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Some results on Hecke and extended Hecke groups

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Abstract: Let $q \geq 3$ be a prime number and let $\overline{H}(\lambda_q)$ be the extended Hecke group associated with q . In this paper, we determine the presentation of the commutator subgroup $(H(\lambda_q)\alpha)'$ of the normal subgroup $H(\lambda_q)\alpha$, where $H(\lambda_q)\alpha$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. Next we discuss the commutator subgroup $(\overline{H}_2)'$ of the principal congruence subgroup $\overline{H}_2(\lambda_q)$ of $\overline{H}(\lambda_q)$. Then we show that some quotient groups of $\overline{H}(\lambda_q)$ are generalized M^* -groups. Finally, we prove some results related to some normal subgroups of $\overline{H}(\lambda_q)$, especially in the case $q = 5$.

Key words: Extended Hecke groups, commutator subgroups, principal congruence subgroups, generalized M^* -groups

1. Introduction

In [14], Hecke introduced the Hecke groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z + \lambda},$$

where λ is a fixed positive real number.

He showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2 \cos \frac{\pi}{q}$, for integer $q \geq 3$, ($\lambda < 2$), or $\lambda \geq 2$. We will focus on the discrete case with $\lambda < 2$ and we denote it by $H(\lambda_q)$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q ,

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q, \quad [10].$$

Let Γ be a subgroup of finite index in $H(\lambda_q)$. Then \mathcal{U}/Γ , where \mathcal{U} is the upper half plane of a Riemann surface. Let g and t be the genus and the number of cusps of \mathcal{U}/Γ , respectively, and let m_1, \dots, m_k be the branching numbers of the branch points on \mathcal{U}/Γ . The signature of Γ is $(g; m_1, \dots, m_k; t)$.

The Hecke group $H(\lambda_q)$ can be thought of as triangle groups having an infinity as one of the parameters. As the signature of $H(\lambda_q)$ is $(0; 2, q, \infty)$, each is an infinite triangle group. Moreover, the quotient space $\mathcal{U}/H(\lambda_q)$ is a sphere with one puncture and two elliptic fixed points of order 2 and q . Hence the surface $\mathcal{U}/H(\lambda_q)$ is an orbifold.

Examples of these groups are $H(\lambda_3) = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2, \mathbb{Z}[\lambda_q])$, for $q \geq 4$.

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In memory of my dear son Can Şahin

The extended modular group, denoted by $\overline{H}(\lambda_3) = \Pi = PGL(2, \mathbb{Z})$, is defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group $H(\lambda_3)$. Then the extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [35] and [39] similar to the extended modular group by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the Hecke group $\overline{H}(\lambda_q)$. Thus the extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (RS)^2 = I \rangle \cong D_2 *_{C_2} D_q. \tag{1}$$

If we take $R_1(z) = \frac{1}{\bar{z}}$, $R_2(z) = -\bar{z}$, $R_3(z) = -\bar{z} - \lambda_q$, where $T = R_2R_1 = R_1R_2$ and $S = R_1R_3$; then we get the alternative presentation

$$\overline{H}(\lambda_q) = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_1R_3)^q = I \rangle.$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. Since the extended Hecke groups $\overline{H}(\lambda_q)$ contain a reflection, they are proper non-Euclidean crystallographic (NEC) groups [28]. Thus the quotient space $\mathcal{U}/\overline{H}(\lambda_q)$ is a Klein surface and $\mathcal{U}/H(\lambda_q)$ is the canonical double cover of $\mathcal{U}/\overline{H}(\lambda_q)$.

The Hecke groups $H(\lambda_q)$, the extended Hecke groups $\overline{H}(\lambda_q)$ and their normal subgroups have been studied for many aspects in the literature (for instances, please see [1, 2, 6, 7, 13, 17, 24, 32, 36, and 45]).

Here the map

$$\alpha : T \rightarrow RT, \quad S \rightarrow S, \quad R \rightarrow R, \tag{2}$$

induces an outer automorphism of $\overline{H}(\lambda_q)$, [17, p. 12]. Thus the group

$$H(\lambda_q)\alpha = \langle RT, S \mid (RT)^2 = S^q = I \rangle \cong C_2 * C_q, \tag{3}$$

is a subgroup of index 2 in $\overline{H}(\lambda_q)$.

Throughout this paper, we identify matrix A in $GL(2, \mathbb{Z}[\lambda_q])$ with $-A$, so that they each represent the same element of $\overline{H}(\lambda_q)$. Thus we can represent the generators of $\overline{H}(\lambda_q)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Next we give some information about the principal congruence subgroups of $\overline{H}(\lambda_q)$.

The *principal congruence subgroups* $\overline{H}_p(\lambda_q)$ of level p , p prime, of $\overline{H}(\lambda_q)$ are defined in [38] (see also [15] and [26]) as

$$\overline{H}_p(\lambda_q) = \left\{ A = \begin{pmatrix} a & b\lambda_q \\ c\lambda_q & d \end{pmatrix} \in \overline{H}(\lambda_q) : a \equiv d \equiv \pm 1, \quad b \equiv c \equiv 0 \pmod{p}, \quad \det A = \pm 1 \right\}.$$

$\overline{H}_p(\lambda_q)$ is always a normal subgroup of finite index in $\overline{H}(\lambda_q)$. It is easily seen that $H_p(\lambda_q) = \overline{H}_p(\lambda_q) \cap H(\lambda_q)$.

