: Vol. 41: No. 6, Article 18. https://doi.org/10.3906/mat-1507-9
Available at: https://dctubitak.researchcommons.org/math/vol41/iss6/18

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2017) 41: 1591 – 1607
© TÜBİTAK
doi:10.3906/mat-1507-9

On Hölder continuity of approximate solution maps to vector equilibrium problems

Lam QUOC ANH[1], **Kien TRUNG NGUYEN**[1], **Tran NGOC TAM**[2,3,*]
[1]Department of Mathematics, Teacher College, Can Tho University, Can Tho, Vietnam
[2]Division of Computational Mathematics and Engineering, Institute for Computational Science,
Ton DucThang University, Ho Chi Minh City, Vietnam
[3]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Received: 02.07.2015 • **Accepted/Published Online:** 07.02.2017 • **Final Version:** 23.11.2017

Abstract: In this article, we consider parametric vector equilibrium problems in normed spaces. Sufficient conditions for Hölder continuity of approximate solution mappings where they are set-valued are established. As applications of these results, the Hölder continuity of the approximate solution mappings for vector optimization problems and vector variational inequalities are derived at the end of the paper. Our results are new and include the existing ones in the literature.

Key words: Hölder continuity, approximate solutions, equilibrium problems, variational inequalities, optimization problems, fixed point problems

1. Introduction

The equilibrium problem [19] has been of great interest since it is the unified framework of many important problems in optimization theory and applications such as the optimization problem, the variational inequality problem, the fixed-point and coincidence problems, and the Nash equilibrium problem. To date, many papers have been devoted to the solution existence, an important issue focusing on the center of any mathematical theory, for the equilibrium problems and related problems [15, 16, 22–25, 28, 29]. The next important issue recently investigated that received growing attention from many researchers is the stability and sensitivity analyses of solution mappings. The stability and sensitivity analyses may be understood in two ways. The first is the semicontinuity, continuity (or Hausdorff continuity) of solution mappings [3, 5, 7, 10, 20, 26, 30, 31] and references therein. The second is the Hölder/Lipschitz continuity of solution mappings [1, 2, 4, 6, 8, 9, 13, 17, 18, 21, 32–34]. Observing that most of the works on the Hölder/Lispchitz continuity of the solutions maps imposed strong monotonicity/convexity properties in the data, the solution sets of the problems will be singleton in the neighborhood of the considered point. However, for a parametric equilibrium problem, in general, the solution mapping is set-valued but not single-valued. Thus, such results are not applicable. One of the most interesting attentions paid by researchers is to study the sufficient conditions for the Hölder/Lipschitz continuity to solution mappings when they are set-valued ones. There are some contributions to this field. In [33, 34], the authors impose an assumption involving the solutions sets. This assumption is hard to verify since when the

*Correspondence: tranngoctam@tdt.edu.vn
2010 *AMS Mathematics Subject Classification:* 49K40, 90C31, 91B50

stability and sensitivity analyses of the problems are studied, it is assumed that the solution sets are unknown. In addition, if the solutions are defined, one often checks directly its desired Hölder property instead of checking many similar Hölder properties of the problem data. In [11], the authors replaced assumptions related to strong monotonicity with convexity/concavity assumptions and they obtained the Hölder continuity for approximate solution mappings of scalar problems. Recently, [12, 14] obtained sufficient conditions for Hölder continuity in the sense of calmness, a term of the weaker Hölder continuity. In these cases, the solution set may be set-valued, except for the considered point. In this paper, we give the sufficient conditions for the approximate solution mappings of the parametric vector equilibrium problems being Hölder continuity when the solution mappings are not singleton. Our results are new and include the existing ones.

The layout of the remainder of the paper is as follows. Section 2 introduces the vector equilibrium problems and recalls some definitions used in the next sections. Next, in Section 3, we establish the sufficient conditions for the Hölder continuity of the solution mapping for the parametric vector equilibrium problem. Finally, Section 4 presents applications of the Hölder continuity of the approximate solution mappings of vector optimization and vector variational inequalities.

2. Preliminaries

Our notations are almost standard. We use $\|.\|$ for the norm in any normed space. $d(x, A)$ is the distance from x to subset A. For a normed space X, X^* is the topological dual space of X and $\mathbb{R}_+$ is the set of nonnegative real numbers. $B(x, r)$ denotes the closed ball of radius $r \geq 0$ and centered at x. $\text{int}A$ stands for the interior of a subset A. The diameter of A is $\text{diam}\, A = \sup_{x,z \in A} \|x - z\|$. For a set-valued map $G : X \rightrightarrows Y$, $\text{gr}G = \{(x, y) \in X \times Y : y \in G(x)\}$ is the graph of G. $L(X, Y)$ is the collection of all continuous linear mappings of X into Y.

Throughout this paper, if not explicitly stated otherwise, let X, Λ, M be normed spaces and $A \subseteq X$ be a nonempty subset. Let Y be a linear normed space and Y^* be the dual space of Y. $C \subset Y$ is a pointed closed convex cone with $\text{int}C \neq \emptyset$. The multifunction $K : \Lambda \rightrightarrows A$ has nonempty bounded convex values and $f : A \times A \times M \to Y$ is a vector-valued function. For $(\lambda, \mu) \in \Lambda \times M$, we consider the following parametric vector equilibrium problem.

(WEP): Find $\bar{x} \in K(\lambda)$ such that, for all $y \in K(\lambda)$,

$$f(\bar{x}, y, \mu) \notin -\text{int}C.$$

To provide the motivations of our problem setting, we consider some special cases of this problem.

(i) If $g : A \times M \to Y$ and $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$, then (WEP) reduces the following vector optimization problem.

(VOP): Find $\bar{x} \in K(\lambda)$ such that, for all $y \in K(\lambda)$, $g(y, \mu) - g(\bar{x}, \mu) \notin -\text{int}C$.

In the case of $Y = \mathbb{R}$, (VOP) is the scalar optimization problem (OP).

(OP): Find $\bar{x} \in K(\lambda)$ such that $g(\bar{x}, \mu) = \min_{y \in K(\lambda)} g(y, \mu)$.

(ii) If $\phi : A \times M \to L(X, Y)$ and set $f(x, y, \mu) = \langle \phi(x, \mu), y - x \rangle$, then (WEP) reduces the vector variational inequality as follows.

(VVI): Find $\bar{x} \in K(\lambda)$ such that, for all $y \in K(\lambda)$, $\langle \phi(\bar{x}, \mu), y - \bar{x} \rangle \notin -\text{int}C$.

If $Y = \mathbb{R}$, then (VVI) is the following variational inequality.

(VI): Find $\bar{x} \in K(\lambda)$ such that, for all $y \in K(\lambda)$, $\langle \phi(\bar{x}, \mu), y - \bar{x} \rangle \geq 0$.

(iii) When $X \equiv Y, M$ are Hilbert spaces, let $\varphi : A \times M \to A$. The fixed point problem is,

(FP): Find $\bar{x} \in A$ such that $\varphi(\bar{x}, \mu) = \bar{x}$. This problem is equivalent to the following special case of (WEP).

(WEP'): Find $\bar{x} \in A$ such that, for all $y \in A$, $\langle x - \varphi(x, \mu), y - x \rangle \notin -\text{int}C$.

Indeed, if $\bar{x}$ is a solution of (FP), then $\langle \bar{x} - \varphi(\bar{x}, \mu), y - \bar{x} \rangle = 0$ for all $y \in A$, and hence $\bar{x}$ solves (WEP'). Conversely, let $\bar{x}$ be a solution of (WEP'), i.e. $\langle \bar{x} - \varphi(\bar{x}, \mu), y - \bar{x} \rangle \notin -\text{int}C$ for all $y \in A$. Taking $y = \varphi(\bar{x}, \mu)$, we have $\langle \bar{x} - \varphi(\bar{x}, \mu), \varphi(\bar{x}, \mu) - \bar{x} \rangle \notin -\text{int}C$ and thus we must have equality, i.e. $\bar{x}$ is a solution of (FP).

For each $(\lambda, \mu) \in \Lambda \times M$, $\varepsilon \geq 0$ and $e \in \text{int}C$, we denote the ε-solution set of (WEP) corresponding to $(\varepsilon, \lambda, \mu)$ by

$$\Pi(\varepsilon, \lambda, \mu) = \{x \in K(\lambda) : f(x, y, \mu) + \varepsilon e \notin -\text{int}C, \forall y \in K(\lambda)\}.$$

Set $C^* = \{\xi \in Y^* : \xi(y) \geq 0, \forall y \in C\}$ as the dual cone of C. Let $e \in \text{int}C$ be given and $B_e^* = \{\xi \in C^* : \xi(e) = 1\}$ be a weak* compact base of C^*.

