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On rectifying curves in Euclidean 3-space

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Abstract: First, we study rectifying curves via the dilation of unit speed curves on the unit sphere S^2 in the Euclidean space $\mathbb{E}^3$. Then we obtain a necessary and sufficient condition for which the centrode $d(s)$ of a unit speed curve $\alpha(s)$ in $\mathbb{E}^3$ is a rectifying curve to improve a main result of [4]. Finally, we prove that if a unit speed curve $\alpha(s)$ in $\mathbb{E}^3$ is neither a planar curve nor a helix, then its dilated centrode $\beta(s) = \rho(s)d(s)$, with dilation factor ρ, is always a rectifying curve, where ρ is the radius of curvature of α.

Key words: Rectifying curve, centrode, spherical curve, dilated centrode

1. Introduction

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with curvature κ and torsion τ (cf. [10, 12]). In order that the curvature κ and the torsion τ of α are at least twice differentiable, let us assume that the curve α is of class C^5 throughout this paper.

Important types of curves in $\mathbb{E}^3$ are helices (characterized by $\tau = c\kappa$, c constant), spherical curves (characterized by $\rho^2 + (\rho'\sigma)^2 =$ constant, with $\rho = \kappa^{-1} =$ *radius of curvature*, $\sigma = \tau^{-1} =$ *radius of torsion*), and rectifying curves (characterized by $\tau/\kappa = as + b$, $a \neq 0$, b are constants).

It has been observed in [2, 4] that for a given unit speed curve $y(t)$ on the unit sphere S^2 centered at the origin and a positive differentiable function $f(t)$, the *dilation* $\alpha(t) = f(t)y(t)$ of $y(t)$ with dilation factor $f(t)$ is a rectifying curve if and only if $f(t) = a\sec(t + t_0)$, for some constants $a > 0$ and t_0. Moreover, it is known that centrodes (i.e. angular velocity vectors) play some important roles in mechanics and joint kinematics (cf. [1, 5, 11, 13, 14]). So far, rectifying curves are constructed via two main sources, namely dilations of unit speed curves on S^2 (cf. [2]) and the centrodes of twisted curves (cf. [4]).

As rectifying curves are important, so is the relation between the Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$ of the rectifying curve $\alpha(t) = a\sec(t + t_0)y(t)$ and that of the unit speed curve $y(t)$. In this paper, we will address this question and derive the Frenet–Serret apparatus of the rectifying curve $\alpha(t)$ in terms of that of the unit speed curve $y(t)$.

Regarding the centrode $d = \tau T + \kappa B$ of a unit speed twisted curve in $\mathbb{E}^3$, it was shown in [4] that the centrode of a unit speed curve $\alpha : I \to \mathbb{E}^3$ with nonzero constant curvature κ and nonconstant torsion τ is

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a rectifying curve and vice versa. It is natural to seek a generalization of this result, or more precisely, seek a necessary and sufficient condition for the centrode of a unit speed curve to be a rectifying curve. In this respect, first we observe that the centrode of any helix cannot be a rectifying curve. Then we answer the above mentioned question for nonhelical curves by showing that a necessary and sufficient for the centrode of a twisted nonhelical curve to be a rectifying curve is that the curvature κ and torsion τ satisfy a nonhomogeneous linear equation in κ, τ, namely $a\kappa - b\tau = c$ for constants a, b, c with $a^2 + b^2 \neq 0$ and $c \neq 0$ (cf. Theorem 4.1).

Finally, for a unit speed curve $\alpha : I \to \mathbb{E}^3$ with curvature $\kappa > 0$, we consider the *dilated centrode* $\beta(s) = \rho(s)d(s)$ of α with dilation factor $\rho = \kappa^{-1}$. We prove that if α is neither a planar curve nor a helix, then β is always a rectifying curve (cf. Theorem 5.1). The last result provides a new way to construct ample examples of rectifying curves associated with unit speed twisted curves via their dilated centrodes.