By [38], we know that if $p \geq 3$ is a prime number, then $\overline{H}_p(\lambda_q) = H_p(\lambda_q)$ and if $p = 2$, then $\overline{H}(\lambda_q)/\overline{H}_2(\lambda_q) \cong H(\lambda_q)/H_2(\lambda_q)$. Thus, the groups $H_2(\lambda_q)$ and $\overline{H}_2(\lambda_q)$ are very important.

The principal congruence subgroups $H_2(\lambda_3) = \Gamma(2)$, $\overline{H}_2(\lambda_3) = \Pi(2)$ and $\Gamma(4) = \Pi(4)$ of Γ and Π , respectively, have been studied extensively in the literature, for example, in relation to number theory, modular forms, modular curves, Belyi's theory, and graph theory (for instance, see [8, 11, 12, 21, 22, 33, and 40]).

Some normal subgroups (the first and the second commutator subgroups $\overline{H}'(\lambda_q)$ and $\overline{H}''(\lambda_q)$, the principal congruence subgroups $\overline{H}_p(\lambda_q)$ and the m -th power subgroups $\overline{H}^m(\lambda_q)$) of $\overline{H}(\lambda_q)$, $q \geq 3$ prime number, have been studied by Ikkardes, Koruoğlu, Sahin, and Bizim in [38, 42, 43, 44]. For $q \geq 3$ a prime number, they proved the following results:

- a) There are exactly 3 normal subgroups of index 2 in $\overline{H}(\lambda_q)$. Namely, $H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q$, $\overline{H}_0(\lambda_q) = \langle R, S, TST \mid R^2 = S^q = (TST)^q = (RS)^2 = (RTST)^2 = I \rangle \cong D_q *_{\mathbb{Z}_2} D_q$, and $H(\lambda_q)\alpha = \langle TR, S \mid (TR)^2 = S^q = I \rangle \cong C_2 * C_q$.
- b) There is exactly one normal subgroup of index 4 in $\overline{H}(\lambda_q)$. Namely, $\overline{H}'(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong C_2 * C_q$.
- c) There are exactly 2 normal subgroups of index $2q$ in $\overline{H}(\lambda_q)$. Namely, $H^q(\lambda_q) = \langle T \rangle * \langle STS^{q-1} \rangle * \dots * \langle S^{q-1}TS \rangle \cong \underbrace{C_2 * C_2 * \dots * C_2}_{q \text{ times}}$, and $\overline{H}_2(\lambda_q) = \langle TR \rangle * \langle RSTS \rangle * \dots * \langle RS^{q-1}TS^{q-1} \rangle \cong \underbrace{C_2 * C_2 * \dots * C_2}_{q \text{ times}}$.
- d) The second commutator subgroup $\overline{H}''(\lambda_q)$ of $\overline{H}(\lambda_q)$ is a normal subgroup of index $4q^2$ in $\overline{H}(\lambda_q)$. Namely $\overline{H}''(\lambda_q)$ is a free group with basis $[S, TST], [S, TS^2T], \dots, [S, TS^{q-1}T], [S^2, TST], [S^2, TS^2T], \dots, [S^2, TS^{q-1}T], \dots, [S^{q-1}, TST], [S^{q-1}, TS^2T], \dots, [S^{q-1}, TS^{q-1}T]$.
- e) The group $(H^2)'(\lambda_q)$ is equal to the second commutator subgroup $\overline{H}''(\lambda_q)$ and it has index q in $H'(\lambda_q)$.
- f) The group $(H^q)'(\lambda_q)$ is a free group of rank $1 + (q - 2)2^{q-1}$, of signature $((q - 3)2^{q-2} + 1; \infty^{(2^{q-1})})$ and of index $2^{q+1}q$ in $\overline{H}(\lambda_q)$.

Using the above results, we get the following subgroup diagram in Figure 1.

On the other hand, when Sahin et al. were studying in [42] some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$, for $q \geq 3$ prime, they came across an interesting general fact. If a bordered surface group Γ is a normal subgroup of finite index in $\overline{H}(\lambda_q)$, then $\overline{H}(\lambda_q)/\Gamma$ is a group of automorphisms of the bordered Klein surface $X = \mathcal{U}/\overline{H}(\lambda_q)$. Moreover, the automorphism groups G of order $\frac{4q}{(q-2)}(g-1)$ that act on compact bordered Klein surfaces X of algebraic genus $g \geq 2$ are finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_q)$, where $q \geq 3$ is an integer. For example, the groups of orders $|G| = 12(g-1)$, $|G| = 8(g-1)$, $|G| = \frac{20}{3}(g-1)$, respectively, are the finite quotient groups of the extended Hecke groups $\overline{H}(\lambda_3)$, $\overline{H}(\lambda_4)$, or $\overline{H}(\lambda_5)$ [41] and [3]. Here the orders of these groups are the highest three among the automorphism groups of the compact Klein surfaces of algebraic genus $g \geq 2$ (see [31, p. 221, proposition 1]). The groups of order $|G| = 12(g-1)$ are M^* -groups. These groups were first introduced in [30], and have been studied in several papers [4] and [5].

