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Two-dimensional generalized discrete Fourier transform and related quasi-cyclic Reed–Solomon codes

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Abstract: Using the concept of the partial Hasse derivative, we introduce a generalization of the classical 2-dimensional discrete Fourier transform, which will be called 2D-GDFT. Begining with the basic properties of 2D-GDFT, we proceed to study its computational aspects as well as the inverse transform, which necessitate the development of a faster way to calculate the 2D-GDFT. As an application, we will employ 2D-GDFT to construct a new family of quasi-cyclic linear codes that can be assumed to be a generalization of Reed–Solomon codes.

Key words: Discrete Fourier transform, partial Hasse derivative, Reed–Solomon codes

1. Introduction

The relationship between one- and two-dimensional Fourier transforms is similar in the discrete domain. Let ω be an nth root of unity in the Galois field F_q, where q is a prime power p^a. Recall that the discrete Fourier transform (DFT) of an n-bit vector $\mathbf{v} = (v_0, v_1, \cdots, v_{n-1}) \in F_q^n$, n coprime with p, is defined as follows:

$$\mathcal{F}\{(v_0, v_1, \cdots, v_{n-1})\} = (V_0, V_1, \cdots, V_{n-1}),$$

where $V_j = \sum_{i=0}^{n-1} v_i \omega^{ij}$, $j = 0, \cdots, n-1$. The vector $\mathbf{v}$ is related to its spectrum $\mathbf{V} = \mathcal{F}\{\mathbf{v}\}$ by

$$v_i = \frac{1}{n} \sum_{j=0}^{n-1} V_j \omega^{-ij}, \ i = 0, \cdots, n-1,$$

where n is interpreted as an integer of the field.

Two-dimensional Fourier transform of an $M \times N$-matrix $A = [a_{ij}] \in (F_q)^{M \times N}$, M and N relatively prime to p, is similarly defined as an $M \times N$-matrix $B = [b_{ij}]$ by

$$B_{kl} = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} A_{ij} \alpha^{ik} \beta^{jl}, \quad k = 0, \cdots, M-1, \quad l = 0, \cdots, N-1,$$

where α and β are respectively an Mth root of unity and an Nth root of unity in some (sufficiently large)

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extension of F_q. In this case, the inverse transform is given by

$$A_{ij} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} B_{kl} \alpha^{-ik} \beta^{-jl}.$$

The importance of two-dimensional DFT arises when we deal with the problem of evaluating the one-dimensional DFT of a vector $\mathbf{v}$ having a large number of elements, on the hypothesis that the working memory of the available processor is not sufficient to handle the vector as a whole. Such a situation can arise in several applications [1,4,5,12], such as Fourier transform spectroscopy or musical sound analysis. In this case it is convenient to fold $\mathbf{v}$ into a matrix A and apply the two-dimensional DFT on the corresponding matrix A.

Some generalizations of the concept of (one and two-dimensional) DFT were given by earlier authors. In [3], (one and two-dimensional) generalized DFT (GFT) was introduced and some basic properties were derived. In particular, it was shown that a given one-dimensional GFT on a vector $\mathbf{v}$ can be performed by means of an infinite number of two-dimensional GFTs on a matrix A whose elements are the elements of $\mathbf{v}$ properly ordered. In [6], multidimensional generalized DFT was introduced and its characteristics were investigated while some general results were derived that included as particular cases the properties previously given in [3].

Here, the key point is that previously introduced two-dimensional GFTs have an inverse only if the characteristic of the field structuring the alphabet was zero or coprime with both M and N, where M and N denote the number of rows and columns of input matrices, respectively. To relax that condition, we shall introduce a new kind of two-dimensional DFT, called the two-dimensional generalized DFT (2D-GDFT), which in turn relies on the concept of the partial Hasse derivative of two-variable polynomials. We will show that the 2D-GDFT enjoys all basic properties of DFT analogously. As an application, using the 2D-GDFT, we will construct a family of linear codes, called quasi-cyclic Reed–Solomon codes.

2. Preliminaries

2.1. Linear codes

Linear codes are widely studied because of their algebraic structure, which makes them easier to describe than nonlinear codes.