2. Preliminaries

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with curvature $\kappa > 0$ and let $\{\kappa, \tau, T, N, B\}$ be the Frenet–Serret apparatus of α. Then α is said to be a *rectifying curve* if it satisfies $\langle \alpha(s), N(s) \rangle = 0$, $s \in I$; in other words, the position vector of $\alpha(s)$ always lies in its rectifying plane spanned by its unit tangent vector T and binormal B.

The distance function $f(s) = \|\alpha(s)\|$ of the rectifying curve satisfies

$$f(s) = \sqrt{s^2 + c_1 s + c_2}, \tag{2.1}$$

where c_1 and c_2 are constants and the converse is also true. Moreover, it is also known that the unit speed curve α is a rectifying curve if and only if the ratio of torsion τ and curvature κ satisfies

$$\frac{\tau}{\kappa} = as + b, \tag{2.2}$$

for some constants $a \neq 0$ and b (see, for instance, [2, 4, 6, 8]).

The *centrode* of $\alpha : I \to \mathbb{E}^3$ is defined by

$$d = \tau T + \kappa B,$$

which is the angular velocity vector of the motion of a mass particle along the curve α and it obeys the laws of motion:

$$T' = d \times T, \quad N' = d \times N, \quad B' = d \times B.$$

The centrodes of unit speed curves were used in [4] to characterize rectifying curves (cf. [4, Theorem 1]). Another major source of examples of rectifying curves is the dilation of a unit speed curve on the unit sphere $S^2 \subset \mathbb{E}^3$ centered at the origin by a positive dilation factor (cf. [2, 4]). In fact, it was proved in [2] that if $y(t)$ is a unit speed curve on S^2 and $f(t)$ is a differentiable function, then $\alpha(t) = f(t)y(t)$ is a rectifying curve if and only if

$$f(t) = a\sec(t + t_0), \tag{2.3}$$

where $a > 0$ and t_0 are constants (cf. [2, Theorem 3]).

We shall find the curvature κ_y of the unit speed curve $y(t)$, which will be used in the subsequent work in this paper. Note that $T_y = y'$ and that $\{y, y', y \times y'\}$ is an orthonormal frame of $\mathbb{E}^3$ and thus using

Frenet–Serret formulae for $y(t)$, and

$$y'' = -y + h\, y \times y', \tag{2.4}$$

with $h = \langle y'', y \times y' \rangle$.

From (2.4) we have

$$T_y = y', \quad N_y = -\frac{1}{\kappa_y} y + \frac{h}{\kappa_y} y \times y' \tag{2.5}$$

It follows from the second equation in (2.5) that

$$\kappa_y = \sqrt{1+h^2}. \tag{2.6}$$

3. Rectifying curves via dilation of curves on S^2

One major source of examples of rectifying curves is provided by the dilated curves $\alpha(t) = a\sec(t+t_0)y(t)$ with constants $a > 0, t_0$, where $y(t)$ is a unit speed curve on $S^2 \subset \mathbb{E}^3$ centered at the origin (cf. [2]). However, if we consider an arc of the great circle $y(t) = (\cos t, 0, \sin t)$ and the curve

$$\alpha(t) = a\sec(t+t_0)y(t) = a\left(\sec(t+t_0)\cos t, 0, \sec(t+t_0)\sin t\right),$$

then we find the speed v_α and the tangent vector filed T_α of α as

$$v_\alpha = a\sec^2(t+t_0),\ T_\alpha = (\sin t_0, 0, \cos t_0)\,,$$

and consequently the curvature κ_α of α is zero. Hence α cannot be a rectifying curve, as the definition of rectifying curve requires that its curvature is positive. Therefore, not all curves that are dilations of unit speed curve $y(t)$ on S^2 of the type $\alpha(t) = a\sec(t+t_0)y(t)$ are rectifying curves. This suggests finding the Frenet–Serret apparatus for the curve $\alpha(t) = a\sec(t+t_0)y(t)$, which will help us to exclude those unit speed curves on S^2, which do not allow α to be a rectifying curve (also pointed out independently in [3]).