Sahin et al. defined the generalized M^* -groups in [41] similar to the M^* -groups. A finite group G is called a generalized M^* -group if it is generated by three distinct nontrivial elements r_1, r_2, r_3 that satisfy the relations

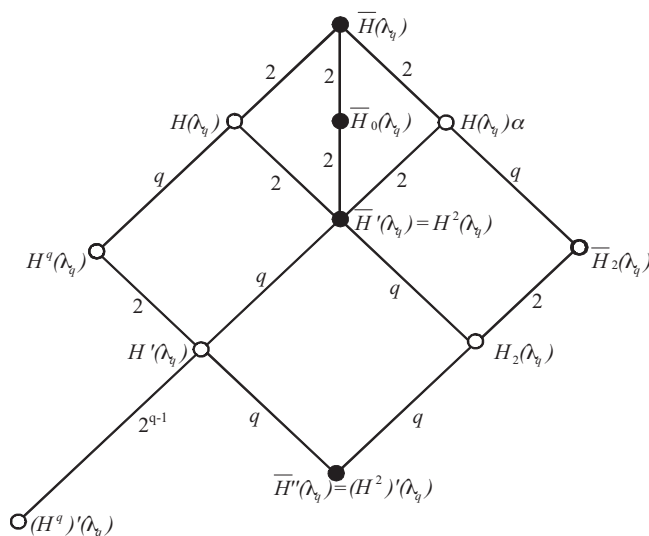


Figure 1.

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_2)^2 = (r_1 r_3)^q = I,$$

$q \geq 3$ prime, and other relations that make the group finite. It is clear that the order of G is $\frac{4q}{(q-2)}(g-1)$, $g \geq 2$ integer.

In [41], Sahin et al. showed that a finite group of order at least $4q$ is a generalized M^* -group if and only if it is the homomorphic image of the extended Hecke group $\overline{H}(\lambda_q)$, $q \geq 3$ prime. Thus, if the index of a normal subgroup N in $\overline{H}(\lambda_q)$ is greater than $2q$, then the quotient group $\overline{H}(\lambda_q)/N$ is a generalized M^* -group.

The aim of this work is to generalize some results known for the extended modular group to the extended Hecke groups $\overline{H}(\lambda_q)$, for $q \geq 3$ prime (especially, $\overline{H}(\lambda_5)$). We obtain the group structures and the signatures of the commutator subgroups $(H(\lambda_q)\alpha)'$ and $(\overline{H}_2)'$ of $\overline{H}(\lambda_q)$. For this, we apply the Reidemeister–Schreier method, the permutation method, and the Riemann–Hurwitz formula. Next, we discuss some normal subgroups and how they are related under the map α introduced in (2). We also give some relations between some normal subgroups of $\overline{H}(\lambda_q)$ and a figure that explain the effect of α . Furthermore, we show that some of the quotient groups of $\overline{H}(\lambda_q)$ are generalized M^* -groups. Finally, we determine the generators of $H_4(\lambda_5)$ and prove that $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$ and $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$.

2. Commutator subgroups of $H(\lambda_q)\alpha$ and $\overline{H}_2(\lambda_q)$

In this section, we study the first commutator subgroups $(H(\lambda_q)\alpha)'$ and $(\overline{H}_2)'$ of $\overline{H}(\lambda_q)$, for $q \geq 3$ a prime number.

Theorem 2.1 *Let $q \geq 3$ be a prime number. Then*

- i) $|H(\lambda_q)\alpha : (H(\lambda_q)\alpha)'| = 2q$,
- ii) the group $(H(\lambda_q)\alpha)'$ is a free group of rank $(q-1)$ with basis $TSTS, TS^2TS^2, \dots, TS^{q-1}TS^{q-1}$.

Proof

i) The quotient group $H(\lambda_q)\alpha/(H(\lambda_q)\alpha)'$ is the group obtained by adding the relation $TRS = STR$ to the relations of $H(\lambda_q)\alpha$ in (3). Then

$$H(\lambda_q)\alpha/(H(\lambda_q)\alpha)' \cong C_2 \times C_q.$$

Therefore, we obtain $|H(\lambda_q)\alpha : (H(\lambda_q)\alpha)'| = 2q$.

ii) We choose $\Sigma = \{I, S, S^2, \dots, S^{q-1}, TR, TRS, TRS^2, \dots, TRS^{q-1}\}$ as a Schreier transversal for $(H(\lambda_q)\alpha)'$. According to the Reidemeister–Schreier method (see [29]), we get the generators of $(H(\lambda_q)\alpha)'$ as the following.

$$\begin{array}{ll} I.TR.(TR)^{-1} = I, & I.S.(S)^{-1} = I, \\ S.TR.(TRS)^{-1} = STRS^{q-1}RT, & S.S.(S^2)^{-1} = I, \\ S^2.TR.(TRS^2)^{-1} = S^2TRS^{q-2}RT, & S^2.S.(S^3)^{-1} = I, \\ \vdots & \vdots \\ S^{q-1}.TR.(TRS^{q-1})^{-1} = S^{q-1}TRSRT, & S^{q-1}.S.(I)^{-1} = I, \\ TR.TR.(I)^{-1} = I, & TR.S.(TRS)^{-1} = I, \\ TRS.TR.(S)^{-1} = TRSTRS^{q-1}, & TRS.S.(TRS^2)^{-1} = I, \\ TRS^2.TR.(S^2)^{-1} = TRS^2TRS^{q-2}, & TRS^2.S.(TRS^3)^{-1} = I, \\ \vdots & \vdots \\ TRS^{q-1}.TR.(S^{q-1})^{-1} = TRS^{q-1}TRS, & TRS^{q-1}.S.(TR)^{-1} = I. \end{array}$$

Here $STRS^{q-1}RT = STST$, $S^2TRS^{q-2}RT = S^2TS^2T$, $S^{q-1}TRSRT = S^{q-1}TS^{q-1}T$, $TRSTRS^{q-1} = TS^{q-1}TS^{q-1}$, $TRS^2TRS^{q-2} = TS^{q-2}TS^{q-2}$ and $TRS^{q-1}TRS = TSTS$, as $TR = RT$ and $SR = RS^{q-1}$. Also as $(STST)^{-1} = TS^{q-1}TS^{q-1}$, $(S^2TS^2T)^{-1} = TS^{q-2}TS^{q-2}$ and $(S^{q-1}TS^{q-1}T)^{-1} = TSTS$, the generators of $(H(\lambda_q)\alpha)'$ are $TSTS$, TS^2TS^2 , ..., $TS^{q-1}TS^{q-1}$.

