

1-1-2018

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Recommended Citation

GRACE, SAID R. and ZAFER, AĞACIK (2018) "On oscillation of integro-differential equations," *Turkish Journal of Mathematics*: Vol. 42: No. 1, Article 18. <https://doi.org/10.3906/mat-1612-21>
Available at: <https://journals.tubitak.gov.tr/math/vol42/iss1/18>

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On oscillation of integro-differential equations

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Received: 05.12.2016

Accepted/Published Online: 08.04.2017

Final Version: 22.01.2018

Abstract: We study the oscillatory behavior of solutions for integro-differential equations of the form

$$x'(t) = e(t) - \int_0^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds, \quad t \geq 0,$$

where $0 < \alpha < 1$. Our method is based on the use of the beta function and asymptotic behavior of nonoscillatory solutions. An example is given to illustrate the main result. Equations of this form include Caputo type fractional differential equations, so the results are applicable to some fractional type differential equations as well.

Key words: Integro-differential equation, oscillation, singular, Volterra equation

1. Introduction

We investigate the oscillatory behavior of solutions of the integro-differential equation

$$x'(t) = e(t) - \int_0^t (t-s)^{\alpha-1} k(t,s) f(s, x(s)) ds, \quad t \geq 0, \quad (1)$$

where $0 < \alpha < 1$ is a real number and the functions e , k , and f are continuous in their domain of definitions. We assume that there exist continuous functions $a, h, m : [0, \infty) \rightarrow [0, \infty)$ and real numbers $\gamma > 0$ and $0 < \lambda < 1$ such that

$$0 \leq k(t,s) \leq a(t)h(s) \text{ for all } t \geq s \geq 0 \quad (2)$$

and

$$0 < xf(t,x) \leq t^{\gamma-1} m(t) |x|^{\lambda+1} \text{ for all } x \neq 0 \text{ and } t \geq 0. \quad (3)$$

Furthermore, there exist real numbers $M_1 > 0$ and M_2 such that

$$|a(t)| \leq M_1 \quad (4)$$

and for every $T \geq 0$,

$$-M_2 \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_T^t e(s) ds \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_T^t e(s) ds \leq M_2. \quad (5)$$

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2010 AMS Mathematics Subject Classification: 34N05, 45D05, 34C10

By an oscillatory solution of (1) we mean a function $x(t)$ having arbitrarily large zeros. That is, there is a sequence $\{t_n\}$ of real numbers such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $x(t_n) = 0$. Therefore, in the present work we only consider those solutions of equations (1) that are nontrivial in the neighborhood of the infinity, and for existence and uniqueness of such solutions we refer to [10]. As usual, the equation is called oscillatory if all its solutions are oscillatory. If a solution is not oscillatory it is said to be nonoscillatory. Clearly, a nonoscillatory solution is either eventually positive or eventually negative.

Our motivation in the present work stems from the fact that equations of type (1) arise in many problems of science and engineering such as mathematical models in ecology [12], spread of epidemics [5, 13], electric-circuit analysis [2, 19], finance [3], mechanics [4], and plasma physics [8]. Therefore, information on the qualitative behavior of the solutions of (1) is crucial in order to better understand the underlying structure. To this end, the oscillation of solutions for such equations seems to be highly important. However, the research on oscillation theory for integro-differential equations is its early stages due to difficulties encountered in adapting the techniques from differential equations. For some limited results we refer the reader in particular to [6, 7, 11, 14–16, 18].

It should also be mentioned that there is a connection with fractional differential equations. Indeed, by setting

$$k(t, s) = \frac{1}{\Gamma(\alpha)}, \quad e(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

we may write from (1) that

$$x'(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [g(s) - f(s, x(s))] ds, \quad 0 < \alpha < 1. \quad (6)$$

Now it is not difficult to see that Eq. (6) is equivalent to the fractional differential equation of the Caputo type

$${}^C D^{\alpha+1} x(t) + f(t, x(t)) = g(t), \quad x_0 = x'(0) \quad (7)$$

under some mild conditions on $f(t, x)$ and $g(t)$; see [9, 17]. Therefore, one can easily rewrite the results of this paper for the fractional differential equation (7). Note that if g is a bounded integrable function on an interval, then $e(t)$ becomes continuous there.

