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Reduction formula of a double binomial sum

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Abstract: A class of double sums with binomial coefficients are evaluated by combining finite differences with partial fraction decompositions.

Key words: Binomial coefficient, finite difference, partial fraction decomposition, telescoping method

1. Introduction and motivation

There has been always a constant interest in finding closed formulae of binomial sums, including double ones (for example, Chu [8]). In the process of evaluating the quadratic moments of binomial products (cf. Chu [9] and Miana–Romero[12]), we encountered the following double sum with the closed formula being detected by *Mathematica* commands:

$$\sum_{j=0}^n \binom{2m-2j}{m-j} \sum_{i=0}^j \binom{2m}{i} \binom{2j-2m}{j-i} \frac{(m-i)^{2n+1}}{m-j} = 0, \quad (1)$$

where m and n are natural numbers with $m > n$ in order to avoid the appearance of zero in denominators.

For an integer k and an indeterminate τ , define the rising and falling factorials respectively by

$$(\tau)_k = \frac{\Gamma(\tau+k)}{\Gamma(\tau)} \quad \text{and} \quad \langle \tau \rangle_k = \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-k)}. \quad (2)$$

Writing the binomial coefficients in terms of shifted factorials

$$\begin{aligned} \binom{2m-2j}{m-j} \frac{1}{m-j} &= \frac{(2m-2j)!}{(m-j)(m-j)!^2} = \frac{2\langle m \rangle_j^2}{\langle 2m \rangle_{2j+1}} \binom{2m}{m}, \\ \binom{2m}{i} &= \frac{\langle 2m \rangle_i}{i!}, \quad \binom{2j-2m}{j-i} = \frac{(-1)^{j-i}}{(j-i)!} \langle 2m-i-j-1 \rangle_{j-i}; \end{aligned}$$

we can reformulate the following binomial product

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$$\begin{aligned} \frac{1}{m-j} \binom{2m-2j}{m-j} \binom{2m}{i} \binom{2j-2m}{j-i} &= 2(-1)^{j-i} \binom{2m}{m} \frac{\langle m \rangle_j \langle m \rangle_j \langle 2m \rangle_i \langle 2m-i-j-1 \rangle_{j-i}}{i!(j-i)! \langle 2m \rangle_{2j+1}} \\ &= 2(-1)^{j-i} \binom{2m}{m} \frac{\langle m \rangle_j \langle m \rangle_j}{i!(j-i)! \langle 2m-i \rangle_{j+1}} \\ &= 2(-1)^{j-i} \binom{2m}{m} \binom{m}{j} \binom{j}{i} \frac{\langle m \rangle_j}{\langle 2m-i \rangle_{j+1}} \end{aligned}$$

that can be used to express the double sum in question equivalently as

$$\sum_{j=0}^n (-1)^j \binom{m}{j} \sum_{i=0}^j (-1)^i \binom{j}{i} \frac{\langle m \rangle_j}{\langle 2m-i \rangle_{j+1}} (m-i)^{2n+1} = 0. \tag{3}$$

Replacing further the integer parameter m by an indeterminate $(-x)$ and introducing an extra integer parameter λ , we shall investigate the following double sum

$$\Omega(\lambda, n) := \sum_{j=0}^n \binom{x+j-1}{j} \sum_{i=0}^j (-1)^i \binom{j}{i} \frac{(x)_j}{(2x+i)_{j+1}} (x+i)^{2\lambda+1}. \tag{4}$$

It turns out that $\Omega(\lambda, n)$ is identical to zero for $0 < \lambda \leq n$ and a polynomial in x when $\lambda > n \geq 0$. This will be accomplished by combining finite differences (cf. Boole [1, Chapter 2]) with partial fraction decompositions (cf. Chu [4]).