Let $q = p^a$ be a prime power and let F_q denote the finite field of order q. A linear code C of length n over F_q is an F_q-vector subspace of F_q^n. The (Hamming) weight of a vector $\mathbf{c} \in (F_q)^n$ is the number $w(\mathbf{c})$ of its nonzero coordinates. For a linear code C, the distance $d(C)$ is defined as the minimum weight of nonzero words. The distance of a code C is important to determine the error correction capability of C (that is, the number of errors that the code can correct) and its error detection capability (that is, the number of errors that the code can detect).

We denote by T the standard shift operator on F_q^n. A (linear) code is said to be quasi-cyclic of index l or l-quasi-cyclic if and only if it is invariant under T^l.

2.2. (Partial) Hasse derivatives

Recall that the uth Hasse derivative ($u = 0, 1, \cdots$) of a polynomial $f(x) = \sum_i a_i x^i \in F_q[x]$ is defined as the polynomial $f^{[u]}(x) = \sum_i \binom{i}{u} a_i x^{i-u}$. Analogously, for a bivariate polynomial $f(x, y) = \sum_{i,j} a_{ij} x^i y^j \in F_q[x, y]$,

the (u,v)th partial Hasse (partial mixed) derivative of f, denoted by $f^{[u,v]}(x,y)$, is defined by

$$f^{[u,v]}(x,y)=\sum_{i,j}\binom{i}{u}\binom{j}{v}a_{ij}x^{i-u}y^{j-v}.$$

Here we use a standard convention for binomial cofficients: $\binom{k}{l}=0$ for all $l>k$, which guarantees that the (u,v)th Hasse derivative is again a polynomial over F_q.

3. Two-dimensional generalized discrete Fourier transform

Let $n=p^am$, where $(m,p)=1$. When $a\geq 1$, n is no longer relatively prime to p, so the classical theory of discrete Fourier transform does not apply to $F_q[x]/\langle x^n-1\rangle$. However, Massey and Serconek [9] introduced a generalized discrete Fourier transform (GDFT) as follows.

Let $\mathbf{c}=\sum_{i=0}^{n-1}c_ix^i\in F_q[x]$, and let ζ be an mth root of unity in some (sufficiently large) extension of F_q. For each $0\leq g\leq p^a-1$ and $0\leq h\leq m-1$, let

$$\hat{c}_{g,h}=\sum_{i=0}^{n-1}\binom{i}{g}c_i\zeta^{h(i-g)}.$$

Note that $\hat{c}_{g,h}=\mathbf{c}^{[g]}(\zeta^h)$.

Then the GDFT of $\mathbf{c}$ can be described in terms of a matrix:

$$\hat{\mathbf{c}}=[\hat{c}_{g,h}]=\begin{bmatrix}\hat{c}_{0,0} & \hat{c}_{0,1} & \cdots & \hat{c}_{0,m-1}\\ \hat{c}_{1,0} & \hat{c}_{1,1} & \cdots & \hat{c}_{1,m-1}\\ \vdots & & & \\ \hat{c}_{p^a-1,0} & \hat{c}_{p^a-1,1} & \cdots & \hat{c}_{p^a-1,m-1}\end{bmatrix}.$$

Motivated by the above definition, we give the following generalization of two-dimensional DFT.

Definition 3.1 *Let $m=p^am'$ and $n=p^bn'$, m' and n' relatively prime to p, and assume that α and β are the m'th root of unity and n'th root of unity in some (sufficiently large) extension of F_q, respectively. Let $\mathbf{c}=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{i,j}x^iy^j\in F_q[x,y]$. The two-dimensional generalized discrete Fourier transform (2D-GDFT, for short) of the bivariate $\mathbf{c}(x,y)$ is a $p^{a+b}\times m'n'$-matrix $\hat{\mathbf{c}}$ whose the rows are indexed by all pairs (g,h), $0\leq g\leq p^a-1$ and $0\leq h\leq p^b-1$, the columns are indexed by all pairs (u,v), $0\leq u\leq m'-1$ and $0\leq v\leq n'-1$, and*

$$\begin{aligned}\hat{c}_{(g,h),(u,v)} &= \mathbf{c}^{[g,h]}(\alpha^u,\beta^v)\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}c_{i,j}\alpha^{u(i-g)}\beta^{v(j-h)}.\end{aligned}$$

To be convenient, we assume that the rows and the columns of the matrix $\hat{\mathbf{c}}$ are ordered lexicographically.

Just as the DFT, the 2D-GDFT enjoys the modulation and translation properties as well as some other nice relations.