Lemma 2.1 (Xem [27]) *If Y is a real topological linear space and C is a convex cone with* $\text{int}C \neq \emptyset$*, then*

$$\text{int}C = \{y \in Y : \langle \xi, y \rangle > 0, \forall \xi \in C^* \setminus \{0\}\}.$$

For every $\xi \in B_e^*$, we denote

$$\Pi_\xi(\varepsilon, \lambda, \mu) = \{x \in K(\lambda) : \xi(f(x, y, \mu)) + \varepsilon \geq 0, \forall y \in K(\lambda)\},$$

which is the ξ-approximate solution set of (WEP).

Now we recall some notions that are needed in the sequel.

Definition 2.2 (a) A vector-valued function $f : X \to Y$ is termed $l.\alpha$-Hölder continuous around $x_0 \in X$, if there is a neighborhood V of x_0 such that, for all $x_1, x_2 \in V$,

$$f(x_1) \in f(x_2) + l\|x_1 - x_2\|^\alpha B(0, 1).$$

(b) A set-valued function $K : \Lambda \rightrightarrows A$ is said to be $l.\alpha$-Hölder continuous around $\lambda_0 \in \Lambda$ if there is a neighborhood U of λ_0 such that, for all $\lambda_1, \lambda_2 \in U$,

$$K(\lambda_1) \subseteq K(\lambda_2) + l\|\lambda_1 - \lambda_2\|^\alpha B(0, 1).$$

The Hölder continuity is called a Lipschitz continuity if $\alpha = 1$.

Definition 2.3 (a) A vector-valued function $f : X \to Y$ is said to be C-convex in convex set $A \subseteq X$ if for any $x_1, x_2 \in A$ and $t \in [0, 1]$,

$$tf(x_1) + (1 - t)f(x_2) \in f(tx_1 + (1 - t)x_2) + C.$$

(b) A vector-valued function $f: X \to Y$ is said to be C-convex-like in set $B \subseteq X$ if for any $x_1, x_2 \in B$ and $t \in [0,1]$, there is $x_3 \in B$ such that

$$tf(x_1) + (1-t)f(x_2) \in f(x_3) + C.$$

As usual, we say that f is C-concave (concave-like, respectively) if $-f$ is C-convex (convex-like, respectively).

Definition 2.4 A mapping $g: X \to L(X,Y)$ is called monotone in $A \subseteq X$ if, for all $x_1, x_2 \in A$,

$$\langle g(x_1) - g(x_2), x_1 - x_2 \rangle \in C.$$

Lemma 2.5 *If for each $\xi \in B_e^*, y \in A$ and $\mu \in M$, $f(\cdot, y, \mu)$ is C-convex (C-concave, respectively) in A then $\xi(f(\cdot, y, \lambda))$ is convex (concave, respectively) in A.*

Proof We omit the proof since it is trivial. □

The following lemma shows the relation between the ε-approximate solution set and ξ-efficient approximate solution sets of (WEP).

Lemma 2.6 *If for each $\lambda \in \Lambda$, $x \in K(\lambda)$ and $\mu \in M$, $f(x, \cdot, \mu)$ is C-convex-like on $K(\lambda)$ then*

$$\Pi(\varepsilon, \lambda, \mu) = \bigcup_{\xi \in B_e^*} \Pi_\xi(\varepsilon, \lambda, \mu).$$

Proof Pick any $x \in \bigcup_{\xi \in B_e^*} \Pi_\xi(\varepsilon, \lambda, \mu)$. Then there exists $\xi' \in B_e^*$ such that $x \in \Pi_{\xi'}(\varepsilon, \lambda, \mu)$. Hence, $\xi'(f(x,y,\mu)) + \varepsilon \geq 0$, for all $y \in K(\lambda)$. If $f(x,y,\mu) + \varepsilon e \in -\text{int}C$, then there is $z \in \text{int}C$ satisfying $z = -f(x,y,\mu) - \varepsilon e$, and thus $\xi'(z) = -\xi'(f(x,y,\lambda)) - \varepsilon \leq 0$. This contradicts Lemma 2.1. Consequently, $f(x,y,\mu) + \varepsilon e \notin -\text{int}C$, i.e. $x \in \Pi(\varepsilon, \lambda, \mu)$.

Conversely, take any $x \in \Pi(\varepsilon, \lambda, \mu)$. Then $x \in K(\lambda)$ and, for all $y \in K(\lambda)$, $f(x,y,\mu) + \varepsilon e \notin -\text{int}C$. Therefore,

$$(f(x, K(\lambda), \mu) + \varepsilon e) \cap (-\text{int}C) = \emptyset,$$

which implies

$$(f(x, K(\lambda), \mu) + C + \varepsilon e) \cap (-\text{int}C) = \emptyset. \tag{2.1}$$

For each $\lambda \in \Lambda$, $x \in K(\lambda)$, and $\mu \in M$, as $f(x, \cdot, \mu)$ is C-convex-like on $K(\lambda)$, one has $f(x, K(\lambda), \mu) + C + \varepsilon e$ is a convex subset of Y. Indeed, take any $z_1, z_2 \in f(x, K(\lambda), \mu) + C + \varepsilon e$ and $t \in [0,1]$. Then there are $y_1, y_2 \in K(\lambda)$ and $c_1, c_2 \in C$ satisfying $z_1 = f(x, y_1, \mu) + c_1 + \varepsilon e$ and $z_2 = f(x, y_2, \mu) + c_2 + \varepsilon e$. By the C-convex-likeness of f on $K(\lambda)$, there is $y_3 \in K(\lambda)$ such that

$$tf(x, y_1, \mu) + (1-t)f(x, y_2, \mu) \in f(x, y_3, \mu) + C.$$

Hence,

$$\begin{aligned} tz_1 + (1-t)z_2 &= t(f(x, y_1, \mu) + c_1 + \varepsilon e) + (1-t)(f(x, y_2, \mu) + c_2 + \varepsilon e) \\ &= (tf(x, y_1, \mu) + (1-t)f(x, y_2, \mu)) + (tc_1 + (1-t)c_2) + \varepsilon e \\ &\in f(x, y_3, \mu) + C + C + \varepsilon e \\ &\subset f(x, K(\lambda), \mu) + C + \varepsilon e. \end{aligned}$$

This means that $f(x, K(\lambda), \mu) + C + \varepsilon e$ is a convex set.

From (2.1), using the separation theorem of Eidelheit, there are a continuous linear function $\bar{\xi} \in Y^* \setminus \{0\}$ and a real number ν such that

$$\bar{\xi}(\hat{c}) < \nu \leq \bar{\xi}(z + c + \varepsilon e),$$

for all $\hat{c} \in -\text{int}C, z \in f(x, K(\lambda), \mu)$, and $c \in C$. Since C is a cone, $\bar{\xi}(\hat{c}) \leq 0, \forall \hat{c} \in -\text{int}C$. Thus, $\bar{\xi}(\bar{c}) \geq 0, \forall \bar{c} \in C$, i.e. $\bar{\xi} \in C^* \setminus \{0\}$. On the other hand, as $c \in C$ and $\hat{c} \in -\text{int}C$ can be chosen arbitrarily close to $0 \in Y$, the continuity of $\bar{\xi}$ gives $\bar{\xi}(z) + \bar{\xi}(\varepsilon e) \geq 0$. It follows from the fact that $e \in \text{int}C$ and $\bar{\xi} \in C^* \setminus \{0\}$, $\bar{\xi}(e) > 0$. By setting $\xi = \dfrac{\bar{\xi}}{\bar{\xi}(e)}$, we see that $\xi \in B_e^*$ and $\xi(z) + \varepsilon\xi(e) = \xi(z) + \varepsilon \geq 0, \forall z \in f(x, K(\lambda), \mu)$. This implies, for all $y \in K(\lambda)$, $\xi(f(x, y, \mu)) + \varepsilon \geq 0$, i.e., $x \in \Pi_\xi(\varepsilon, \lambda, \mu) \subset \bigcup_{\xi \in B_e^*} \Pi_\xi(\varepsilon, \lambda, \mu)$. This completes the proof. □

For $A, B \subseteq X$, the Hausdorff distance between A and B is defined by

$$H(A, B) = \max\{H^*(A, B), H^*(B, A)\},$$

where $H^*(A, B) = \sup\limits_{a \in A} d(a, B)$ and $d(x, A) = \inf\limits_{y \in A} d(x, y)$. Note that $H(\cdot, \cdot)$ may not be a metric in the space of the subsets of X, since it can take the value ∞.