Using the permutation method (see [46]) and the Riemann–Hurwitz formula, we get the signature of $(H(\lambda_q)\alpha)'$ as $(0; \underbrace{\infty, \infty, \dots, \infty}_{q \text{ times}}) = (0; \infty^{(q)})$. □

It is clear that the group $(H(\lambda_q)\alpha)'$ is a subgroup of $H(\lambda_q)$. From [27], there are only two normal subgroups of index $2q$ in $H(\lambda_q)$, for $q \geq 3$ prime. Namely, $H'(\lambda_q)$ and $H_2(\lambda_q)$. As the signature of $H_2(\lambda_q)$ is $(0; \infty^{(q)})$ (see [16]), we get the following result.

Corollary 2.2 *The subgroup $(H(\lambda_q)\alpha)'$ is equal to the principal congruence subgroup $H_2(\lambda_q)$ of $H(\lambda_q)$, i.e. $(H(\lambda_q)\alpha)' = H_2(\lambda_q)$.*

Theorem 2.3 *Let $q \geq 3$ be a prime number.*

- i) $|\overline{H}_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)| = 2^q$.
- ii) The group $(\overline{H}_2)'(\lambda_q)$ is a free group of rank $1 + (q - 2)2^{q-1}$.
- iii) The group $(\overline{H}_2)'(\lambda_q)$ is of index 2^{q-1} in $H_2(\lambda_q)$.

Proof

i) If we take $k_1 = TR, k_2 = RSTS, k_3 = RS^2TS^2, \dots, k_q = RS^{q-1}TS^{q-1}$ as the generators of $\overline{H}_2(\lambda_q)$, then the quotient group $\overline{H}_2(\lambda_q)/(\overline{H}_2)'(\lambda_q)$ is the group obtained by adding the relation $k_i k_j = k_j k_i$ to the relations of $\overline{H}_2(\lambda_q)$, for $i \neq j$ and $i, j \in \{1, 2, \dots, q\}$. Thus we have

$$\overline{H}_2(\lambda_q)/(\overline{H}_2)'(\lambda_q) \cong \underbrace{C_2 \times C_2 \times \dots \times C_2}_{q \text{ times}}$$

Therefore, we obtain $|\overline{H}_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)| = 2^q$.

ii) Let $\Sigma = \{I, k_1, k_2, \dots, k_q, k_1 k_2, k_1 k_3, \dots, k_1 k_q, k_2 k_3, k_2 k_4, \dots, k_2 k_q, \dots, k_{q-1} k_q, k_1 k_2 k_3, k_1 k_2 k_4, \dots, k_1 k_2 k_q, \dots, k_1 k_2 \dots k_q\}$ be a Schreier transversal for $(\overline{H}_2)'(\lambda_q)$. Using the Reidemeister-Schreier method, we obtain the generators of $(\overline{H}_2)'(\lambda_q)$ as the following.

There are $C(q, 2) = \binom{q}{2}$ generators of the form $k_i k_j k_i k_j$, where $i < j$ and $i, j \in \{1, 2, \dots, q\}$. There are

$2 \times \binom{q}{3}$ generators of the form $k_i k_j k_t k_j k_t k_i$, or $k_i k_j k_t k_i k_t k_j$, where $i < j < t$ and $i, j, t \in \{1, 2, \dots, q\}$.

There are $3 \times \binom{q}{4}$ generators of the form $k_i k_j k_t k_u k_i k_u k_t k_j$, or $k_i k_j k_t k_u k_j k_u k_t k_i$, or $k_i k_j k_t k_u k_t k_u k_j k_i$,

where $i < j < t < u$ and $i, j, t, u \in \{1, 2, \dots, q\}$. Similarly, there are $(q-1) \times \binom{q}{q}$ generators of the

form $k_1 k_2 \dots k_q k_1 k_q k_{q-1} \dots k_2$, or $k_1 k_2 \dots k_q k_2 k_q k_{q-1} \dots k_3 k_1$, or \dots , or $k_1 k_2 \dots k_q k_{q-1} k_q k_{q-2} \dots k_2 k_1$.

Totally, there are $1 + (q-2)2^{q-1}$ generators of $(\overline{H}_2)'(\lambda_q)$.

iii) We know that $|H(\lambda_q) : (\overline{H}_2)'(\lambda_q)| = 2^q \cdot q$ and $|H(\lambda_q) : H_2(\lambda_q)| = 2q$. Therefore we get $|H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)| = 2^{q-1}$.

Finally, we find the signature of $(\overline{H}_2)'(\lambda_q)$ as $(q2^{q-3} - 2^{q-1} + 1; \underbrace{\infty, \infty, \dots, \infty}_{q \cdot 2^{(q-2)} \text{ times}}) = ((q-4)2^{q-3} + 1;$

$\infty^{(q \cdot 2^{q-2})}$).

□

Corollary 2.4 We have $H_2(\lambda_q) = (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)$.

