

1-1-2003

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Recommended Citation

MAVROMATIS, HARRY A. (2003) "Clebsch-Gordan Equalities that imply the Vanishing of Particular $6j$ Symbols," *Turkish Journal of Physics*: Vol. 27: No. 3, Article 5. Available at: <https://journals.tubitak.gov.tr/physics/vol27/iss3/5>

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Clebsch-Gordan Equalities that imply the Vanishing of Particular $6j$ Symbols

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Received 08.01.2003

Abstract

Three Clebsch-Gordan equalities and three individual Clebsch-Gordan relations are obtained by studying the non-accidental vanishing of certain $6j$ symbols.

Key Words: Clebsch-Gordan coefficients, $6j$ symbols, Regge symmetries, angular momenta.

1. Introduction

In two interesting papers [1, 2] involving nuclear shell model calculations, Robinson and Zamick show that certain $6j$ and $9j$ symbols vanish and that this vanishing is not accidental. In particular, they point out that

$$\left\{ \begin{matrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{matrix} \right\} = 0,$$

for both integer and half integer values of j . By using Regge [3] symmetries they also show this particular $6j$ is isomorphic with other $6j$ symbols which are therefore also zero.

2. Formalism

In the present note we examine what the result

$$\left\{ \begin{matrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{matrix} \right\} = 0$$

implies at the level of Clebsch-Gordan coefficients and show that unexpected equalities, involving these coefficients, arise.

The $6j$ coefficient

$$\left\{ \begin{matrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{matrix} \right\}$$

is related to the sum over four Clebsch-Gordan coefficients as follows:

$$\begin{aligned} & \left\{ \begin{matrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{matrix} \right\} = \frac{1}{\sqrt{(4j-5)(4j-1)}} \\ & \times \sum_{m_a, m_b} (j \ j \ m_a \ m_b \mid 2j-3 \ m_a+m_b) (2j-3 \ j \ m_a+m_b \ M-m_a-m_b \mid 3j-4 \ M) \\ & \times (j \ j \ m_b \ M-m_a-m_b \mid 2j-1 \ M-m_a) (j \ 2j-1 \ m_a \ M-m_a \mid 3j-4 \ M), \end{aligned} \quad (1)$$

where $-|3j-4| \leq M \leq 3j-4$. Since this expression is independent of M , in what follows we choose $M = 3j-4$. To understand why the double sum expression (1) is zero, at first seems a daunting task. For an arbitrary j , $(2j+1)^2$ terms are involved. Thus, for $j = 19/2$ this amounts to 400 terms.

A careful analysis however shows that, independent of j , only six non-zero terms arise in expression (1). The Clebsch-Gordan coefficient $(2j-3 \ j \ m_a+m_b \ 3j-4-m_a-m_b \ | \ 3j-4 \ 3j-4)$ requires that $m_a + m_b$ must equal $2j-3$ or $2j-4$. Otherwise it is zero. This is only possible if $m_a = j, m_b = j-3$; $m_a = j-1, m_b = j-2$; $m_a = j-2, m_b = j-1$; $m_a = j-3, m_b = j$; or $m_a = j, m_b = j-4$; $m_a = j-1, m_b = j-3$; $m_a = j-2, m_b = j-2$; $m_a = j-3, m_b = j-1$; and $m_a = j-4, m_b = j$.

Thus, there are only nine combinations that in principle can contribute to expression (1). Of these, two are accidentally zero, namely $m_a = j-2, m_b = j-2$, since $(j \ j \ j-2 \ j-2 \ | \ 2j-3 \ 2j-4)$ is zero, and $m_a = j-2, m_b = j-1$ since $(j \ j \ j-1 \ j-1 \ | \ 2j-1 \ 2j-2)$ is zero, and the combination $m_a = j-4, m_b = j$, is zero since $(j \ j \ j \ j \ | \ 2j-1 \ 2j)$ involves a projection $2j$ for an angular momentum $2j-1$. Thus, for any j , only six pairs (m_a, m_b) contribute, namely: $(j, j-3)$, $(j, j-4)$, $(j-1, j-2)$, $(j-1, j-3)$, $(j-3, j)$, $(j-3, j-1)$.

An analysis of the first two pairs $(j, j-3)$, $(j, j-4)$ in the sum of Eqn. (1) leads to the following expression:

$$\begin{aligned} & [(j \ j \ j \ j-3 \ | \ 2j-3 \ 2j-3) (2j-3 \ j \ 2j-3 \ j-1 \ | \ 3j-4 \ 3j-4) (j \ j \ j-3 \ j-1 \ | \ 2j-1 \ 2j-4) \\ & + (j \ j \ j \ j-4 \ | \ 2j-3 \ 2j-4) (2j-3 \ j \ 2j-4 \ j \ | \ 3j-4 \ 3j-4) (j \ j \ j-4 \ j \ | \ 2j-1 \ 2j-4)] \\ & \times (j \ 2j-1 \ j \ 2j-4 \ | \ 3j-4 \ 3j-4) . \end{aligned} \quad (2)$$

But it can be shown that the product

$$\begin{aligned} & (j \ j \ j \ j-3 \ | \ 2j-3 \ 2j-3) (2j-3 \ j \ 2j-3 \ j-1 \ | \ 3j-4 \ 3j-4) (j \ j \ j-3 \ j-1 \ | \ 2j-1 \ 2j-4) \\ & = -\frac{j}{(4j-3)} \sqrt{\frac{2j-3}{6(j-1)}} \\ & = -(j \ j \ j \ j-4 \ | \ 2j-3 \ 2j-4) (2j-3 \ j \ 2j-4 \ j \ | \ 3j-4 \ 3j-4) (j \ j \ j-4 \ j \ | \ 2j-1 \ 2j-4) . \end{aligned} \quad (3)$$

Hence the sum of the contributions of this pair vanishes.