3. Hölder continuity of approximate solutions mappings

In this section, we present the main results of the paper. First, we give the sufficient conditions for the Hölder continuity of the ξ-approximate solution mappings. Then we apply this result to establish the sufficient conditions for the Hölder property of the approximate solution mappings to equilibrium problems.

Theorem 3.1 *For* (WEP), *assume that for each* $\xi \in B_e^*$, *the* ξ*-approximate solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$. *Furthermore, assume that the following conditions hold.*

(i) *K is $l.\alpha$-Hölder continuous around λ_0, i.e. there exists a neighborhood U of λ_0 such that for all $\lambda_1, \lambda_2 \in U$,*

$$K(\lambda_1) \subseteq K(\lambda_2) + l\|\lambda_1 - \lambda_2\|^\alpha B(0, 1).$$

(ii) *There is a neighborhood V of μ_0 such that for each $y \in K(U)$ and $\mu \in V$, $f(\cdot, y, \mu)$ is C-concave in $K(U)$.*

(iii) *For $x, y \in K(U)$, $f(x, y, \cdot)$ is $h.\beta$-Hölder continuous in V.*

(iv) *For $\mu \in V$ and $x \in K(U), f(x, \cdot, \mu)$ is $q.\delta$-Hölder in $K(U)$.*

Then, for any $\bar{\varepsilon} > 0$ and $\bar{\xi} \in B_e^*$, there exist open neighborhoods $N(\bar{\xi})$ of $\bar{\xi}$, $N_{\bar{\xi}}(\lambda_0)$ of λ_0 and $N_{\bar{\xi}}(\mu_0)$ of μ_0 such that the ξ-approximate solution mapping $\Pi_\xi(\cdot, \cdot, \cdot)$ satisfies the following Hölder property in $[\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$, i.e.

$$H(\Pi_\xi(\varepsilon_1, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2)) \leq k_1|\varepsilon_1 - \varepsilon_2| + k_2||\mu_1 - \mu_2||^\beta + k_3||\lambda_1 - \lambda_2||^{\alpha\delta},$$

where $\xi \in N(\bar{\xi}), (\varepsilon_i, \lambda_i, \mu_i) \in [\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$, $i = 1, 2$, and k_1, k_2, k_3 are positive and depend on $\bar{\varepsilon}$, l, α, h, β, etc.

Proof Let $\bar{\xi} \in B_e^*$, let $N(\bar{\xi}) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0) \subset B_e^* \times U \times V$ be open. Let $(\varepsilon_1, \lambda_1, \mu_1), (\varepsilon_2, \lambda_2, \mu_2) \in [\bar{\varepsilon}, +\infty) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$ be arbitrarily given. Without loss of generality, we always assume that $\varepsilon_1 < \varepsilon_2$.

Step 1 For $0 < \varepsilon_1 < \varepsilon_2$ with $\bar{\varepsilon} < \varepsilon_2$, $(\lambda_1, \mu_1) \in N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$ and $\xi \in N(\bar{\xi})$, we provide an estimation of $H(\Pi_\xi(\varepsilon_1, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2, \lambda_1, \mu_1))$.
Put $\rho := \text{diam}K(\lambda_0) + 2l(\text{diam}(U))^\alpha$. We first show that for all $(\lambda, \mu) \in U \times V$ and $\xi \in B_e^*$,

$$H(\Pi_\xi(\varepsilon_1, \lambda, \mu), \Pi_\xi(\varepsilon_2, \lambda, \mu)) \le \frac{\rho}{\varepsilon_2}|\varepsilon_1 - \varepsilon_2|. \tag{3.1}$$

Obviously, we see that $\Pi_\xi(\varepsilon_1, \lambda, \mu) \subseteq \Pi_\xi(\varepsilon_2, \lambda, \mu)$. Therefore,

$$H^*(\Pi_\xi(\varepsilon_1, \lambda, \mu), \Pi_\xi(\varepsilon_2, \lambda, \mu)) = 0. \tag{3.2}$$

Take any $x_2 \in \Pi_\xi(\varepsilon_2, \lambda, \mu)$, $x_0 \in \Pi_\xi(0, \lambda, \mu)$. Then for all $y \in K(\lambda)$, we have

$$\min\{\xi(f(x_2, y, \mu)) + \varepsilon_2,\ \xi(f(x_0, y, \mu))\} \ge 0.$$

This inequality leads to, for $y \in K(\lambda)$,

$$\frac{\varepsilon_1}{\varepsilon_2}\xi(f(x_2, y, \mu)) + \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}\xi(f(x_0, y, \mu)) \ge 0.$$

By the linearity of ξ, one has

$$\xi\left(\frac{\varepsilon_1}{\varepsilon_2} f(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} f(x_0, y, \mu)\right) + \varepsilon_1 \ge 0. \tag{3.3}$$

On the other hand, due to assumption (ii), we get

$$\frac{\varepsilon_1}{\varepsilon_2} f(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} f(x_0, y, \mu) \in f(\frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}x_0, y, \mu) - C.$$

Hence, there exists $c_1 \in C$, such that

$$\frac{\varepsilon_1}{\varepsilon_2} f(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} f(x_0, y, \mu) = f(\frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}x_0, y, \mu) - c_1,$$

which implies

$$\xi\left(\frac{\varepsilon_1}{\varepsilon_2} f(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} f(x_0, y, \mu)\right) = \xi\left(f(\frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}x_0, y, \mu)\right) - \xi(c_1).$$

As $c_1 \in C$, $\xi(c_1) \ge 0$. Thus,

$$\xi\left(\frac{\varepsilon_1}{\varepsilon_2} f(x_2, y, \mu) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} f(x_0, y, \mu)\right) \le \xi\left(f(\frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}x_0, y, \mu)\right). \tag{3.4}$$

It follows from (3.3) and (3.4), $\xi\left(f(\frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2}x_0, y, \mu)\right) + \varepsilon_1 \ge 0$. Consequently,

$$x_1 := \frac{\varepsilon_1}{\varepsilon_2}x_2 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2}x_0 \in \Pi_\xi(\varepsilon_1, \lambda, \mu).$$

Then

$$||x_2 - x_1|| = \frac{|\varepsilon_1 - \varepsilon_2|}{\varepsilon_2}||x_2 - x_0||.$$

We observe that, for all $\lambda \in U$, $K(\lambda) \subseteq K(\lambda_0) + lB(0, \|\lambda - \lambda_0\|^\alpha)$, one has $\mathrm{diam}K(\lambda) \leq \rho$. Therefore, $\|x_2 - x_1\| \leq \frac{\rho}{\varepsilon_2}|\varepsilon_1 - \varepsilon_2|$, and thus

$$H^*(\Pi_\xi(\varepsilon_2, \lambda, \mu), \Pi_\xi(\varepsilon_1, \lambda, \mu)) \leq \frac{\rho}{\varepsilon_2}|\varepsilon_1 - \varepsilon_2|. \tag{3.5}$$

From (3.2) and (3.5), we have (3.1) proved. Since $\varepsilon_2 \in [\bar{\varepsilon}, +\infty)$, (3.1) derives

$$\begin{aligned} H(\Pi_\xi(\varepsilon_1, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2, \lambda_1, \mu_1)) &\leq \frac{\rho}{\bar{\varepsilon}}|\varepsilon_1 - \varepsilon_2| \\ &:= k_1|\varepsilon_1 - \varepsilon_2|. \end{aligned}$$

Step 2 Now we estimate $H(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2, \lambda_1, \mu_2))$ with $\bar{\varepsilon} < \varepsilon_2$, $\lambda_1 \in N_{\bar{\xi}}(\lambda_0)$, $\xi \in N(\bar{\xi})$ and $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$ such that $\mu_1 \neq \mu_2$.