In the present work we establish two theorems. Theorem 1 deals with the asymptotic behavior of nonoscillatory solutions, whereas Theorem 2 provides sufficient conditions for oscillation of all solutions of Eq. (1). The novelty of the oscillation theorem lies in the use of Theorem 1.

The following inequality is essentially the Young inequality [1].

Lemma 1 *If X and Y are nonnegative real numbers, then we have*

$$X^\beta - (1-\beta)Y^\beta - \beta XY^{\beta-1} \leq 0 \quad \text{for } 0 < \beta < 1,$$

where the equality holds if and only if $X = Y$.

We will also make use of the simple identity

$$\int_0^t s^{a-1} (t-s)^{b-1} ds = t^{a+b+1} B(a, b), \quad (8)$$

where

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

is the well-known beta function.

2. The main results

For a given continuous function $\eta : [0, \infty) \rightarrow (0, \infty)$, we define

$$C(t, \eta) = \eta^{\frac{\lambda}{1-\lambda}}(t) m^{\frac{1}{1-\lambda}}(t) h^{\frac{1}{1-\lambda}}(t), \quad t \geq 0, \quad (1)$$

where λ , m , and h are as in the previous section.

We first show that every nonoscillatory solution of Eq. (1) satisfies

$$x(t) = O(t), \quad t \rightarrow \infty$$

under some very mild conditions.

Theorem 1 *Let q be a conjugate number of $p > 1$, i.e. $q = p/(p-1)$. Suppose that $p < 1/(1-\alpha)$, and the conditions (2)–(5) hold with $\gamma = 1 - \alpha + 1/q$.*

If there exists continuous function $\eta : [0, \infty) \rightarrow [0, \infty)$ for which $t\eta(t), C(t, \eta(t)) \in L^q[0, \infty)$, then every nonoscillatory solution $x(t)$ of Eq. (1) satisfies

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t} < \infty. \quad (2)$$

Proof Let $x(t)$ be a nonoscillatory solution, say $x(t) > 0$, for all $t \geq t_1$ for some $t_1 \geq 1$. Put

$$k_1 := \max\{|f(t, x(t))| : t \in [0, t_1]\} \geq 0 \quad \text{and} \quad k_2 := k_1 \int_0^{t_1} (t_1 - s)^{\alpha-1} h(s) ds \geq 0. \quad (3)$$

In view of (2), (3), and (3), we may write from Eq. (1) that

$$\begin{aligned} x'(t) &= e(t) - \int_0^{t_1} (t-s)^{\alpha-1} k(t, s) f(s, x(s)) ds - \int_{t_1}^t (t-s)^{\alpha-1} k(t, s) f(s, x(s)) ds \\ &\leq e(t) + \int_0^{t_1} (t_1-s)^{\alpha-1} k(t, s) |f(s, x(s))| ds + \int_{t_1}^t (t-s)^{\alpha-1} k(t, s) f(s, x(s)) ds \\ &\leq e(t) + k_1 a(t) \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) m(s) x^\lambda(s) ds \\ &\leq e(t) + k_2 a(t) + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \left(h(s) m(s) x^\lambda(s) - \eta(s) x(s) \right) ds \\ &\quad + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \eta(s) x(s) ds, \quad t \geq t_1. \end{aligned} \quad (4)$$