Proof As $(H^2)'(\lambda_q)$ and $(\overline{H}_2)'(\lambda_q)$ are normal subgroups of $H'(\lambda_q)$, we obtain the chains

$$(H^2)'(\lambda_q) \subset (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \subset H_2(\lambda_q) \text{ and } (\overline{H}_2)'(\lambda_q) \subset (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \subset H_2(\lambda_q).$$

Then we get the index $|H_2(\lambda_q) : (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)|$ divides both of q and 2^{q-1} . Since $(q, 2^{q-1}) = 1$, we have $|H_2(\lambda_q) : (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)| = 1$. Thus we get $H_2(\lambda_q) = (H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q)$. □

Corollary 2.5 We have

- a) $|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))| = 2^q \cdot q^2$
- b) $|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (H^q)'(\lambda_q))| = 2^q \cdot q^2$

Proof a) $(H^2)'(\lambda_q)$ and $(\overline{H}_2)'(\lambda_q)$ are normal subgroups of $H(\lambda_q)$. By one of the isomorphism theorems of the groups, we have that

$$((H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q))/(H^2)'(\lambda_q) \cong (\overline{H}_2)'(\lambda_q)/((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q)).$$

As $(H^2)'(\lambda_q)(\overline{H}_2)'(\lambda_q) \cong H_2(\lambda_q)$, we find

$$H_2(\lambda_q)/(H^2)'(\lambda_q) \cong (\overline{H}_2)'(\lambda_q)/((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q)).$$

Then

$$|H_2(\lambda_q) : (H^2)'(\lambda_q)| = |(\overline{H}_2)'(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))|.$$

As $|H_2(\lambda_q) : ((H^2)'(\lambda_q))'| = q$, we get

$$|(\overline{H}_2)'(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))| = q.$$

Thus, we have

$$|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))| = |H(\lambda_q) : H_2(\lambda_q)| \cdot |H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)| \cdot |(\overline{H}_2)'(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))|.$$

As $|H_2(\lambda_q) : (\overline{H}_2)'(\lambda_q)| = 2^{q-1}$, we obtain

$$|H(\lambda_q) : ((H^2)'(\lambda_q) \cap (\overline{H}_2)'(\lambda_q))| = 2^q \cdot q^2.$$

b) The proof is similar to a). □

Remark 2.6 Under the map α in (2), any subgroup of $\overline{H}(\lambda_q)$ is mapped to a subgroup of $\overline{H}(\lambda_q)$ similar to the extended modular group in [19] and [20]. Indeed one finds

$$\begin{array}{ccc} H(\lambda_q) & \xrightarrow{\alpha} & H(\lambda_q)\alpha \\ H^q(\lambda_q) & \leftrightarrow & \overline{H}_2(\lambda_q) \\ H'(\lambda_q) & \leftrightarrow & H_2(\lambda_q) \\ (H^q)'(\lambda_q) & \leftrightarrow & (\overline{H}_2)'(\lambda_q). \end{array}$$

Of course, if we know the generators of any one of these subgroups, then we find the generators of its image under α . The subgroups $\overline{H}(\lambda_q)$, $\overline{H}_0(\lambda_q)$, $\overline{H}'(\lambda_q)$, and $\overline{H}''(\lambda_q)$ of $\overline{H}(\lambda_q)$ are α -invariant and hence they are characteristic subgroups. Figure 2 summarizes these results.

As shown in [18], if \mathcal{M} is a regular or orientably regular hypermap corresponding to a normal subgroup M of $\overline{H}(\lambda_q)$, then $\mathcal{M}\alpha$ is the hypermap corresponding to the normal subgroup $M\alpha$.

Corollary 2.7 Let $q \geq 3$ be a prime number.

- i) The quotient groups $\overline{H}(\lambda_q)/H'(\lambda_q)$ and $\overline{H}(\lambda_q)/H_2(\lambda_q)$ are generalized M^* -groups. These quotient groups act on surfaces of topological type $((q-1), 1, +)$ and $((q-1), q, +)$ respectively, where in the triple (g, k, ϵ) , g is the algebraic genus, k is the number of boundary components, and ϵ describes the orientability of a bordered Klein surface.

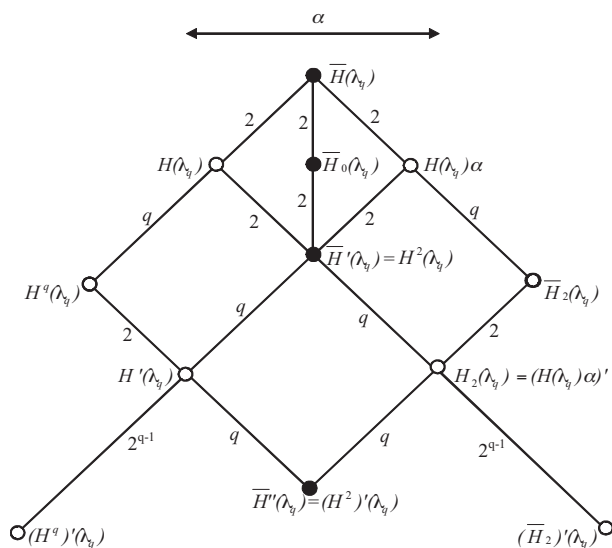


Figure 2. \circ Normal subgroups of $\overline{H}(\lambda_q)$ \bullet Characteristic subgroups of $\overline{H}(\lambda_q)$.

ii) The quotient group $\overline{H}(\lambda_q)/H''(\lambda_q)$ is a generalized M^* -group. This quotient group acts on surfaces of topological type $(q.(q - 2) + 1, q, +)$.

iii) The quotient groups $\overline{H}(\lambda_q)/(H^q)'(\lambda_q)$ and $\overline{H}(\lambda_q)/(\overline{H}_2)'(\lambda_q)$ are generalized M^* -groups. These quotient groups act on surfaces of topological type $(2^{q-1}.(q - 2) + 1, 2^{q-1}, +)$ and $(2^{q-1}.(q - 2) + 1, q.2^{q-2}, +)$ respectively.