Proposition 3.2 *If* $\mathbf{c} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j}x^iy^j \leftrightarrow \hat{\mathbf{c}} = [\hat{c}_{(g,h),(u,v)}]$ *is a 2D-GDFT pair, then the following are 2D-GDFT pairs:*

(1) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \alpha^{il} c_{i,j}x^iy^j \leftrightarrow [\alpha^{gl}\hat{c}_{(g,h),(l+u,v)}]$,

(2) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \beta^{jk} c_{i,j}x^iy^j \leftrightarrow [\beta^{hk}\hat{c}_{(g,h),(u,k+v)}]$,

(3) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j-l}x^iy^j \leftrightarrow [\sum_{k=0}^{l}\binom{l}{k}\beta^{v(l-k)}\hat{c}_{(g,h-k),(u,v)}]$,

(4) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i-l,j}x^iy^j \leftrightarrow [\sum_{k=0}^{l}\binom{l}{k}\alpha^{u(l-k)}\hat{c}_{(g-k,h),(u,v)}]$,

(5) $\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{l-i,j}x^iy^j \leftrightarrow [\sum_{k=0}^{l}\sum_{r=0}^{g-k-1}(-1)^{g-k}\binom{l}{k}\binom{g-k-1}{r}\alpha^{u(-2g+l+k)+r}\hat{c}_{(g-k-r,h),(-u,v)}]$,

where $k, l \geq 0$ *are integers and all indices are calculated modulo appropriate* $t \in \{m, n, p^a, p^b\}$.

Proof Let $\mathbf{c}' = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\alpha^{il}c_{i,j}x^iy^j$. Then

$$\begin{aligned}
\hat{c'}_{(g,h),(u,v)} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}\alpha^{il}c_{i,j}\alpha^{u(i-g)}\beta^{v(j-h)} \\
&= \alpha^{gl}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}c_{i,j}\alpha^{(u+l)(i-g)}\beta^{v(j-h)} \\
&= \alpha^{gl}\hat{c}_{(g,h),(u+l,v)}.
\end{aligned}$$

The proof of the second equality is similar to (1). To prove (3) (and similarly (4)), let $\mathbf{s} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j-l}x^iy^j$. Then

$$\begin{aligned}
\hat{s}_{(g,h),(u,v)} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}c_{i,j-l}\alpha^{u(i-g)}\beta^{v(j-h)} \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\left(\sum_{k=0}^{l}\binom{l}{k}\binom{j-l}{h-k}\right)c_{i,j-l}\alpha^{u(i-g)}\beta^{v(j-h)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{l}\binom{l}{k}\left(\sum_{i=0}^{m-1}\sum_{r=0}^{n-1}\binom{i}{g}\binom{r}{h-k}c_{i,r}\alpha^{u(i-g)}\beta^{v(r+l-h)}\right)\\
&= \sum_{k=0}^{l}\binom{l}{k}\left(\sum_{i=0}^{m-1}\sum_{r=0}^{n-1}\binom{i}{g}\binom{r}{h-k}c_{i,r}\alpha^{u(i-g)}\beta^{v(r-(h-k))}\right)\beta^{v(l-k)}\\
&= \sum_{k=0}^{l}\binom{l}{k}\beta^{v(l-k)}\hat{c}_{(g,h-k),(u,v)},
\end{aligned}
$$

showing that the translation property holds.

Finally, let $\mathbf{w}=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{l-i,j}x^iy^j$. Then

$$
\begin{aligned}
\hat{w}_{(g,h),(u,v)} &= \sum_{i,j}\binom{i}{g}\binom{j}{h}c_{l-i,j}\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{i,j}\binom{j}{h}(\sum_{k=0}^{l}\binom{l}{k}\binom{i-l}{g-k})c_{l-i,j}\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}\binom{l}{k}\sum_{i,j}\binom{i-l}{g-k}\binom{j}{h}c_{l-i,j}\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}\binom{l}{k}\sum_{s,j}\binom{-s}{g-k}\binom{j}{h}c_{s,j}\alpha^{u(l-s-g)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}\binom{l}{k}\sum_{s,j}(-1)^{g-k}\binom{s+g-k-1}{g-k}\binom{j}{h}c_{s,j}\alpha^{u(l-s-g)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}(-1)^{g-k}\binom{l}{k}\sum_{t,j}\binom{t}{g-k}\binom{j}{h}c_{t-g+k+1,j}\alpha^{u(l-t-k-1)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}(-1)^{g-k}\binom{l}{k}\alpha^{u(l-g-1)}\sum_{t,j}\binom{t}{g-k}\binom{j}{h}c_{t-g+k+1,j}\alpha^{-u(t-g+k)}\beta^{v(j-h)}\\
&= \sum_{k=0}^{l}(-1)^{g-k}\binom{l}{k}\alpha^{u(l-g-1)}\sum_{r=0}^{g-k-1}\binom{g-k-1}{r}\alpha^{-u(g-k-1-r)}\hat{c}_{(g-k-r,h),(-u,v)}\\
&= \sum_{k=0}^{l}\sum_{r=0}^{g-k-1}(-1)^{g-k}\binom{l}{k}\binom{g-k-1}{r}\alpha^{u(-2g+l+k)+r}\hat{c}_{(g-k-r,h),(-u,v)},
\end{aligned}
$$