By Lemma 1 with the choices

$$X = (hm)^{1/\lambda} x, \quad Y = (\eta(hm)^{-1/\lambda} \lambda^{-1})^{1/(\lambda-1)}, \quad \beta = \lambda,$$

we see that

$$h(s)m(s)x^\lambda(s) - \eta(s)x(s) \leq \lambda_0 C(s, \eta(s)), \quad s \geq t_1,$$

where $\lambda_0 = (1 - \lambda)\lambda^{\lambda/(1-\lambda)}$ and $C(t, \eta)$ is defined in (1). Therefore, inequality (4) gives

$$\begin{aligned} x'(t) \leq & e(t) + k_2 a(t) + \lambda_0 a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} C(s, \eta(s)) \, ds \\ & + a(t) \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \eta(s)x(s) \, ds, \quad t \geq t_1, \end{aligned}$$

and hence by (4),

$$\begin{aligned} x'(t) \leq & e(t) + k_2 M_1 + \lambda_0 M_1 \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} C(s, \eta(s)) \, ds \\ & + M_1 \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \eta(s)x(s) \, ds, \quad t \geq t_1. \end{aligned} \tag{5}$$

Integrating (5) over $[t_1, t]$ and then changing the order of integration, we have

$$\begin{aligned} x(t) \leq & x(t_1) + k_2 M_1 t + \int_{t_1}^t e(s) \, ds + \frac{\lambda_0 M_1 t}{\alpha} \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} C(s, \eta(s)) \, ds \\ & + \frac{M_1 t}{\alpha} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \eta(s)x(s) \, ds, \quad t \geq t_1. \end{aligned} \tag{6}$$

By using the Hölder inequality and the identity (8), we see that

$$\begin{aligned} \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} C(s, \eta(s)) \, ds & \leq \left(\int_0^t (t-s)^{p(\alpha-1)} s^{p(\gamma-1)} \, ds \right)^{1/p} \left(\int_0^t C^q(s, \eta(s)) \, ds \right)^{1/q} \\ & \leq t^{\theta/p} B^{1/p} (p(\gamma-1) + 1, p(\alpha-1) + 1) \left(\int_0^t C^q(s, \eta(s)) \, ds \right)^{1/q} \\ & \leq b_1 \left(\int_0^\infty C^q(s, \eta(s)) \, ds \right)^{1/q} = b_1 \|C\|_q \end{aligned} \tag{7}$$

where $b_1 = B^{1/p} (p(\gamma-1) + 1, p(\alpha-1) + 1)$, and by the definition of γ , $\theta = p(\alpha + \gamma - 2) + 1 = 0$. Similarly, we estimate the second integral as

$$\begin{aligned} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} \eta(s)x(s) \, ds & \leq \left(\int_0^t (t-s)^{p(\alpha-1)} s^{p(\gamma-1)} \, ds \right)^{1/p} \left(\int_{t_1}^t \eta^q(s)x^q(s) \, ds \right)^{1/q} \\ & \leq b_1 \left(\int_{t_1}^t \eta^q(s)x^q(s) \, ds \right)^{1/q}. \end{aligned} \tag{8}$$

In view of (7) and (8), from (6) we get

$$\frac{x(t)}{t} \leq \lambda_1 + \lambda_2 \left(\int_{t_1}^t \eta^q(s)x^q(s) \, ds \right)^{1/q}, \quad t \geq t_1, \tag{9}$$

where

$$\lambda_1 = \frac{x(t_1)}{t_1} + k_2 M_1 + M_2 + \frac{\lambda_0 b_1 M_1}{\alpha} \|C\|_q, \quad \lambda_2 = \frac{b_1 M_1}{\alpha}.$$

Now we employ the elementary inequality $(a + b)^q \leq 2^{q-1}(a^q + b^q)$ for $a, b \geq 0$ in (9) to arrive at

$$w(t) \leq 2^{q-1} \lambda_1^q + 2^{q-1} \lambda_2^q \int_{t_1}^t s^q \eta^q(s) w(s) ds, \quad t \geq t_1, \tag{10}$$

where $w(t) = x^q(t)/t^q$. Finally, if we apply the Gronwall-Bellman inequality in (10) and then use the assumption that $t\eta(t) \in L^q[0, \infty)$, we obtain

$$\limsup_{t \rightarrow \infty} \frac{x(t)}{t} < \infty \tag{11}$$

as desired.