Similarly, the sum of the pairs $(j-1, j-2)$ and $(j-1, j-3)$ vanishes because of the identity

$$\begin{aligned} & (j \ j \ j-1 \ j-2 \ | \ 2j-3 \ 2j-3) (2j-3 \ j \ 2j-3 \ j-1 \ | \ 3j-4 \ 3j-4) (j \ j \ j-2 \ j-1 \ | \ 2j-1 \ 2j-3) \\ & = \frac{\sqrt{j(2j-3)}}{2(4j-3)} \\ & = -(j \ j \ j-1 \ j-3 \ | \ 2j-3 \ 2j-4) (2j-3 \ j \ 2j-4 \ j \ | \ 3j-4 \ 3j-4) (j \ j \ j-3 \ j \ | \ 2j-1 \ 2j-3) ; \end{aligned} \quad (4)$$

and the sum of the pairs $(j-3, j)$ and $(j-3, j-1)$ vanishes because of the identity:

$$\begin{aligned} & (j \ j \ j-3 \ j \ | \ 2j-3 \ 2j-3) (2j-3 \ j \ 2j-3 \ j-1 \ | \ 3j-4 \ 3j-4) (j \ j \ j \ j-1 \ | \ 2j-1 \ 2j-1) \\ & = -\frac{1}{2} \sqrt{\frac{j(2j-3)}{3(j-1)(4j-3)}} \\ & = -(j \ j \ j-3 \ j-1 \ | \ 2j-3 \ 2j-4) (2j-3 \ j \ 2j-4 \ j \ | \ 3j-4 \ 3j-4) (j \ j \ j-1 \ j \ | \ 2j-1 \ 2j-1) . \end{aligned} \quad (5)$$

General results, involving relations between single Clebsch-Gordan Coefficients, follow by combining expressions (3), (4), and (5). In particular, combining Eqns. (3) and (5), one obtains

$$(j \ j \ j-3 \ j-1 \ | \ 2j-1 \ 2j-4) = -\sqrt{\frac{2j}{(4j-3)}}(j \ j \ j \ j-1 \ | \ 2j-1 \ 2j-1), \quad (6)$$

while combining Eqns. (4) and (5) yields

$$(j \ j \ j-3 \ j \ | \ 2j-1 \ 2j-3) = -\sqrt{\frac{3(j-1)}{(4j-3)}}(j \ j \ j \ j-1 \ | \ 2j-1 \ 2j-1). \quad (7)$$

Combining the results of Eqns. (6) and (7) one obtains

$$(j \ j \ j-3 \ j-1 \ | \ 2j-1 \ 2j-4) = \sqrt{\frac{2j}{3(j-1)}}(j \ j \ j-3 \ j \ | \ 2j-1 \ 2j-3). \quad (8)$$

It is of interest to compare these results with those obtained using the symmetries of Regge's elegant expression [4].

Written in terms of $3-j$'s Eqn. (8) becomes:

$$\begin{pmatrix} j & j & 2j-1 \\ j-3 & j-1 & -2j+4 \end{pmatrix} = \sqrt{\frac{2j}{3(j-1)}} \begin{pmatrix} j & j & 2j-1 \\ j-3 & j & -2j+3 \end{pmatrix}.$$

From Regge's symmetries (aside from the standard relations one obtains if one interchanges rows, or changes the signs of the projections), one obtains additionally

$$\begin{pmatrix} j & j & 2j-1 \\ j-3 & j-1 & -2j+4 \end{pmatrix} = \begin{pmatrix} 2j-2 & 2j-1 & 2 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} j & 2j-2 & j+1 \\ 1-j & 2j-3 & 2-j \end{pmatrix}.$$

These involve different angular momenta, whereas Eqns. (6)–(8) involve different projections for the same j 's. The three relations, Eqns. (6)–(8), can be verified by construction, starting with the state $\psi_{2j}^{2j} = \phi_j^j \eta_j^j$, using lowering operators to obtain the state $\psi_{2j-1}^{2j} = \sqrt{\frac{1}{2}}(\phi_j^j \eta_{j-1}^j + \phi_{j-1}^j \eta_j^j)$, constructing the orthogonal state ψ_{2j-1}^{2j-1} , and finally using lowering operators to obtain ψ_{2j-2}^{2j-1} , ψ_{2j-3}^{2j-1} , and ψ_{2j-4}^{2j-1} .

3. Conclusions

It has been shown that three unexpected Clebsch-Gordan equalities, namely Eqns. (3), (4), (5) lead to the non-accidental vanishing of the $6j$ symbol:

$$\left\{ \begin{matrix} j & j & 2j-3 \\ j & 3j-4 & 2j-1 \end{matrix} \right\}$$

that can generally be expressed as a sum of six terms. These three relations are also of some interest by themselves and lead to general relations between individual Clebsch-Gordan coefficients namely Eqns. (6), (7) and (8).

The author would like to thank KFUPM for its support in this research.

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