which proves (5). □

Corollary 3.3 *If* $\mathbf{c}=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{i,j}x^iy^j \leftrightarrow \hat{\mathbf{c}}=[\hat{c}_{(g,h),(u,v)}]$ *is a 2D-GDFT pair, then, for any* $l,k\geq 0$, *the following is a 2D-GDFT pair:*

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i-l,j-k}x^iy^j \leftrightarrow [\sum_{r=0}^{l}\sum_{s=0}^{k}\binom{l}{r}\binom{k}{s}\alpha^{u(l-r)}\beta^{v(k-s)}\hat{c}_{(g-r,h-s),(u,v)}].$$

Let $A=[a_{ij}]$ and $B=[b_{ij}]$ be $m\times n$-matrices over F_q. The convolution product $A\star B$ is defined as an $m\times n$-matrix C whose

$$C_{i,j}=\sum_{l=0}^{m-1}\sum_{k=0}^{n-1}A_{i-l,j-k}B_{lk},$$

where the indices are calculated modulo appropriate $t\in\{m,n\}$. The following theorem describes what the 2D-GDFT will do with the convolution product.

Theorem 3.4 *If* $\mathbf{c}\leftrightarrow\hat{\mathbf{c}}$ *and* $\mathbf{d}\leftrightarrow\hat{\mathbf{d}}$ *are 2D-GDFT pairs, then* $\mathbf{e}=\mathbf{c}\star\mathbf{d}\leftrightarrow\hat{\mathbf{e}}$ *is a 2D-GDFT pair, where for each* $0\le g\le p^a-1$, $0\le h\le p^b-1$, $0\le u\le m'-1$, *and* $0\le v\le n'-1$,

$$\hat{e}_{(g,h),(u,v)}=\sum_{r=0}^{g}\sum_{s=0}^{h}\hat{c}_{(g-r,h-s),(u,v)}\hat{d}_{(r,s),(u,v)}.$$

Proof By definition, we have

$$\begin{aligned}
\hat{e}_{(g,h),(u,v)} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}e_{i,j}\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\binom{i}{g}\binom{j}{h}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{n-1}c_{i-l,j-k}d_{l,k}\right)\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sum_{l=0}^{m-1}\sum_{k=0}^{n-1}\binom{i}{g}\binom{j}{h}c_{i-l,j-k}d_{l,k}\alpha^{u(i-g)}\beta^{v(j-h)}\\
&= \sum_{i,j,l,k}\left(\sum_{r=0}^{g}\binom{l}{r}\binom{i-l}{g-r}\right)\left(\sum_{s=0}^{h}\binom{k}{s}\binom{j-k}{h-s}\right)c_{i-l,j-k}d_{l,k}\alpha^{u(i-g)}\beta^{v(j-h)}
\end{aligned}$$

$$=\sum_{l,k}\sum_{r,s}\binom{l}{r}\binom{k}{s}\left(\sum_{i,j}\binom{i-l}{g-r}\binom{j-k}{h-s}c_{i-l,j-k}\alpha^{u(i-l-g+r)}\beta^{v(j-k+s-h)}\right)d_{l,k}\alpha^{u(l-r)}\beta^{v(k-s)}$$

$$\begin{aligned}
&= \sum_{l=0}^{m-1}\sum_{r=0}^{g}\sum_{k=0}^{n-1}\sum_{s=0}^{h}\binom{l}{r}\binom{k}{s}\hat{c}_{(g-r,h-s),(u,v)}d_{l,k}\alpha^{u(l-r)}\beta^{v(k-s)}\\
&= \sum_{r=0}^{g}\sum_{s=0}^{h}\hat{c}_{(g-r,h-s),(u,v)}\hat{d}_{(r,s),(u,v)},
\end{aligned}$$