Remark 2.8 If $q = 3$, then $\overline{H}(\lambda_3)/H'(\lambda_3)$ and $\overline{H}(\lambda_3)/H_2(\lambda_3)$ act on surfaces of topological type $(2, 1, +)$ and $(2, 3, +)$; $\overline{H}(\lambda_3)/H''(\lambda_3)$ acts on surfaces of topological type $(4, 3, +)$; $\overline{H}(\lambda_3)/(H^3)'(\lambda_3)$ and $\overline{H}(\lambda_3)/(\overline{H}_2)'(\lambda_3)$ act on surfaces of topological type $(5, 4, +)$ and $(5, 6, +)$. If $q = 5$, then $\overline{H}(\lambda_5)/H'(\lambda_5)$ and $\overline{H}(\lambda_5)/H_2(\lambda_5)$ act on surfaces of topological type $(4, 1, +)$ and $(4, 5, +)$; $\overline{H}(\lambda_5)/H''(\lambda_5)$ acts on surfaces of topological type $(16, 5, +)$; $\overline{H}(\lambda_5)/(H^5)'(\lambda_5)$ and $\overline{H}(\lambda_5)/(\overline{H}_2)'(\lambda_5)$ act on surfaces of topological type $(49, 16, +)$ and $(49, 40, +)$. All these results coincide with some results given in [3].

In the following, we focus on the Hecke group $H(\lambda_5)$. We know from [37] that the subgroup $(\overline{H}_2)'(\lambda_3)$ is equal to the congruence subgroup $H_4(\lambda_3)$. We want to derive a similar equation for $H(\lambda_5)$. For this we start with the special example $q = 5$ in ii) of the proof of Theorem 2.3.

Example 2.9 Let $q = 5$. Then $|\overline{H}_2(\lambda_5) : (\overline{H}_2)'(\lambda_5)| = 32$. We choose $\Sigma = \{I, k_1, k_2, k_3, k_4, k_5, k_1k_2, k_1k_3, k_1k_4, k_1k_5, k_2k_3, k_2k_4, k_2k_5, k_3k_4, k_3k_5, k_4k_5, k_1k_2k_3, k_1k_2k_4, k_1k_2k_5, k_1k_3k_4, k_1k_3k_5, k_1k_4k_5, k_2k_3k_4, k_2k_3k_5, k_2k_4k_5, k_3k_4k_5, k_1k_2k_3k_4, k_1k_2k_3k_5, k_1k_2k_4k_5, k_1k_3k_4k_5, k_2k_3k_4k_5, k_1k_2k_3k_4k_5\}$ as a Schreier transversal for $(\overline{H}_2)'(\lambda_5)$. Using the Reidemeister-Schreier method, we get the following generators of $(\overline{H}_2)'(\lambda_5)$ (here $\lambda = \lambda_5 = \frac{1+\sqrt{5}}{2}$ and λ is a root of the polynomial $\lambda^2 - \lambda - 1 = 0$). There are 10 generators of the form,

$$\begin{aligned}
 k_1k_2k_1k_2 &= \begin{pmatrix} 1 & 4\lambda \\ 0 & 1 \end{pmatrix} & k_2k_4k_2k_4 &= \begin{pmatrix} 48\lambda + 29 & 40\lambda + 24 \\ -20\lambda - 12 & -16\lambda - 11 \end{pmatrix} \\
 k_1k_3k_1k_3 &= \begin{pmatrix} 16\lambda + 9 & 20\lambda + 12 \\ 12\lambda + 8 & 16\lambda + 9 \end{pmatrix} & k_2k_5k_2k_5 &= \begin{pmatrix} 4\lambda + 13 & 4\lambda + 8 \\ -4\lambda - 8 & -4\lambda - 3 \end{pmatrix} \\
 k_1k_4k_1k_4 &= \begin{pmatrix} 16\lambda + 9 & 12\lambda + 8 \\ 20\lambda + 12 & 16\lambda + 9 \end{pmatrix} & k_3k_4k_3k_4 &= \begin{pmatrix} 8\lambda + 5 & 8\lambda + 4 \\ -8\lambda - 4 & -8\lambda - 3 \end{pmatrix} \\
 k_1k_5k_1k_5 &= \begin{pmatrix} 1 & 0 \\ 4\lambda & 1 \end{pmatrix} & k_3k_5k_3k_5 &= \begin{pmatrix} 48\lambda + 29 & 20\lambda + 12 \\ -40\lambda - 24 & -16\lambda - 11 \end{pmatrix} \\
 k_2k_3k_2k_3 &= \begin{pmatrix} 4\lambda + 5 & 8\lambda + 4 \\ -4\lambda & -4\lambda - 3 \end{pmatrix} & k_4k_5k_4k_5 &= \begin{pmatrix} 4\lambda + 5 & 4\lambda \\ -8\lambda - 4 & -4\lambda - 3 \end{pmatrix}
 \end{aligned}$$