First, we show that for $x, y \in K(U)$, $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$ and $\xi \in N(\bar{\xi})$,

$$|\xi(f(x, y, \mu_1)) - \xi(f(x, y, \mu_2))| \leq h\|\mu_1 - \mu_2\|^\beta. \tag{3.6}$$

Indeed, by virtue of assumption (iii), for any $x, y \in K(U)$ and $\mu_1, \mu_2 \in N_{\bar{\xi}}(\mu_0)$,

$$f(x, y, \mu_1) \in f(x, y, \mu_2) + h\|\mu_1 - \mu_2\|^\beta B(0, 1).$$

Then, for each $\xi \in N(\bar{\xi})$,

$$\begin{aligned} |\xi(f(x, y, \mu_1)) - \xi(f(x, y, \mu_2))| &\leq h\|\mu_1 - \mu_2\|^\beta \sup\{\xi(\bar{e}) : \bar{e} \in B(0, 1)\} \\ &\leq h\|\mu_1 - \mu_2\|^\beta. \end{aligned}$$

Thus, we get (3.6).

Now we divide Step 2 into two cases.

Case 1. $h\|\mu_1 - \mu_2\|^\beta \leq \varepsilon_2$. Let $r = h\|\mu_1 - \mu_2\|^\beta$. Then $0 < r \leq \varepsilon_2$. We conclude that $\Pi_\xi(\varepsilon_2 - r, \lambda_1, \mu_1) \subseteq \Pi_\xi(\varepsilon_2, \lambda_1, \mu_2)$. Indeed, let $\bar{x} \in \Pi_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)$. Then, for all $y \in K(\lambda_1)$,

$$\xi\left(f(\bar{x}, y, \mu_2)\right) + \xi\left(f(\bar{x}, y, \mu_1)\right) - \xi\left(f(\bar{x}, y, \mu_2)\right) + \varepsilon_2 - r \geq 0.$$

This inequality together with (3.6) implies that $\xi(f(\bar{x}, y, \mu_2)) + \varepsilon_2 \geq 0$ for all $y \in K(\lambda_1)$, i.e. $\bar{x} \in \Pi_\xi(\varepsilon_2, \lambda_1, \mu_2)$. Hence, using the result of Step 1, it results in the following estimates:

$$\begin{aligned} H^*(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2, \lambda_1, \mu_2)) &\leq H^*(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)) \\ &\leq H(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_1), \Pi_\xi(\varepsilon_2 - r, \lambda_1, \mu_1)) \\ &\leq \frac{\rho r}{\varepsilon_2} \\ &\leq \frac{\rho h}{\bar{\varepsilon}}\|\mu_1 - \mu_2\|^\beta. \end{aligned}$$

In the same way, we get

$$H^*(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2),\Pi_\xi(\varepsilon_2,\lambda_1,\mu_1)) \le \frac{\rho h}{\bar{\varepsilon}}\|\mu_1-\mu_2\|^\beta.$$

Hence,

$$H(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_1),\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2)) \quad \le \quad \frac{\rho h}{\bar{\varepsilon}}\|\mu_1-\mu_2\|^\beta.$$

Case 2. $h\|\mu_1-\mu_2\|^\beta > \varepsilon_2$. Let $\rho_1 = \mathrm{diam}V$. Then there exists a natural number n such that $\frac{\rho_1}{n} \le \left(\frac{\bar{\varepsilon}}{h}\right)^{1/\beta}$. Let $P_1 = \{v_1 = \mu_1, v_2, ..., v_{n+1} = \mu_2\}$ be a partition of segment $[\mu_1,\mu_2]$ such that

$$\|v_i - v_{i+1}\| = \frac{\|\mu_1-\mu_2\|}{n}.$$

Then we have $\|v_i-v_{i+1}\|^\beta = \left(\frac{\|\mu_1-\mu_2\|}{n}\right)^\beta \le \left(\frac{\rho_1}{n}\right)^\beta \le \frac{\bar{\varepsilon}}{h} \le \frac{\varepsilon_2}{h}$. Hence, using the result of Case 1, one gets

$$\begin{aligned}
H(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_1),\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2)) \quad &\le \quad \sum_{i=1}^{n} H(\Pi_\xi(\varepsilon_2,\lambda_1,v_i),\Pi_\xi(\varepsilon_2,\lambda_1,v_{i+1}))\\
&\le \quad \frac{\rho_1 h}{\bar{\varepsilon}}\sum_{i=1}^{n}\|v_i-v_{i+1}\|^\beta\\
&\le \quad \frac{n\rho_1 h}{\bar{\varepsilon}}\|\mu_1-\mu_2\|^\beta\\
&:= \quad k_2\|\mu_1-\mu_2\|^\beta.
\end{aligned}$$

Step 3 We estimate $H(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2),\Pi_\xi(\varepsilon_2,\lambda_2,\mu_2))$ for $\lambda_1 \neq \lambda_2$. We have also two cases.

Case 1. $\|\lambda_1-\lambda_2\|^\alpha \le \frac{1}{l}(\frac{\varepsilon_2}{q})^{1/\delta}$. Let $r' = l^\delta q||\lambda_1-\lambda_2||^{\alpha\delta}$. Then $0 < r' \le \varepsilon_2$. We first claim that $\Pi_\xi(\varepsilon_2 - r',\lambda_1,\mu_2) \subseteq \Pi_\xi(\varepsilon_2,\lambda_2,\mu_2)$. Indeed, for any $x' \in \Pi_\xi(\varepsilon_2-r',\lambda_1,\mu_2)$ and $y_2 \in K(\lambda_2)$, there is $y_1 \in K(\lambda_1)$ such that

$$||y_1-y_2|| \le l||\lambda_1-\lambda_2||^\alpha,$$

and

$$\xi(f(x',y_2,\mu_2)) + \xi(f(x',y_1,\mu_2)) - \xi(f(x',y_2,\mu_2)) + \varepsilon_2 - r' \ge 0.$$

It follows from (iv) that

$$f(x',y_1,\mu_2)) \in f(x',y_2,\mu_2)) + q\|y_1-y_2\|^\delta B(0,1).$$

Thus,

$$\begin{aligned}
|\xi(f(x',y_1,\mu_2)) - \xi(f(x',y_2,\mu_2))| \quad &\le \quad q\|y_1-y_2\|^\delta \sup\{\xi(e') : e' \in B(0,1)\}\\
&\le \quad q\|y_1-y_2\|^\delta\\
&\le \quad ql^\delta\|\lambda_1-\lambda_2\|^{\alpha\delta}\\
&\le \quad r'.
\end{aligned}$$

Hence, for all $y_2 \in K(\lambda_2)$,

$$\xi(f(x', y_2, \mu_2)) + \varepsilon_2 \geq 0,$$

i.e. $x' \in \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2)$, resulting in

$$\Pi_\xi(\varepsilon_2 - r', \lambda_1, \mu_2) \subseteq \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2).$$

Applying this and the result of Step 1, one has the following estimates:

$$\begin{aligned} H^*(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_2), \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2)) &\leq H(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_2), \Pi_\xi(\varepsilon_2 - r', \lambda_1, \mu_2)) \\ &\leq \frac{\rho q l^\delta}{\varepsilon_2} \|\lambda_1 - \lambda_2\|^{\alpha\delta} \\ &\leq \frac{\rho q l^\delta}{\bar{\varepsilon}} \|\lambda_1 - \lambda_2\|^{\alpha\delta}. \end{aligned}$$

Similarly,

$$H^*(\Pi_\xi(\varepsilon_2, \lambda_2, \mu_2), \Pi_\xi(\varepsilon_2, \lambda_1, \mu_2)) \leq \frac{\rho q l^\delta}{\bar{\varepsilon}} ||\lambda_1 - \lambda_2||^{\alpha\delta}.$$

Thus,

$$H(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_2), \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2)) \leq \frac{\rho q l^\delta}{\bar{\varepsilon}} \|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

Case 2. $\|\lambda_1 - \lambda_2\|^\alpha > \frac{1}{l}\left(\frac{\varepsilon_2}{q}\right)^{1/\delta}$. Let $\varepsilon_0 = \left[\frac{1}{l}\left(\frac{\bar{\varepsilon}}{q}\right)^{1/\delta}\right]^{1/\alpha}$ and $\wp = \operatorname{diam}(U) < +\infty$. There exists a natural number $\mathcal{N}$ such that $\frac{\wp}{\mathcal{N}} \leq \varepsilon_0$. Let P_2 be a partition of segment $[\lambda_1, \lambda_2]$ with $\mathcal{N}+1$ nodes $u_1, ..., u_{\mathcal{N}+1}$ such that

$$u_1 = \lambda_1, \qquad u_{\mathcal{N}+1} = \lambda_2, \quad \|u_j - u_{j+1}\| = \frac{\|\lambda_1 - \lambda_2\|}{\mathcal{N}}.$$