Theorem 3.1 *Let $y(t)$ be a unit speed curve on the unit sphere S^2 centered at the origin $o \in \mathbb{E}^3$ and let $\alpha(t) = a\sec(t+t_0)y(t)$ be a dilation of α. Then we have*

$$\kappa_\alpha = \frac{1}{a}\cos^3(t+t_0)\sqrt{\kappa_y^2 - 1}, \quad \tau_\alpha = \frac{1}{a}\cos^2(t+t_0)\sin(t+t_0)\sqrt{\kappa_y^2-1},$$

$$T_\alpha = \sin(t+t_0)y + \cos(t+t_0)y',\ N_\alpha = y \times y',\ B_\alpha = \cos(t+t_0)y - \sin(t+t_0)y',$$

where κ_y is the curvature of the unit speed curve $y(t)$ and $\{\kappa_\alpha, \tau_\alpha, T_\alpha, N_\alpha, B_\alpha\}$ is the Frenet–Serret apparatus of α.

Proof Under the hypothesis of the theorem, it is easy to verify that the speed of α is given by $v_\alpha = a\sec^2(t+t_0)$, and consequently we get

$$T_\alpha = \sin(t+t_0)y + \cos(t+t_0)y'.$$

Recall that $\{y, y', y \times y'\}$ is an orthonormal frame of $\mathbb{E}^3$ along $y(s)$. Let s be the arc-length parameter for α; then we have

$$\frac{ds}{dt} = a\sec^2(t+t_0).$$

By differentiating the expression for T_α and using equation (2.4) and Frenet–Serret formulae, we find

$$a\kappa_\alpha \sec^2(t+t_0)N_\alpha = hy \times y'$$

with $h = \langle y'', y \times y' \rangle$. Hence, by using equation (2.6), we derive

$$\kappa_\alpha = \frac{1}{a}\cos^3(t+t_0)\sqrt{\kappa_y^2 - 1} \quad \text{and} \quad N_\alpha = y \times y'.$$

Now, using $B_\alpha = T_\alpha \times N_\alpha$, we get

$$B_\alpha = \cos(t+t_0)y - \sin(t+t_0)y'.$$

After differentiating the equation above and using equation (2.4) and Frenet–Serret formulae we obtain

$$-a\tau_\alpha \sec^2(t+t_0)N_\alpha = -\sin(t+t_0)hy \times y',$$

and it leads to

$$\tau_\alpha = \frac{1}{a}\cos^2(t+t_0)\sin(t+t_0)\sqrt{\kappa_y^2 - 1}.$$

This completes the proof of the theorem. □

Remark 3.1 *Since by Theorem A (cf. [4, page 80]) the curve $\alpha(t) = a\sec(t+t_0)y(t)$ is a rectifying curve that requires $\kappa_\alpha > 0$. Hence we have the following result according to Theorem 3.1.*

Corollary 3.1 *Let $y(t)$ be a unit speed curve on the unit sphere S^2 that is not an arc of the great circle; then $\alpha(t) = a\sec(t+t_0)y(t)$ is a rectifying curve.*

Since great circles need to be excluded in Theorem 3 of [2]. Consequently, Theorem 3 of [2] can be restated as follows.

Theorem A *Let $y : I \to S^2$ be a unit speed curve on the unit sphere S^2 centered at the origin $o \in \mathbb{E}^3$ and $f(t)$ a positive differentiable function defined on open interval I. Then $\alpha(t) = f(t)y(t)$ is a rectifying curve if and only if $f(t) = a\sec(t+t_0)$ for some constants $a > 0$ and t_0 such that there exists no subinterval $J \subset I$ with $y(J) \subset S^2 \cap \mathbb{E}^2$ for any 2-plane $\mathbb{E}^2$ containing the origin of $\mathbb{E}^3$.*