as we claimed. □

4. 2D-GDFT is invertible

In this section, we are going to describe the inverse 2D-GDFT clearly. For each $0 \leq i \leq p^a-1$ and $0 \leq g \leq p^b-1$, let

$$\mathbf{c}_{(i,g)}(x,y) = \sum_{r=0}^{m'-1} \sum_{s=0}^{n'-1} c_{i+rp^a,g+sp^b} X^r Y^s.$$

Let $\lambda = \alpha^{p^a}$ and $\mu = \beta^{p^b}$, so that λ and μ are again m'th and n'th roots of unity, respectively. By the classical two-dimensional DFT (with λ and μ as the chosen m'th and n'th roots of unity), we have

$$c_{i+rp^a,g+sp^b} = \frac{1}{m'n'} \sum_{u=0}^{m'-1} \sum_{v=0}^{n'-1} \mathbf{c}_{(i,g)}(\lambda^u, \mu^v)(\lambda^{-r})^u(\mu^{-s})^v.$$

Definition 4.1 *The partial Hasse matrix $H(X,Y)$ is the $p^{a+b} \times p^{a+b}$-matrix whose rows and columns are indexed (and ordered lexicographically) by all pairs (r,s), $0 \leq r \leq p^a-1$ and $0 \leq s \leq p^b-1$, and the $(i,g),(j,h)$ th entry is $\binom{j}{i}\binom{h}{g}X^{j-i}Y^{h-g}$ (this is the (i,g) th partial Hasse derivative of the monomial X^jY^h in $F_q[X,Y]$).*

By definition, we have

$$\begin{aligned}\Big(H(X,Y)H(-X,-Y)\Big)_{(i,g),(j,h)} &= \sum_{k=0}^{p^a-1}\sum_{l=0}^{p^b-1}\binom{k}{i}\binom{l}{g}X^{k-i}Y^{l-g}\binom{j}{k}\binom{h}{l}(-X)^{j-k}(-Y)^{h-l} \\ &= X^{j-i}Y^{h-g}\Big(\sum_k \binom{k}{i}\binom{j}{k}(-1)^{j-k}\Big)\Big(\sum_l \binom{l}{g}\binom{h}{l}(-1)^{h-l}\Big)\end{aligned}$$

$$= \binom{j}{i}\binom{h}{g}X^{j-i}Y^{h-g}\Big(\sum_k (-1)^{j-k}\binom{j-i}{j-k}\Big)\Big(\sum_l (-1)^{h-l}\binom{h-g}{h-l}\Big).$$

Now, from the binomial expansion

$$(1-1)^w = \sum_{u \leq v} \binom{w}{u}(-1)^u = 0,$$

applied to the off-diagonal terms in the product $H(X,Y)H(-X,-Y)$, we see that the inverse of the partial Hasse matrix $H(X,Y)$ is $H(-X,-Y)$.

Before going on, we need the following simple lemma.

Lemma 4.2 *Let $q = p^m$ be a prime power and F_q be a field of order q. For each $i,a,b,c \geq 0$ we have*

$$\binom{a}{i} = \binom{a+bp^c}{i},$$

where $\binom{a}{i}$ and $\binom{a+bp^c}{i}$ are interpreted as integers of the field F_q.

Proof Just note that the field F_q has characteristics p. Hence, $a+bp^c$ equals a when all the quantities involved are integers. Thus, the result is obvious. □