Hence,

$$\|u_j - u_{j+1}\|^\alpha \leq (\frac{\wp}{\mathcal{N}})^\alpha \leq \frac{1}{l}\left(\frac{\bar{\varepsilon}}{q}\right)^{1/\delta} \leq \frac{1}{l}\left(\frac{\varepsilon_2}{q}\right)^{1/\delta}.$$

Applying the result of Case 1, one has

$$H(\Pi_\xi(\varepsilon_2, \lambda_1, \mu_2), \Pi_\xi(\varepsilon_2, \lambda_2, \mu_2)) \leq \frac{\rho q l^\delta}{\bar{\varepsilon}} \sum_{j=1}^{\mathcal{N}} ||u_j - u_{j+1}||^{\alpha\delta}$$

$$\leq \frac{\mathcal{N}\rho q l^\delta}{\bar{\varepsilon}} \left[\frac{1}{l}\left(\frac{\varepsilon_2}{q}\right)^{1/\delta}\right]^\delta \leq \frac{\mathcal{N}\rho q l^\delta}{\bar{\varepsilon}} \|\lambda_1 - \lambda_2\|^{\alpha\delta} := k_3 \|\lambda_1 - \lambda_2\|^{\alpha\delta}.$$

Step 4 Now we are ready to complete the proof. Combining the results of the preceding three steps, we obtain

$$\begin{aligned} H(\Pi_\xi(\varepsilon_1,\lambda_1,\mu_1),\Pi_\xi(\varepsilon_2,\lambda_2,\mu_2)) &\leq H(\Pi_\xi(\varepsilon_1,\lambda_1,\mu_1),\Pi_\xi(\varepsilon_2,\lambda_1,\mu_1)) \\ &\quad +H(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_1),\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2)) \\ &\quad +H(\Pi_\xi(\varepsilon_2,\lambda_1,\mu_2),\Pi_\xi(\varepsilon_2,\lambda_2,\mu_2)) \\ &\leq k_1|\varepsilon_1-\varepsilon_2|+k_2\|\mu_1-\mu_2\|^\beta \\ &\quad +k_3\|\lambda_1-\lambda_2\|^{\alpha\delta}. \end{aligned}$$

□

By using Theorem 3.1 and a suitable technique, we obtain the Hölder continuity of the approximate solution mappings to (WEP). Concretely, we have the following result.

Theorem 3.2 *For* (WEP)*, assume that for each $\xi \in B_e^*$, the ξ-approximate solutions exist in a neighborhood of the considered point $(\lambda_0,\mu_0) \in \Lambda \times M$. Furthermore, assume that the following conditions hold.*

(i) *K is $l.\alpha$-Hölder continuous around λ_0, i.e. there exists a neighborhood U of λ_0 such that for all $\lambda_1,\lambda_2 \in U$,*

$$K(\lambda_1) \subseteq K(\lambda_2) + l\|\lambda_1-\lambda_2\|^\alpha B(0,1).$$

(ii) *There is a neighborhood V of μ_0 such that for each $y \in K(U)$ and $\mu \in V$, $f(\cdot,y,\mu)$ is C-concave in $K(U)$.*

(iii) *For $x,y \in K(U)$, $f(x,y,\cdot)$ is $h.\beta$-Hölder continuous in V.*

(iv) *For $\mu \in V$ and $x \in K(U), f(x,\cdot,\mu)$ is C-convex as well as $q.\delta$-Hölder in $K(U)$.*

Then, for any $\bar\varepsilon > 0$, there exist open neighborhoods $N'(\lambda_0)$ of λ_0 and $N'(\mu_0)$ of μ_0 such that the approximate solution set $\Pi(\cdot,\cdot,\cdot)$ satisfies the following Hölder property in $[\bar\varepsilon,+\infty) \times N'(\lambda_0) \times N'(\mu_0)$, i.e.

$$H(\Pi(\varepsilon_1,\lambda_1,\mu_1),\Pi(\varepsilon_2,\lambda_2,\mu_2)) \leq k_1|\varepsilon_1-\varepsilon_2| + k_2||\mu_1-\mu_2||^\beta + k_3||\lambda_1-\lambda_2||^{\alpha\delta},$$

where $(\varepsilon_i,\lambda_i,\mu_i) \in [\bar\varepsilon,+\infty) \times N'(\lambda_0) \times N'(\mu_0)$, $i=1,2$, and k_1,k_2,k_3 are positive and depend on $\bar\varepsilon$, l,α,h,β, etc.

Proof We see that B_e^* is a weak* compact set and in Theorem 3.1 the system of $\{N(\bar\xi)\}_{\bar\xi\in B_e^*}$ is an open covering of B_e^*. Consequently, there is a finite number of points $\xi_i \in B_e^*(\, i=1,...,n)$ satisfying

$$B_e^* \subset \bigcup_{i=1}^n N(\xi_i). \tag{3.7}$$

Let $N'(\mu_0)=\bigcap_{i=1}^n N_{\xi_i}(\mu_0)$ and $N'(\lambda_0)=\bigcap_{i=1}^n N_{\xi_i}(\lambda_0)$. Take arbitrarily $(\lambda,\mu) \in N'(\lambda_0) \times N'(\mu_0)$. Thanks to (3.7), for any $\xi \in B_e^*$, there is $i_0 \in \{1,...,n\}$ such that $\xi \in N(\xi_{i_0})$. From the construction of the neighborhoods $N'(\lambda_0)$ and $N'(\mu_0)$, we have $(\lambda,\mu) \in N_{\xi_{i_0}}(\lambda_0) \times N_{\xi_{i_0}}(\mu_0)$. Thus, $N'(\lambda_0)$ and $N'(\mu_0)$ are desired neighborhoods of λ_0 and μ_0, respectively.

Due to assumption (iv) and in view of Lemma 2.6, one has

$$\Pi(\varepsilon,\lambda,\mu)=\bigcup_{\xi\in B_e^*}\Pi_\xi(\varepsilon,\lambda,\mu).$$

For all $(\lambda_1,\mu_1),(\lambda_2,\mu_2)\in N'(\lambda_0)\times N'(\mu_0)$, we now claim that

$$H(\Pi(\varepsilon_1,\lambda_1,\mu_1),\Pi(\varepsilon_2,\lambda_2,\mu_2))\leq k_1|\varepsilon_1-\varepsilon_2|+k_2\|\mu_1-\mu_2||^\beta+k_3||\lambda_1-\lambda_2||^{\alpha\delta}. \tag{3.8}$$

Indeed, for each $x_1\in\Pi(\varepsilon_1,\lambda_1,\mu_1)=\bigcup_{\xi\in B_e^*}\Pi_\xi(\varepsilon_1,\lambda_1,\mu_1)$, there is $\hat{\xi}\in B_e^*$ such that $x_1\in\Pi_{\hat{\xi}}(\varepsilon_1,\lambda_1,\mu_1)$. As $\Pi_{\hat{\xi}}(\varepsilon_2,\lambda_2,\mu_2)\subseteq\Pi(\varepsilon_2,\lambda_2,\mu_2)$ and applying Theorem 3.1, one has

$$\begin{aligned}
d(x_1,\Pi(\varepsilon_2,\lambda_2,\mu_2)) &\leq d(x_1,\Pi_{\hat{\xi}}(\varepsilon_2,\lambda_2,\mu_2))\\
&\leq H^*(\Pi_{\hat{\xi}}(\varepsilon_1,\lambda_1,\mu_1),\Pi_{\hat{\xi}}(\varepsilon_2,\lambda_2,\mu_2))\\
&\leq H(\Pi_{\hat{\xi}}(\varepsilon_1,\lambda_1,\mu_1),\Pi_{\hat{\xi}}(\varepsilon_2,\lambda_2,\mu_2)).\\
&\leq k_1|\varepsilon_1-\varepsilon_2|+k_2\|\mu_1-\mu_2||^\beta+k_3\|\lambda_1-\lambda_2\|^{\alpha\delta},
\end{aligned}$$

where k_1,k_2,k_3 are positive and depend on $\bar{\varepsilon}$, l,α,h,β, etc.
Therefore,

$$H^*(\Pi(\varepsilon_1,\lambda_1,\mu_1),\Pi(\varepsilon_2,\lambda_2,\mu_2))\leq k_1|\varepsilon_1-\varepsilon_2|+k_2\|\mu_1-\mu_2||^\beta+k_3\|\lambda_1-\lambda_2\|^{\alpha\delta}. \tag{3.9}$$

In a similar way, we also obtain that

$$H^*(\Pi(\varepsilon_2,\lambda_2,\mu_2),\Pi(\varepsilon_1,\lambda_1,\mu_1))\leq k_1|\varepsilon_1-\varepsilon_2|+k_2\|\mu_1-\mu_2\|^\beta+k_3\|\lambda_1-\lambda_2\|^{\alpha\delta}. \tag{3.10}$$

From (3.9) and (3.10), we have (3.8) proved. Hence the proof is complete. □

We now provide some examples to illustrate the essentialness of assumptions in Theorem 3.2. Firstly, the concavity assumption of f cannot be dropped.