2. Main theorem and proof

Rewriting the binomial coefficients

$$\binom{x+j-1}{j} \binom{j}{i} = \binom{x+i-1}{i} \binom{x+j-1}{j-i}$$

and interchanging the order of double sums, we can state $\Omega(\lambda, n)$ equivalently as

$$\begin{aligned} \Omega(\lambda, n) &= \sum_{i=0}^n (-1)^i \binom{x+i-1}{i} (x+i)^{2\lambda+1} \\ &\quad \times \sum_{j=i}^n \binom{x+j-1}{j-i} \frac{(x)_j}{(2x+i)_{j+1}}. \end{aligned} \tag{5}$$

For the sequence σ_j defined below, it is trivial to check its difference

$$\sigma_j = \frac{(x+i)_{j-i} (x)_j}{\Gamma(j-i) (2x+i)_j} \text{ and } \sigma_{j+1} - \sigma_j = (x+i)^2 \binom{x+j-1}{j-i} \frac{(x)_j}{(2x+i)_{j+1}}.$$

In view of the fact that “ $\frac{1}{\Gamma(0)} = 0$ ”, the inner sum with respect to j can be evaluated by telescoping (cf. [7, 13])

$$\begin{aligned} \sum_{j=i}^n \binom{x+j-1}{j-i} \frac{(x)_j}{(2x+i)_{j+1}} &= \sum_{j=i}^n \frac{\sigma_{j+1} - \sigma_j}{(x+i)^2} = \frac{\sigma_{n+1}}{(x+i)^2} \\ &= \frac{(x+i)_{1+n-i} (x)_{n+1}}{(x+i)^2 (n-i)! (2x+i)_{n+1}}. \end{aligned}$$

Substituting this equality into (5), we reduce it, after some simplification, to the following single sum.

Lemma 1 For the two natural numbers λ and n , there holds the following identity:

$$\Omega(\lambda, n) = \frac{(x)_{n+1}^2}{n!} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(x+i)^{2\lambda-1}}{(2x+i)_{n+1}}.$$

Now we are in a position to prove the following interesting theorem, which confirms, in particular for $\lambda = n$, the double sum identities (1) and (3).

Theorem 2 Let n and λ be two natural numbers subject to $0 < \lambda \leq n$. Then, for the double sum defined by (4), we have the following identity $\Omega(\lambda, n) = 0$.

Proof For the rational function in the variable i , by decomposing it into partial fractions (cf. [2, 3])

$$\frac{(x+i)^\lambda}{(2x+i)_{n+1}} = \frac{(-1)^\lambda}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(x+j)^\lambda}{2x+i+j}, \tag{6}$$

we can express the sum in Lemma 1 as the following double sums

$$\Omega(\lambda, n) = \frac{(x)_{n+1}^2}{(n!)^2} \sum_{i=0}^n \sum_{j=0}^n (-1)^{i+j+\lambda} \binom{n}{i} \binom{n}{j} \frac{(x+i)^{\lambda-1} (x+j)^\lambda}{2x+i+j}, \tag{7}$$

$$\Omega(\lambda, n) = \frac{(x)_{n+1}^2}{(n!)^2} \sum_{i=0}^n \sum_{j=0}^n (-1)^{i+j+\lambda} \binom{n}{i} \binom{n}{j} \frac{(x+i)^\lambda (x+j)^{\lambda-1}}{2x+i+j}, \tag{8}$$

where the last one is justified by interchanging the summation indices i and j . Adding these two equalities together, we derive the following symmetric expression

$$\begin{aligned} 2\Omega(\lambda, n) &= \frac{(x)_{n+1}^2}{(n!)^2} \sum_{i=0}^n \sum_{j=0}^n (-1)^{i+j+\lambda} \binom{n}{i} \binom{n}{j} (x+i)^{\lambda-1} (x+j)^{\lambda-1} \\ &= (-1)^\lambda \frac{(x)_{n+1}^2}{(n!)^2} \left\{ \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} (x+i)^{\lambda-1} \right\}^2. \end{aligned}$$

The rightmost sum vanishes because it results in the n th differences of a polynomial with degree $\lambda - 1$ less than n . This completes the proof of Theorem 2. □

3. Convolution expression

When $\lambda = 0$, the last sum can be evaluated by (6) as

$$\sum_{i=0}^n \binom{n}{i} \frac{(-1)^i}{x+i} = \frac{n!}{(x)_{n+1}}.$$

From this formula, we can retrieve the respective particular case $\lambda = 0$ for both Lemma 1 and Theorem 2