Using the previous lemma, we can write

$$\begin{aligned}
\sum_{j,h}\binom{j}{i}\binom{h}{g}\alpha^{u(j-i)}\beta^{v(h-g)}\mathbf{c}_{(j,h)}(\lambda^u,\mu^v) &= \sum_{j,h}\sum_{r,s}\binom{j}{i}\binom{h}{g}c_{j+rp^a,h+sp^b}\alpha^{u(j+rp^a-i)}\beta^{v(h+sp^b-g)}\\
&= \sum_{h,s}\left(\sum_{j,r}\binom{j+rp^a}{i}c_{j+rp^a,h+sp^b}\alpha^{u(j+rp^a-i)}\right)\binom{h}{g}\beta^{v(h+sp^b-g)}\\
&= \sum_{h,s}\left(\sum_{k=0}^{m-1}\binom{k}{i}c_{k,h+sp^b}\alpha^{u(k-i)}\right)\binom{h}{g}\beta^{v(h+sp^b-g)}\\
&= \sum_{k=0}^{m-1}\binom{k}{i}\alpha^{u(k-i)}\left(\sum_{h,s}\binom{h+sp^b}{g}c_{k,h+sp^b}\beta^{v(h+sp^b-g)}\right)\\
&= \sum_{k=0}^{m-1}\binom{k}{i}\alpha^{u(k-i)}\left(\sum_{l=0}^{n-1}\binom{l}{g}c_{k,l}\beta^{v(l-g)}\right)\\
&= \sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\binom{k}{i}\binom{l}{g}c_{k,l}\alpha^{u(k-i)}\beta^{v(l-g)}\\
&= \hat{c}_{(i,g),(u,v)}.
\end{aligned}$$

Hence, we have

$$H(\alpha^u,\beta^v)\begin{bmatrix}
\mathbf{c}_{(0,0)}(\lambda^u,\mu^v)\\
\mathbf{c}_{(0,1)}(\lambda^u,\mu^v)\\
\vdots\\
\mathbf{c}_{(0,p^b-1)}(\lambda^u,\mu^v)\\
\mathbf{c}_{(1,0)}(\lambda^u,\mu^v)\\
\mathbf{c}_{(1,1)}(\lambda^u,\mu^v)\\
\vdots\\
\mathbf{c}_{(1,p^b-1)}(\lambda^u,\mu^v)\\
\vdots\\
\mathbf{c}_{(p^a-1,0)}(\lambda^u,\mu^v)\\
\mathbf{c}_{(p^a-1,1)}(\lambda^u,\mu^v)\\
\vdots\\
\mathbf{c}_{(p^a-1,p^b-1)}(\lambda^u,\mu^v)
\end{bmatrix}=\begin{bmatrix}
\hat{c}_{(0,0),(u,v)}\\
\hat{c}_{(0,1),(u,v)}\\
\vdots\\
\hat{c}_{(0,p^b-1),(u,v)}\\
\hat{c}_{(1,0),(u,v)}\\
\vdots\\
\hat{c}_{(1,p^b-1),(u,v)}\\
\vdots\\
\hat{c}_{(p^a-1,0),(u,v)}\\
\hat{c}_{(p^a-1,1),(u,v)}\\
\vdots\\
\hat{c}_{(p^a-1,p^b-1),(u,v)}
\end{bmatrix}.$$

Since the partial Hasse matrix is invertible, the above equality can be rewritten as

$$\begin{bmatrix} \mathbf{c}_{(0,0)}(\lambda^u,\mu^v) \\ \mathbf{c}_{(0,1)}(\lambda^u,\mu^v) \\ \vdots \\ \mathbf{c}_{(0,p^b-1)}(\lambda^u,\mu^v) \\ \mathbf{c}_{(1,0)}(\lambda^u,\mu^v) \\ \mathbf{c}_{(1,1)}(\lambda^u,\mu^v) \\ \vdots \\ \mathbf{c}_{(1,p^b-1)}(\lambda^u,\mu^v) \\ \vdots \\ \mathbf{c}_{(p^a-1,0)}(\lambda^u,\mu^v) \\ \mathbf{c}_{(p^a-1,1)}(\lambda^u,\mu^v) \\ \vdots \\ \mathbf{c}_{(p^a-1,p^b-1)}(\lambda^u,\mu^v) \end{bmatrix} = H(-\alpha^u,-\beta^v) \begin{bmatrix} \hat{c}_{(0,0),(u,v)} \\ \hat{c}_{(0,1),(u,v)} \\ \vdots \\ \hat{c}_{(0,p^b-1),(u,v)} \\ \hat{c}_{(1,0),(u,v)} \\ \vdots \\ \hat{c}_{(1,p^b-1),(u,v)} \\ \vdots \\ \hat{c}_{(p^a-1,0),(u,v)} \\ \hat{c}_{(p^a-1,1),(u,v)} \\ \vdots \\ \hat{c}_{(p^a-1,p^b-1),(u,v)} \end{bmatrix}.$$

Consequently,

$$c_{i+rp^a,g+sp^b} = \frac{1}{m'n'} \sum_{u=0}^{m'-1} \sum_{v=0}^{n'-1} \left(\sum_{j=0}^{p^a-1} \sum_{h=0}^{p^b-1} \binom{j}{i} \binom{h}{g} (-\alpha^u)^{j-i} (-\beta^v)^{h-g} \hat{c}_{(j,h),(u,v)} \right) (\lambda^{-r})^u (\mu^{-s})^v.$$

Therefore, the 2D-GDFT is invertible.