20 generators of the form,

$$\begin{aligned}
 k_1k_2k_3k_1k_3k_2 &= \begin{pmatrix} -24\lambda - 15 & -48\lambda - 28 \\ -12\lambda - 8 & -24\lambda - 15 \end{pmatrix} & k_1k_2k_3k_2k_3k_1 &= \begin{pmatrix} 4\lambda + 5 & -8\lambda - 4 \\ 4\lambda & -4\lambda + 5 \end{pmatrix} \\
 k_1k_2k_4k_1k_4k_2 &= \begin{pmatrix} -42\lambda - 25 & -128\lambda - 80 \\ -20\lambda - 12 & -52\lambda - 33 \end{pmatrix} & k_1k_2k_4k_2k_4k_1 &= \begin{pmatrix} 48\lambda + 29 & -40\lambda - 24 \\ 20\lambda + 12 & -16\lambda - 11 \end{pmatrix} \\
 k_1k_2k_5k_1k_5k_2 &= \begin{pmatrix} -8\lambda - 3 & -20\lambda - 16 \\ -4\lambda & -8\lambda - 7 \end{pmatrix} & k_1k_2k_5k_2k_5k_1 &= \begin{pmatrix} 4\lambda + 13 & -4\lambda - 8 \\ 4\lambda + 8 & -4\lambda - 3 \end{pmatrix} \\
 k_1k_3k_4k_1k_4k_3 &= \begin{pmatrix} -36\lambda - 23 & -72\lambda - 44 \\ -32\lambda - 20 & -64\lambda - 39 \end{pmatrix} & k_1k_3k_4k_3k_4k_1 &= \begin{pmatrix} -4\lambda - 3 & 4\lambda + 2 \\ -4\lambda - 2 & 4\lambda + 1 \end{pmatrix} \\
 k_1k_3k_5k_1k_5k_3 &= \begin{pmatrix} -48\lambda - 31 & -60\lambda - 36 \\ -40\lambda - 24 & -48\lambda - 41 \end{pmatrix} & k_1k_3k_5k_3k_5k_1 &= \begin{pmatrix} 48\lambda + 29 & -20\lambda - 12 \\ 40\lambda + 24 & -16\lambda - 11 \end{pmatrix} \\
 k_1k_4k_5k_1k_5k_4 &= \begin{pmatrix} -24\lambda - 15 & -20\lambda - 8 \\ -32\lambda - 20 & -28\lambda - 15 \end{pmatrix} & k_1k_4k_5k_4k_5k_1 &= \begin{pmatrix} 4\lambda + 5 & -4\lambda \\ 8\lambda + 4 & -4\lambda - 3 \end{pmatrix} \\
 k_2k_3k_4k_2k_4k_3 &= \begin{pmatrix} -100\lambda - 63 & -112\lambda - 68 \\ 48\lambda + 28 & 52\lambda + 33 \end{pmatrix} & k_2k_3k_4k_3k_4k_2 &= \begin{pmatrix} -16\lambda - 11 & -40\lambda + 44 \\ 8\lambda + 4 & 16\lambda + 13 \end{pmatrix} \\
 k_2k_3k_5k_2k_5k_3 &= \begin{pmatrix} -80\lambda - 51 & -92\lambda - 60 \\ 40\lambda + 24 & 48\lambda + 25 \end{pmatrix} & k_2k_3k_5k_3k_5k_2 &= \begin{pmatrix} -80\lambda - 51 & -228\lambda - 140 \\ 40\lambda + 24 & 112\lambda + 69 \end{pmatrix} \\
 k_2k_4k_5k_2k_5k_4 &= \begin{pmatrix} -20\lambda - 11 & -12\lambda \\ 8\lambda + 4 & 4\lambda + 1 \end{pmatrix} & k_2k_4k_5k_4k_5k_2 &= \begin{pmatrix} -20\lambda - 11 & -52\lambda - 32 \\ -8\lambda - 4 & 20\lambda + 13 \end{pmatrix} \\
 k_3k_4k_5k_3k_5k_4 &= \begin{pmatrix} -206\lambda - 127 & -92\lambda - 56 \\ 182\lambda + 108 & 80\lambda + 49 \end{pmatrix} & k_3k_4k_5k_4k_5k_3 &= \begin{pmatrix} -44\lambda - 27 & -52\lambda - 32 \\ 40\lambda + 20 & 44\lambda + 29 \end{pmatrix}
 \end{aligned}$$