Example 3.3 Let $\bar{\varepsilon}=\frac{1}{3},X=A=\mathbb{R},\Lambda\equiv M=[0,1],K(\lambda)=[-3,3],Y=\mathbb{R}^2,C=\mathbb{R}^2_+,e=(1,1)\in\text{int}C$ and

$$f(x,y,\mu)=\begin{cases}(0,0), & x=0,\\ (-1,-1), & x\neq 0.\end{cases}$$

Then we see that (i), (iii), (iv), and (v) of Theorem 3.2 are satisfied. Some direct computations give the approximate solution set

$$\Pi(\varepsilon)=\begin{cases}[-3,3], & \varepsilon\geq 1,\\ \{0\}, & \frac{1}{3}\leq\varepsilon<1.\end{cases}$$

which is not Hölder continuous at $\varepsilon_0=1$. The reason is that the concavity assumption with the first variable of the objective function f is violated (for instance, take $x_1=0,x_2=1$ and $t=\frac{1}{2}$).

The following example shows that the convexity of the constrained set cannot be dispensed.

Example 3.4 Let $X = A = \mathbb{R}, \Lambda \equiv M = [0,4], K(\lambda) = [1,3] \cup \{4\}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, e = (1,1) \in \mathrm{int}C$ and $f(x,y,\mu) = (\mu - x, \mu - x)$. It is not hard to check that all assumptions of Theorem 3.2 are fulfilled. Obviously, K is not convex. By direct computations, we have $\Pi(\varepsilon,\mu) = (-\infty, \mu + \varepsilon] \cap ([1,3] \cup \{4\})$. We see that Π is not Hölder continuous at $\mu_0 = 3$ for $\varepsilon_0 = 1$.

The next example confirms that the boundedness of K is essential.

Example 3.5 Let $X = A = \mathbb{R}, \Lambda \equiv M = [0,1], K(\lambda) = [0,+\infty), Y = \mathbb{R}^2, C = \mathbb{R}^2_+, e = (1,1) \in \mathrm{int}C$ and $f(x,y,\mu) = (-\mu x, -\mu x)$. Then all assumptions of Theorem 3.2 hold. Direct computations give $\Pi(\varepsilon,0) = [0,+\infty)$ and $\Pi(\varepsilon,\mu) = \left[0, \frac{\varepsilon}{\mu}\right], \quad \mu \neq 0$. It is easy to verify that Π is not Hölder continuous at $\mu_0 = 0$ for $\varepsilon_0 = 1$.

4. Applications

Since the vector equilibrium problem contains many optimization related problems as special cases, we will apply the results presented in Section 3 to obtain sufficient conditions for Hölder continuity of approximate solution sets of these particular cases. In this section, we only take the vector optimization problems, the vector variational inequalities, and fixed-point problems as examples.

4.1. Vector optimization problems

For $(\varepsilon,\lambda,\mu) \in \mathbb{R}_+ \times \Lambda \times M$, denote the approximate solution set of (VOP) by $\Pi_1(\varepsilon,\lambda,\mu)$. Then, applying Theorem 3.2, we get the following result.

Corollary 4.1 *For* (VOP), *let* $\Pi_1(\varepsilon,\lambda,\mu)$ *is nonempty for small* $\varepsilon > 0$ *in a neighborhood of the considered point* $(\lambda_0,\mu_0) \in \Lambda \times M$*. Suppose that the following conditions hold:*

(a) K *is* $l_1.\alpha_1$*-Hölder continuous around* λ_0*, i.e. there is a neighborhood* U *of* λ_0 *such that, for all* $\lambda_1, \lambda_2 \in U$*,*

$$K(\lambda_1) \subseteq K(\lambda_2) + l_1 B(0, ||\lambda_1 - \lambda_2||^{\alpha_1}).$$

(b) *There is a neighborhood* V *of* μ_0 *such that, for each* $\mu \in V, g(\cdot,\mu)$ *is* C*-convex and* $q_1.\delta_1$*-Hölder continuous in* $K(U)$*.*

(c) *For every* $x \in K(U), g(x,\cdot)$ *is* $h_1.\beta_1$*-Hölder continuous in* V*.*

Then, for each $\bar{\varepsilon} > 0$*,* Π_1 *is Hölder continuous in* $[\bar{\varepsilon},+\infty) \times U \times V$*, i.e. for all* $((\varepsilon_1,\lambda_1,\mu_1),(\varepsilon_2,\lambda_2,\mu_2)) \in [\bar{\varepsilon},+\infty) \times U \times V$*,*

$$H(\Pi_1(\varepsilon_1,\lambda_1,\mu_1),\Pi_1(\varepsilon_2,\lambda_2,\mu_2)) \leq \kappa_1|\varepsilon_1 - \varepsilon_2| + \kappa_2\|\mu_1 - \mu_2\|^{\beta_1} + \kappa_3\|\lambda_1 - \lambda_2\|^{\alpha_1\delta_1},$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ *and depend on* $\bar{\varepsilon}$*,* $l_1, \alpha_1, h_1, \beta_1$*, etc.*

Proof We will prove this corollary by checking the validity of Assumptions (ii), (iii), and (iv) of Theorem 3.2. We first verify (ii). Take arbitrarily $x_1, x_2 \in K(U), t \in [0,1]$, $y \in K(U)$ and $\mu \in V$. We have

$$\begin{aligned} &f(tx_1 + (1-t)x_2, y, \mu) - tf(x_1, y, \mu) - (1-t)f(x_2, y, \mu) \\ &= g(y,\mu) - g(tx_1 + (1-t)x_2, \mu) - t[g(y,\mu) - g(x_1,\mu)] - (1-t)[g(y,\mu) - g(x_2,\mu)] \\ &= tg(x_1,\mu) + (1-t)g(x_2,\mu) - g(tx_1 + (1-t)x_2, \mu) \in C, \end{aligned}$$

which is true due to the convexity of $g(\cdot, \mu)$. Hence (ii) holds. Passing to (iii), taking any $\mu_1, \mu_2 \in V$ and $x, y \in K(U)$, one sees that

$$\begin{aligned} \|f(x,y,\mu_1) - f(x,y,\mu_2)\| &= \|(g(y,\mu_1) - g(y,\mu_2)) + (g(x,\mu_2) - g(x,\mu_1))\| \\ &\leq h_1\|\mu_1 - \mu_2\|^{\beta_1} + h_1\|\mu_1 - \mu_2\|^{\beta_1} \\ &\leq 2h_1||\mu_1 - \mu_2||^{\beta_1}. \end{aligned}$$

Thus, (iii) is satisfied with $h = 2h_1$ and $\beta = \beta_1$.

For (iv), the convexity of f with the second component is checked similarly to the concavity with the first one. Next, for any $y_1, y_2 \in K(U)$, $x \in K(U)$ and $\mu \in V$, we have

$$\begin{aligned} \|f(x,y_1,\mu) - f(x,y_2,\mu)\| &= \|g(y_1,\mu) - g(y_2,\mu)\| \\ &\leq q_1||y_1 - y_2||^{\delta_1}. \end{aligned}$$

Thus, the Hölder continuity in (vi) holds with $q = q_1$ and $\delta = \delta_1$. □

Remark 4.2 Recently, there have been many papers devoted to the Hölder continuity for optimization problems [5, 9, 35]. However, since the imposed assumptions relate to strong monotonicity/convexity, the solution sets are unique. Therefore, Corollary 4.1 is a new result.

In the case of $Y = \mathbb{R}$, we have the following result for (OP).