5. A family of quasi-cyclic codes

Reed–Solomon codes (RS codes) are a class of error-correcting cyclic codes proposed by Reed and Solomon in their original paper [10]. RS codes have optimal parameters and can be efficiently decoded [7,11,13].

Considering a vector space of polynomials f such that $f(m) = 0$ for all m in the set $B = \{\alpha^{r_0}, \alpha^{r_0+1}, \cdots, \alpha^{r_0+n-k-1}\}$, we can define an RS code of length n and dimension k over the finite field F_q. Here α can be any element in F_q of multiplicative order at least n where n is a divisor of $q-1$. The key point here is that we can construct the RS codes from another fruitful method, the DFT approach ([2], Section 6), which enables us to introduce our generalization of such codes.

Definition 5.1 *Let $d \geq 2$, $m = p^a m'$, and $n = p^b n'$, where $a, b \geq 0$ are integers and m', n' are relatively prime to p. Consider the subspace $C^\star$ consisting of all matrices $\mathbf{c} \in (F_q)^{m \times n}$ whose $\hat{c}_{(g,h),(u,v)} = 0$ for all pairs (g,h) and (u,v) in which $0 \leq v \leq n'-2$. A generalized RS code C of block length mn over F_q, denoted $GRS_{m,n,d}$, will be defined as the set of all words $\mathbf{c} \in C^\star$ whose $\hat{c}_{(g,h),(u,n'-1)} = 0$ for all pairs (g,h) and all pairs $(u,n'-1)$ in which u belongs to a specified block of $d-1$ consecutive integers, denoted $\{z_0, z_0+1, ..., z_0+d-2\}$, i.e. $0 \leq z_0 \leq u \leq z_0+d-2 \leq m'-1$.*

Note that, by definition, we obtain a code whose elements are matrices, which can be viewed as vectors of length mn, by reading them column by column. It is easy to verify that $\mathrm{GRS}_{m,n,d}$ is an $[mn, p^{a+b}(m'-d+1)]$-linear code.

In the following, $\mathfrak{B}_{z_0,d}$ stands for the set

$$\{(u,n'-1) \mid z_0 \leq u \leq z_0+d-2\} \cup \{(u,v) \mid 0 \leq u \leq m'-1 \ , \ 0 \leq v \leq n'-2\}$$

and will be called the defining set of the code $\mathrm{GRS}_{m,n,d}$.

Proposition 5.2 *The code $GRS_{m,n,d}$ is a quasi-cyclic code of index m.*

Proof Suppose that $\mathbf{c} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j}x^iy^j$ is a word of $\mathrm{GRS}_{m,n,d}$. Hence, $\hat{c}_{(g,h),(u,v)} = 0$ for all pairs (g,h) and for each pair $(u,v) \in \mathfrak{B}_d$. Thus,

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\beta^{v(n-1-k)}\hat{c}_{(g,h-k),(u,v)} = 0$$

for all pairs (g,h) and for each pair $(u,v) \in \mathfrak{B}_d$. Therefore, by proposition 3.2(3), the 2D-GDFT of the word $\mathbf{c}' = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j-1}x^iy^j$ is equal to zero in those columns (u,v) in which $(u,v) \in \mathfrak{B}_d$, proving that $\mathbf{c}'$ is a word of $\mathrm{GRS}_{m,n,d}$, as desired. □

Next, the minimum distance of the code $\mathrm{GRS}_{m,n,d}$ is going to be discussed.