If $x(t)$ is eventually negative, we can set $y = -x$ to see that y satisfies Eq. (1) with $e(t)$ replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. It follows in a similar manner that

$$\limsup_{t \rightarrow \infty} \frac{-x(t)}{t} < \infty. \tag{12}$$

□

Now we give our oscillation theorem.

Theorem 2 *In addition to the hypothesis of Theorem 1, suppose that*

$$\lim_{t \rightarrow \infty} a(t) = 0. \tag{13}$$

If for every $\mu \in (0, 1)$ we have

$$\liminf_{t \rightarrow \infty} \left[\mu t + \int_0^t e(s) ds \right] = -\infty, \quad \limsup_{t \rightarrow \infty} \left[\mu t - \int_0^t e(s) ds \right] = \infty, \tag{14}$$

then Eq. (1) is oscillatory.

Proof Suppose on the contrary that there is a nonoscillatory solution $x(t)$ of Eq. (1). We may assume that $x(t)$ is eventually positive, i.e. there exists a sufficiently large $t_1 > 1$ such that $x(t) > 0$ for all $t \geq t_1$.

Proceeding as in the proof of Theorem 1 we arrive at (6), so

$$x(t) \leq x(t_1) + k_2 M_1 t + \int_{t_1}^t e(s) ds + t \frac{\lambda_0 M_1 b_1 \|C\|_q}{\alpha} + t \frac{M_1 b_1}{\alpha} \left(\int_{t_1}^t \eta^q(s) x^q(s) ds \right)^{1/q}. \tag{15}$$

In view of $t\eta(t) \in L^q[0, \infty)$ and (2) we see that the last integral in (15) is bounded, and because of (13) we can make M_1 as small as we please by increasing the size of t_1 if necessary. Therefore, it follows from (15) that

$$x(t) \leq x(t_1) - \int_0^{t_1} e(s) ds + \int_0^t e(s) ds + t/2, \quad t \geq t_2 \tag{16}$$

for some $t_2 \geq t_1$. Taking \liminf as $t \rightarrow \infty$ in (16) and using (14) results in a contradiction with the fact that $x(t)$ is eventually positive.

The proof when $x(t)$ is eventually negative is similar. □

Example 1 Consider the integro-differential equation

$$x'(t) = t \sin t - \int_0^t (t-s)^{-1/3} \frac{s}{t+s+1} x^{1/3}(s) ds, \quad t \geq 0. \quad (17)$$

Comparing with (1) we may write that

$$e(t) = t \sin t, \quad \alpha = 2/3,$$

$$f(t, x) = x^{1/3} = t^{-1/3} t^{1/3} x^{1/3}, \quad \lambda = 1/3,$$

$$k(t, s) = \frac{s}{t+s+1} \leq \frac{s}{t+1}.$$

Letting $q = 3$ and taking $a(t) = 1/(t+1)$, $h(t) = t$, $m(t) = t^{1/3}$, and $\eta(t) = 1/(t+1)^6$, we calculate that

$$p = 3/2, \quad \gamma = 2/3, \quad C(t, \eta) = \frac{t^2}{(t+1)^3}.$$

Clearly, $p < 1/(1-\alpha)$, (2)-(5) hold, $t\eta(t), C(t, \eta(t)) \in L^3$, $\lim_{t \rightarrow \infty} a(t) = 0$, and, in view of

$$\int_{t_1}^t e(s) ds = \sin t - t \cos t - \sin t_1 + t_1 \cos t_1,$$

the conditions (5) and (14) are satisfied. Since all the conditions of Theorem 2 are satisfied, we may conclude that Eq. (17) is oscillatory. We should note that to the best of our knowledge none of the results in the literature are applicable to Eq. (17).

Acknowledgment

The authors would like to thank the referees for their valuable comments.

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