Corollary 4.3 *For* (OP)*, let* $\Pi_1(\varepsilon, \lambda, \mu)$ *be nonempty for small* $\varepsilon > 0$ *in a neighborhood of the considered point* $(\lambda_0, \mu_0) \in \Lambda \times M$*. Suppose that the following conditions hold:*

(a) K *is* $l_1.\alpha_1$*-Hölder continuous around* λ_0*, i.e. there is a neighborhood* U *of* λ_0 *such that, for all* $\lambda_1, \lambda_2 \in U$*,*

$$K(\lambda_1) \subseteq K(\lambda_2) + l_1 B(0, ||\lambda_1 - \lambda_2||^{\alpha_1}).$$

(b) *There is a neighborhood* V *of* μ_0 *such that, for each* $\mu \in V, g(\cdot,\mu)$ *is convex and* $q_1.\delta_1$*-Hölder continuous in* $K(U)$*.*

(c) *For every* $x \in K(U), g(x,\cdot)$ *is* $h_1.\beta_1$*-Hölder continuous in* V*.*

Then, for each $\bar{\varepsilon} > 0$*,* Π_1 *is Hölder continuous in* $[\bar{\varepsilon}, +\infty) \times U \times V$*, i.e. for all* $((\varepsilon_1, \lambda_1, \mu_1), (\varepsilon_2, \lambda_2, \mu_2)) \in [\bar{\varepsilon}, +\infty) \times U \times V$*,*

$$H(\Pi_1(\varepsilon_1, \lambda_1, \mu_1), \Pi_1(\varepsilon_2, \lambda_2, \mu_2)) \leq \kappa_1|\varepsilon_1 - \varepsilon_2| + \kappa_2\|\mu_1 - \mu_2\|^{\beta_1} + \kappa_3\|\lambda_1 - \lambda_2\|^{\alpha_1\delta_1},$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ *and depend on* $\bar{\varepsilon}$*,* $l_1, \alpha_1, h_1, \beta_1$*, etc.*

4.2. Vector variational inequalities

For $(\varepsilon, \lambda, \mu) \in \mathbb{R}_+ \times \Lambda \times M$, we denote the approximate solution set of (VVI) by $\Pi_2(\varepsilon, \lambda, \mu)$.

The following result is a consequence of Theorem 3.2.

Corollary 4.4 *For* (VVI), *assume that* $\Pi_2(\varepsilon, \lambda, \mu)$ *is nonempty for small* $\varepsilon > 0$. *Suppose that the following conditions hold:*

(a) K *is* $l_2.\alpha_2$*-Hölder continuous around* λ_0, *i.e. there is a neighborhood* U *of* λ_0 *such that, for all* $\lambda_1, \lambda_2 \in U$,
$$K(\lambda_1) \subseteq K(\lambda_2) + l_2 B(0, ||\lambda_1 - \lambda_2||^{\alpha_2}).$$

(b) *There is a neighborhood* V *of* μ_0 *such that, for each* $\mu \in V$, $\phi(\cdot, \mu)$ *is bounded (i.e. there exists* $q_2 > 0$ *such that* $\|\phi(x, \mu)\| \le q_2, \ \forall x \in K(U)$*), monotone and affine in* $K(U)$.

(c) *For* $x \in K(U), \phi(x, \cdot)$ *is* $h_2.\beta_2$*-Hölder continuous in* V.

Then, for each $\bar{\varepsilon} > 0$, *the approximate solution* Π_2 *satisfies the following Hölder property in* $[\bar{\varepsilon}, +\infty) \times U \times V$:
$$H(\Pi_2(\varepsilon_1, \lambda_1, \mu_1), \Pi_2(\varepsilon_2, \lambda_2, \mu_2)) \le \kappa_1 |\varepsilon_1 - \varepsilon_2| + \kappa_2 \|\mu_1 - \mu_2\|^{\beta_2} + \kappa_3 \|\lambda_1 - \lambda_2\|^{\alpha_2},$$
where $\kappa_1, \kappa_2, \kappa_3 > 0$ *and depend on* $\bar{\varepsilon}$, l_2, *etc.*

Proof Similarly to the proof of Corollary 4.1, we also examine Assumptions (ii), (iii), (iv), and (iv) of Theorem 3.2. For (ii), taking arbitrarily $x_1, x_2 \in K(U), t \in [0,1]$, and $y \in K(U)$, by (b) we have
$$\begin{aligned}
& f(tx_1 + (1-t)x_2, y, \mu) - tf(x_1, y, \mu) - (1-t)f(x_2, y, \mu) \\
&= \langle t\phi(x_1, \mu) + (1-t)\phi(x_2, \mu), y - tx_1 - (1-t)x_2 \rangle \\
&\qquad - t\langle \phi(x_1, \mu), y - x_1 \rangle - (1-t)\langle \phi(x_2, \mu), y - x_2 \rangle \\
&= t\langle \phi(x_1, \mu), (1-t)x_1 - (1-t)x_2 \rangle + (1-t)\langle \phi(x_2, \mu), -tx_1 + tx_2 \rangle \\
&= t(1-t)\langle \phi(x_1, \mu) - \phi(x_2, \mu), x_1 - x_2 \rangle \in C.
\end{aligned}$$

Thus, (ii) of Theorem 3.2 is satisfied. Turning to (iii) of Theorem 3.2, take any $\mu_1, \mu_2 \in V$ and $x, y \in K(U)$ to see that
$$\begin{aligned}
\|f(x, y, \mu_1) - f(x, y, \mu_2)\| &= \|\langle \phi(x, \mu_1) - \phi(x, \mu_2), y - x \rangle\| \\
&\le \|\phi(x, \mu_1) - \phi(x, \mu_2)\| \|x - y\| \\
&\le h_2 \rho_2 \|\mu_1 - \mu_2\|^{\beta_2},
\end{aligned}$$
where $\rho_2 = \text{diam} K(U)$. Thus, assumption (iii) is fulfilled with $h = h_2 \rho_2$ and $\beta = \beta_2$.

The Hölder continuity in (iv) holds with $q = q_2$ and $\delta = 1$, since, for $y_1, y_2 \in K(U)$, $x \in K(U)$ and $\mu \in V$, we have
$$\begin{aligned}
\|f(x, y_1, \mu) - f(x, y_2, \mu)\| &= \|\langle \phi(x, \mu), y_1 - x \rangle - \langle \phi(x, \mu), y_2 - x \rangle\| \\
&= \|\langle \phi(x, \mu), y_1 - y_2 \rangle\| \\
&\le \|\phi(x, \mu)\| ||y_1 - y_2|| \\
&\le q_2 ||y_1 - y_2||.
\end{aligned}$$

With the same arguments as for the concavity assumption in (i), we also get the convexity assumption in (iv).

Therefore, the conclusion of Corollary 4.4 is implied from Theorem 3.2. □

Remark 4.5 In the literature, the Hölder continuity of solution mappings to variational inequality was investigated intensively [citeyen1, yenlee. Most of these works imposed assumptions related to strong monotonicity/convexity and the solution sets were unique. It is worth noticing that, in Corollary 4.4, although the function g is assumed to be monotone, the solutions sets may be not unique. This is illustrated by the following example.

Example 4.6 Let $X = \mathbb{R}, \Lambda \equiv M = [0,1], A = [0,1], K(\lambda) = [0,1]$, $Y = \mathbb{R}^2, C = \mathbb{R}^2_+$ and $\phi(x,\mu) = (0,0)$. Then all assumptions of Corollary 4.4 are fulfilled. Hence, Corollary 4.4 derives the Hölder continuity of Π_2. By direct calculations we have, for all $(\varepsilon,\mu) \in \mathbb{R}_+ \times M$, $\Pi_2(\varepsilon,\mu) = [0,1]$. Obviously, Π_2 is not a singleton.

If $Y = \mathbb{R}$, we also have the result for (VI) as follows.

Corollary 4.7 *For* (VI), *assume that* $\Pi_2(\varepsilon,\lambda,\mu)$ *is nonempty for small* $\varepsilon > 0$. *Suppose that the following conditions hold:*

(a) K *is* $l_2.\alpha_2$*-Hölder continuous around* λ_0, *i.e. there is a neighborhood* U *of* λ_0 *such that, for all* $\lambda_1, \lambda_2 \in U$,

$$K(\lambda_1) \subseteq K(\lambda_2) + l_2 B(0, ||\lambda_1 - \lambda_2||^{\alpha_2}).$$

(b) *There is a neighborhood* V *of* μ_0 *such that, for each* $\mu \in V$, $\phi(\cdot,\mu)$ *is bounded (i.e. there exists* $q_2 > 0$ *such that* $\|\phi(x,\mu)\| \le q_2, \ \forall x \in K(U)$*), monotone and affine in* $K(U)$.