Proposition 5.3 *The minimum distance of the code $GRS_{m,n,d}$ satisfies*

$$n'd \le d_{\min}(\mathrm{GRS}_{m,n,d}) \le p^{a+b}(m'n' - m' + d - 1) + 1.$$

Proof Without loss of generality, we can suppose $z_0 = m' - d + 1$. Otherwise, use proposition 3.2(1) to translate the defining set $\mathfrak{B}_{z_0,d}$ to $\mathfrak{B}_{m'-d+1,d}$, thereby multiplying each codeword component by a power of α, which does not change the weight of a codeword because components that were nonzero remain nonzero. Suppose that $\mathbf{c} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} c_{i,j}x^iy^j$ is a nonzero word of $\mathrm{GRS}_{m,n,d}$. For any $0 \le i \le p^a - 1$ and $0 \le g \le p^b - 1$, let

$$C_{(i,g)}(x) = \sum_{u=0}^{m'-1} \mathbf{c}_{(i,g)}(\lambda^u, \mu^{n'-1})x^u,$$

where $\mathbf{c}_{(i,g)}$, λ, and μ are defined as in Section 4. Recall that

$$\mathbf{c}_{(i,g)}(\lambda^u, \mu^v) = \sum_{k=0}^{p^a-1}\sum_{l=0}^{p^b-1}\binom{k}{i}\binom{l}{g}(-\alpha^u)^{k-i}(-\beta^v)^{l-g}\hat{c}_{(k,l),(u,v)}.$$

On the other hand, $\hat{c}_{(k,l),(u,v)} = 0$ for all $0 \le k \le p^a - 1$, $0 \le l \le p^b - 1$, and $(u,v) \in \mathfrak{B}_{m'-d+1,d}$, showing that $\mathbf{c}_{(i,g)}(\lambda^u, \mu^v) = 0$ for each pair $(u,v) \in \mathfrak{B}_{m'-d+1,d}$. Therefore, the polynomial $C_{(i,g)}(x)$ is either zero or has degree at most $m' - d$. Since $\mathbf{c} \neq 0$, we can find a nonzero polynomial $C_{(i,g)}(x)$ for some $0 \le i \le p^a - 1$ and $0 \le g \le p^b - 1$. Some of the components of the codeword $\mathbf{c}$ are $c_{i+rp^a,g+sp^b} = \frac{(\mu^{-s})^{n'-1}}{m'n'}C_{(i,g)}(\lambda^{-r})$, $r = 0, \cdots, m' - 1$, and $s = 0, \cdots, n' - 1$. Since $C_{(i,g)}(x)$ is a polynomial of degree at most $m' - d$, it can have at most $m' - d$ zeros. Hence, for any $0 \le s \le n' - 1$, there will be at least d index r such that $c_{i+rp^a,g+sp^b} \neq 0$. Consequently, $w(\mathbf{c}) \ge td$ where t is the number of those pairs (i,g) whose $C_{(i,g)}(x) \neq 0$. Thus, $d_{\min}(\mathrm{GRS}_{m,n,d}) \ge m'n' - (m' - d)n' = n'd$. The right side of the inequality will be obtained from the Singleton bound for linear codes. This completes the proof. □

Example 5.4 *Let $q=4$, $m=6$, and $n=5$. Choosing α^5 and α^3 as the fifth and third roots of unity in the Galois field $F_4=\{a_0+a_1\alpha+a_2\alpha^2+a_3\alpha^3 \mid a_i\in F_2,\ \alpha^4=\alpha+1\}$, the 2D-GDFT of a bivariate polynomial $\mathbf{c}=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}c_{i,j}x^iy^j$ is given by the matrix $\hat{\mathbf{c}}$ whose*

$$\begin{aligned}\hat{c}_{(g,h),(u,v)} &= \mathbf{c}^{[g,h]}(\alpha^{5u},\alpha^{3v})\\ &= \sum_{i=0}^{5}\sum_{j=0}^{4}\binom{i}{g}\binom{j}{h}c_{i,j}\alpha^{5u(i-g)+3v(j-h)}.\end{aligned}$$

Now, let $d=1$. Then the code $GRS_{6,5,1}$ is a linear $[30,9]$-quasi-cyclic code of minimum distance 16 (http://www.codetables.de). This shows that good quasi-cyclic codes can be constructed via our algebraic approach, as in [9], where such codes have been constructed using integer linear programming and a heuristic combinatorial optimization algorithm based on a greedy local search.

6. Conclusion

We generalized and studied the 2D-GDFT, which enables us to apply the powerful concept of 2D-DFT on data matrices for which the number of rows or columns is not necessarily coprime with the field characteristic. Our generalized 2D-DFT enjoys the basic properties of the original one. As an application, we introduced a family of quasi-cyclic linear codes, denoted by $\mathrm{GRS}_{m,n,d}$, which are a natural generalization of the classical Reed–Solomon codes, and the code parameters were described.

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