15 generators of the form,

$$\begin{aligned}
 k_1k_2k_3k_4k_1k_4k_3k_2 &= \begin{pmatrix} 68\lambda + 41 & 84\lambda + 52 \\ 32\lambda + 20 & 40\lambda + 25 \end{pmatrix} & k_1k_2k_3k_4k_2k_4k_3k_1 &= \begin{pmatrix} -100\lambda - 63 & 112\lambda + 68 \\ -48\lambda - 28 & 52\lambda + 33 \end{pmatrix} \\
 k_1k_2k_3k_4k_3k_4k_2k_1 &= \begin{pmatrix} -16\lambda - 11 & 40\lambda + 20 \\ -8\lambda - 4 & 16\lambda + 13 \end{pmatrix} & k_1k_2k_3k_5k_1k_5k_3k_2 &= \begin{pmatrix} 80\lambda + 49 & 160\lambda + 100 \\ 40\lambda + 24 & 80\lambda + 49 \end{pmatrix} \\
 k_1k_2k_3k_5k_2k_5k_3k_1 &= \begin{pmatrix} -80\lambda - 51 & 92\lambda + 60 \\ -40\lambda - 24 & 48\lambda + 25 \end{pmatrix} & k_1k_2k_3k_5k_3k_5k_2k_1 &= \begin{pmatrix} -80\lambda - 51 & 128\lambda + 80 \\ -40\lambda - 24 & 64\lambda + 37 \end{pmatrix} \\
 k_1k_2k_4k_5k_1k_5k_4k_2 &= \begin{pmatrix} 80\lambda + 49 & 200\lambda + 120 \\ 32\lambda + 20 & 80\lambda + 49 \end{pmatrix} & k_1k_2k_4k_5k_2k_5k_4k_1 &= \begin{pmatrix} -20\lambda - 11 & 12\lambda \\ -8\lambda - 4 & 4\lambda + 1 \end{pmatrix} \\
 k_1k_2k_4k_5k_4k_5k_2k_1 &= \begin{pmatrix} -20\lambda - 11 & 52\lambda + 32 \\ -8\lambda - 4 & 20\lambda + 13 \end{pmatrix} & k_1k_3k_4k_5k_1k_5k_4k_3 &= \begin{pmatrix} 120\lambda + 73 & 136\lambda + 84 \\ 108\lambda + 60 & 120\lambda + 73 \end{pmatrix} \\
 k_1k_3k_4k_5k_3k_5k_4k_1 &= \begin{pmatrix} -128\lambda - 79 & 92\lambda + 56 \\ -112\lambda - 68 & 80\lambda + 49 \end{pmatrix} & k_1k_3k_4k_5k_4k_5k_3k_1 &= \begin{pmatrix} -44\lambda - 27 & 52\lambda + 32 \\ -40\lambda - 20 & 44\lambda + 29 \end{pmatrix} \\
 k_2k_3k_4k_5k_2k_5k_4k_3 &= \begin{pmatrix} 368\lambda + 221 & 604\lambda + 372 \\ -172\lambda - 112 & -288\lambda - 179 \end{pmatrix} & k_2k_3k_4k_5k_3k_5k_4k_2 &= \begin{pmatrix} 232\lambda + 145 & 588\lambda + 360 \\ -112\lambda - 68 & -280\lambda - 175 \end{pmatrix} \\
 k_2k_3k_4k_5k_4k_5k_3k_2 &= \begin{pmatrix} 56\lambda + 33 & 164\lambda + 96 \\ -40\lambda - 20 & -76\lambda - 51 \end{pmatrix} & &
 \end{aligned}$$

and 4 generators of the form

$$\begin{aligned} k_1 k_2 k_3 k_4 k_5 k_1 k_5 k_4 k_3 k_2 &= \begin{pmatrix} -192\lambda - 119 & -372\lambda - 228 \\ -92\lambda - 56 & -176\lambda - 111 \end{pmatrix} \\ k_1 k_2 k_3 k_4 k_5 k_2 k_5 k_4 k_3 k_1 &= \begin{pmatrix} 368\lambda + 221 & -604\lambda - 372 \\ 172\lambda + 112 & -288\lambda - 179 \end{pmatrix} \\ k_1 k_2 k_3 k_4 k_5 k_3 k_5 k_4 k_2 k_1 &= \begin{pmatrix} 232\lambda + 145 & -588\lambda - 360 \\ 112\lambda + 68 & -280\lambda - 175 \end{pmatrix} \\ k_1 k_2 k_3 k_4 k_5 k_4 k_5 k_3 k_2 k_1 &= \begin{pmatrix} 76\lambda + 53 & -164\lambda - 96 \\ 40\lambda + 20 & -76\lambda - 51 \end{pmatrix} \end{aligned}$$

Therefore, the subgroup $(\overline{H}_2)'(\lambda_5)$ is a free group of rank 49 and of signature $(5; \infty^{(40)})$.

Corollary 2.10 *The subgroup $(\overline{H}_2)'(\lambda_5)$ of $H(\lambda_5)$ is equal to the congruence subgroup $H_4(\lambda_5)$, i.e. $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$.*

Proof From Theorem 2.3, the group $(\overline{H}_2)'(\lambda_5)$ is a normal subgroup of index 160 in $H(\lambda_5)$. Moreover, the congruence subgroup $H_4(\lambda_5)$ is a normal subgroup of index 160 in $H(\lambda_5)$ (see [16] and [23]). Indeed, there are 4 normal subgroups of index 160 in $H(\lambda_5)$ (see [9]). However, from the previous example, all generators of the group $(\overline{H}_2)'(\lambda_5)$ are congruent to the $\pm I \pmod{4}$. Thus $(\overline{H}_2)'(\lambda_5) \subseteq H_4(\lambda_5)$ and we get $(\overline{H}_2)'(\lambda_5) = H_4(\lambda_5)$. \square

On the other hand, in [34], Newman and Smart showed that

$$H_6(\lambda_3) = (H^2)'(\lambda_3) \cap (H^3)'(\lambda_3).$$

Now we show that this equality is not true for the Hecke group $H(\lambda_5)$.

Corollary 2.11 $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$.

Proof Since $(H^2)'(\lambda_5) \subset H'(\lambda_5)$ and $(H^5)'(\lambda_5) \subset H'(\lambda_5)$, we have $(H^2)'(\lambda_5) \cap (H^5)'(\lambda_5) \subset H'(\lambda_5)$. If $H_{10}(\lambda_5) = (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$, then $H_{10}(\lambda_5) \subset H'(\lambda_5)$. However, this is impossible since the commutator subgroup $H'(\lambda_5)$ is not congruence, from [25]. Then we get $H_{10}(\lambda_5) \neq (H^2)'(\lambda_5) \cap (H^5)'(\lambda_5)$. \square

Finally, we formulate the following conjectures. It seems to us difficult to prove them.

Conjecture 2.12 *i) For all $q \geq 3$ prime, $(\overline{H}_2(\lambda_q))' = H_4(\lambda_q)$.*

ii) For all $q \geq 3$ prime, $H_{2q}(\lambda_q) \neq (H^2)'(\lambda_q) \cap (H^q)'(\lambda_q)$.

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