(c) *For* $x \in K(U), \phi(x,\cdot)$ *is* $h_2.\beta_2$*-Hölder continuous in* V.

Then, for each $\bar{\varepsilon} > 0$, *the approximate solution* Π_2 *satisfies the following Hölder property in* $[\bar{\varepsilon}, +\infty) \times U \times V$:

$$H(\Pi_2(\varepsilon_1,\lambda_1,\mu_1), \Pi_2(\varepsilon_2,\lambda_2,\mu_2)) \le \kappa_1|\varepsilon_1 - \varepsilon_2| + \kappa_2\|\mu_1 - \mu_2\|^{\beta_2} + \kappa_3\|\lambda_1 - \lambda_2\|^{\alpha_2},$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ *and depend on* $\bar{\varepsilon}$, l_2, *etc.*

4.3. Fixed-point problems

For $\varepsilon \ge 0$ and $\mu \in M$, we denote the ε-solution set of (FP) by $\Pi_3(\varepsilon,\mu)$, i.e.

$$\Pi_3(\varepsilon,\mu) = \{\bar{x} \in A : \|\bar{x} - g(\bar{x},\mu)\| \le \varepsilon\}.$$

Corollary 4.8 *Assume that* $\Pi_3(\varepsilon,\mu)$ *is nonempty for small* $\varepsilon > 0$ *in a neighborhood of the considered point* $\mu_0 \in M$. *Suppose that the following conditions hold:*

(a) *there is a neighborhood* U *of* μ_0 *such that, for each* $\mu \in U, \varphi(\cdot,\mu)$ *is affine and* $x \mapsto x - \varphi(x,\mu)$ *is monotone in* A;

(b) *for every* $x \in A$, $\varphi(x,\cdot)$ *is* $h_3.\beta_3$*-Hölder continuous in* A.

Then, for each $\bar{\varepsilon} > 0$, Π_3 *satisfies the following Hölder condition in* $[\sqrt{\bar{\varepsilon}}, +\infty) \times U$:

$$H(\Pi_3(\varepsilon_1,\mu_1), \Pi_3(\varepsilon_2,\mu_2)) \le k_1|\varepsilon_1 - \varepsilon_2| + k_2\|\mu_1 - \mu_2\|^{\beta_3},$$

where $k_1, k_2 > 0$ *and depend on* $\bar{\varepsilon}$, h_3, *and* β_3.

Proof We omit the proof because it is similar to the proof of Corollary 4.1 and 4.4. □

Acknowledgments

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2017.18. The authors would like to thank the anonymous referees for their valuable remarks and suggestions, which have improved the paper.

References

[1] Ait Mansour M, Riahi H. Sensitivity analysis for abstract equilibrium problems. J Math Anal Appl 2005; 306: 684-691.

[2] Ait Mansour M, Scrimali L. Hölder continuity of solutions to elastic traffic network models. J Global Optim 2008; 40: 175-184.

[3] Anh LQ, Khanh PQ. Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems. J Math Anal Appl 2004; 294: 699-711.

[4] Anh LQ, Khanh PQ. On the Hölder continuity of solutions to multivalued vector equilibrium problems. J Math Anal Appl 2006; 321: 308-315.

[5] Anh LQ, Khanh PQ. On the stability of the solution sets of general multivalued vector quasiequilibrium problems. J Optim Theory Appl 2007; 135: 271-284.

[6] Anh LQ, Khanh PQ. Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces. J Global Optim 2007; 37: 449-465.

[7] Anh LQ, Khanh PQ. Various kinds of semicontinuity and solution sets of parametric multivalued symmetric vector quasiequilibrium problems. J Global Optim 2008; 41: 539-558.

[8] Anh LQ, Khanh PQ. Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions. J Global Optim 2008; 42: 515-531.

[9] Anh LQ, Khanh PQ. Hölder continuity of the unique solution to quasiequilibrium problems in metric Spaces. J Optim Theory Appl 2009; 141: 37-54.

[10] Anh LQ, Khanh PQ. Continuity of solution maps of parametric quasiequilibrium problems. J Global Optim 2010; 46: 247-259.

[11] Anh LQ, Khanh PQ, Tam TN. On Hölder continuity of approximate solutions to parametric equilibrium problems. Nonlinear Anal 2012; 75: 2293-2303.

[12] Anh LQ, Khanh PQ, Tam TN, Van DTM. On Hölder calmness and Hölder well-posedness of vector quasi-equilibrium problems. Vietnam J Math 2013; 41: 507-517.

[13] Anh LQ, Khanh PQ, Tam TN. On Hölder continuity of solution maps of parametric primal and dual Ky Fan inequalities. TOP 2015; 23: 151-167.

[14] Anh LQ, Kruger AY, Thao NH. On Hölder calmness of solution mappings in parametric equilibrium problems. TOP 2014; 22: 331-342.

[15] Bianchi M, Schaible S. Generalized monotone bifunctions and equilibrium problems. J Optim Theory Appl 1996; 90: 31-43.

[16] Bianchi M, Hadjisavvas N, Schaible S. Equilibrium problems with generalized monotone bifunctions. J Optim Theory Appl 1997; 92: 527-542.

[17] Bianchi M, Pini R. A note on stability for parametric equilibrium problems. Oper Res Lett 2003; 31: 445-450.

[18] Bianchi M, Pini R. Sensitivity for parametric vector equilibria. Optimization 2006; 55: 221-230.

[19] Blum E, Oettli W. From optimization and variational inequalities to equilibrium problems. Math Student 1994; 63: 123-145.

[20] Chen CR, Li SJ, Teo KL. Solution semicontinuity of parametric generalized vector equilibrium problems. J Glob Optim 2009; 45: 309-318.

[21] Chen CR, Li SJ, Zeng J, Li XB. Error analysis of approximate solutions to parametric vector quasiequilibrium problems. Optim Lett 2011; 5: 85-98.

[22] Giannessi F (ed). Vector variational inequalities and vector equilibria, Mathematical Theories. Dordrecht, Netherlands: Kluwer Academic, 2000.

[23] Hai NX, Khanh PQ. Existence of solutions to general quasiequilibrium problems and applications. J Optim Theory Appl 2007; 133: 317-327.

[24] Hai NX, Khanh PQ. The solution existence of general variational inclusion problems. J Math Anal Appl 2007; 328: 1268-1277.

[25] Hai NX, Khanh PQ, Quan NH. On the existence of solutions to quasivariational inclusion problems. J Glob Optim 2009; 45: 565-581.

[26] Huang NJ, Li J, Thompson HB. Stability for parametric implicit vector equilibrium problems. Math Comput Modelling 2006; 43: 1267-1274.

[27] Jahn J. Mathematical vector optimization in partially ordered linear spaces, Peter Lang, Frankfurt, 1986.

[28] Khanh PQ, Quan NH. Generic stability and essential components of generalized KKM points and applications. J Optim Theory Appl 2011; 148: 488-504.

[29] Khanh PQ, Long VST. Invariant-point theorems and existence of solutions to optimization-related problems. J Glob Optim 2014; 58: 545-564.

[30] Kimura K, Yao JC. Semicontinuity of solution mappings of parametric generalized vector equilibrium problems. J Optim Theory Appl 2008; 138: 429-443.

[31] Kimura K, Yao JC. Sensitivity analysis of solution mappings of parametric vector quasi-equilibrium problems. J Glob Optim 2008; 41: 187-202.

[32] Li SJ, Li XB. Hölder continuity of solutions to parametric weak generalized Ky Fan inequality. J Optim Theory Appl 2011; 149: 540-553.

[33] Li SJ, Li XB, Wang LN, Teo KL. The Hölder continuity of solutions to generalized vector equilibrium problems. European J Oper Res 2009; 199: 334-338.

[34] Li SJ, Chen CR, Li XB, Teo KL. Hölder continuity and upper estimates of solutions to vector quasiequilibrium problems. European J Oper Res 2011; 210: 148-157.

[35] Li XB, Li SJ. Hölder continuity of perturbed solution set for convex optimization problems. Appl Math Comput 2014; 232: 908-918.

[36] Yen ND. Hölder continuity of solutions to parametric variational inequalities. Appl Math Optim 1995; 31: 245-255.

[37] Yen ND, Lee GM. Solution sensitivity of a class of variational inequalities. J Math Anal Appl 1997; 215: 48-55.