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{(x+i)(2x+i)_{n+1}} = \frac{n!}{2(x)_{n+1}^2}, \tag{9}$$

$$\sum_{j=0}^n \binom{x+j-1}{j} \sum_{i=0}^j (-1)^i \binom{j}{i} \frac{(x+i)(x)_j}{(2x+i)_{j+1}} = \frac{1}{2}. \tag{10}$$

When $\lambda > n$, we need the following equality, instead of (6)

$$\frac{(x+i)^n}{(2x+i)_{n+1}} = \frac{(-1)^n}{n!} \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(x+j)^n}{2x+i+j}.$$

Substituting this into the equation displayed in Lemma 1, we have

$$\begin{aligned} 2\Omega(\lambda, n) &= (-1)^n \frac{(x)_{n+1}^2}{(n!)^2} \sum_{i,j=0}^n (-1)^{i+j} \binom{n}{i} \binom{n}{j} \\ &\times \frac{(x+i)^{2\lambda-n-1}(x+j)^n + (x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j}. \end{aligned}$$

Rewriting the last fraction by

$$\begin{aligned} &\frac{(x+i)^{2\lambda-n-1}(x+j)^n + (x+j)^{2\lambda-n-1}(x+i)^n}{2x+i+j} \\ &= (x+i)^n(x+j)^n \frac{(x+i)^{2\lambda-2n-1} + (x+j)^{2\lambda-2n-1}}{(x+i) + (x+j)} \\ &= \sum_{k=1+n-\lambda}^{\lambda-n-1} (-1)^{\lambda-n-k-1} (x+i)^{\lambda+k-1} (x+j)^{\lambda-k-1}, \end{aligned}$$

we derive the following polynomial expression.

Proposition 3 *Let n and λ be two natural numbers subject to $\lambda > n$. Then for the double sum defined by (4), we have the following convolution formula*

$$\Omega(\lambda, n) = \frac{(x)_{n+1}^2}{2(n!)^2} \sum_{k=1+n-\lambda}^{\lambda-n-1} (-1)^{\lambda-k-1} P_{\lambda+k-1,n}(x) P_{\lambda-k-1,n}(x),$$

where $P_{m,n}(x)$ is a polynomial defined by

$$P_{m,n}(x) = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} (x+i)^m \quad \text{for } m, n \in \mathbb{N} \quad \text{with } m > n.$$

Observing that $P_{m,n}(x)$ is essentially the n th differences of a monomial with the degree m such that $m > n$, we assert (cf. [5, 6]) that $P_{m,n}(x)$ results in a polynomial of degree $m - n$ in x . In particular, we have $P_{m,n}(0) = n! S(m, n)$, where $S(m, n)$ is the Stirling number of the second kind (cf. [10, §5.1] and [11, §6.1]).

According to Proposition 3, we can evaluate the double sum $\Omega(\lambda, n)$ for $\lambda > n$. The first three examples

are displayed below:

$$\Omega(1+n, n) = (-1)^n \frac{(x)_{n+1}^2}{2}, \tag{11}$$

$$\Omega(2+n, n) = (-1)^n \frac{(n+1)(x)_{n+1}^2}{12} \{2n^2 + 6xn + n + 6x^2\}, \tag{12}$$

$$\begin{aligned} \Omega(3+n, n) = (-1)^n \frac{(n+1)(n+2)(x)_{n+1}^2}{720} \{ & 180x^2(n+x)^2 \\ & + 120nx(n+1)(n+x) + n(2n+1)(2n+3)(5n-1) \}. \end{aligned} \tag{13}$$

In view of Lemma 1, they correspond to the following binomial identities:

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(x+i)^{2n+1}}{(2x+i)_{n+1}} = (-1)^n \frac{n!}{2}, \tag{14}$$

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(x+i)^{2n+3}}{(2x+i)_{n+1}} = (-1)^n \frac{(n+1)!}{12} \{2n^2 + 6xn + n + 6x^2\}, \tag{15}$$

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(x+i)^{2n+5}}{(2x+i)_{n+1}} = (-1)^n \frac{(n+2)!}{720} \{120nx(n+1)(n+x) \tag{16}$$

$$+ 180x^2(n+x)^2 + n(2n+1)(2n+3)(5n-1)\}. \tag{17}$$

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