4. Centrodes of unit speed curves

Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$ and let $d : I \to \mathbb{E}^3$ be the centrode of α defined by

$$d(t) = \tau(t)T(t) + \kappa(t)B(t). \tag{4.1}$$

In view of Frenet–Serret formulae, equation (4.1) yields

$$d' = \tau' T + \kappa' B.$$

Consequently, the speed v_d of centrode d is given by

$$v_d = \sqrt{\tau'^2 + \kappa'^2}. \tag{4.2}$$

Thus the unit tangent vector field T_d of the centrode is given by

$$T_d = \frac{\tau'}{v_d}T + \frac{\kappa'}{v_d}B. \tag{4.3}$$

Let s be the arc-length parameter and let κ_d denote the curvature of the centrode. Then, by differentiating equation (4.3), we find

$$\kappa_d v_d N_d = \left(\frac{\tau'}{v_d}\right)' T + \left(\frac{\tau'\kappa - \kappa'\tau}{v_d}\right) N + \left(\frac{\kappa'}{v_d}\right)' B. \tag{4.4}$$

We give the following easy result, which provides a simple characterization of helices in terms of centrode.

Proposition 4.1 *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve whose curvature κ and torsion τ satisfy $\kappa > 0$ and $\tau'^2 + \kappa'^2 \neq 0$. Then α is a helix if and only if its centrode is a line segment.*

Proof If the centrode of α is a line segment, then equation (4.4) gives $\tau'\kappa = \kappa'\tau$ and this shows that τ/κ is a constant. Hence α is a helix.

Conversely, if α is a helix, then we have $\tau = ck$ for some constant $c \neq 0$, and thus equation (4.4) gives $k_d = 0$. Hence α is a line segment. □

Next, we prove the following result, which can be regarded as a generalization of [4, Theorems 1 and 2].

Theorem 4.1 *We have:*

(a) Let $\alpha = \alpha(t)$ be a unit speed curve in $\mathbb{E}^3$ whose curvature κ and torsion τ satisfy $\kappa, \tau \neq 0$ and $\tau'^2 + \kappa'^2 \neq 0$. If α is not a helix, then the centrode $d = \tau T + \kappa B$ of α is a rectifying curve if and only if κ and τ satisfy a nonhomogeneous linear equation $a\kappa - b\tau = c$, so that a, b, c are constants with $a^2 + b^2 \neq 0$ and $c \neq 0$.

(b) If $\kappa' \neq 0$ and if the centrode $d(t)$ of $\alpha(t)$ is a rectifying curve, then the Frenet–Serret apparatus $\{\kappa_d, \tau_d, T_d, N_d, B_d\}$ of the centrode satisfies

$$\kappa_d = \frac{\hat{c}\kappa - \tau}{\kappa'(1+\hat{c}^2)}, \quad \tau_d = \frac{\kappa + \hat{c}\tau}{\kappa'(1+\hat{c}^2)}, \quad T_d = \frac{\hat{c}}{\sqrt{1+\hat{c}^2}}T + \frac{1}{\sqrt{1+\hat{c}^2}}B$$

$$N_d = N, \quad B_d = -\frac{1}{\sqrt{1+\hat{c}^2}}T + \frac{\hat{c}}{\sqrt{1+\hat{c}^2}}B,$$

where $\hat{c} = a/b$ and a, b are defined as in statement (a).

(c) If $\tau' \neq 0$ and if the centrode $d(t)$ is a rectifying curve, then the Frenet–Serret apparatus $\{\kappa_d, \tau_d, T_d, N_d, B_d\}$ of the centrode satisfies

$$\kappa_d = \frac{\kappa - \bar{c}\tau}{\tau'(1+\bar{c}^2)}, \quad \tau_d = \frac{\bar{c}\kappa + \tau}{\tau'(1+\bar{c}^2)}, \quad T_d = \frac{1}{\sqrt{1+\bar{c}^2}}T + \frac{\bar{c}}{\sqrt{1+\bar{c}^2}}B$$

$$N_d = N, \quad B_d = -\frac{\bar{c}}{\sqrt{1+\bar{c}^2}}T + \frac{1}{\sqrt{1+\bar{c}^2}}B,$$

where $\bar{c} = b/a$ and a, b are defined as in statement (a).

Note that $\{\kappa, \tau, T, N, B\}$ *in statements* (b) *and* (c) *denotes the Frenet–Serret apparatus of the unit speed curve* α.

Proof Let $\alpha = \alpha(t)$ be a unit speed curve in $\mathbb{E}^3$ whose curvature κ and torsion τ satisfy $\kappa, \tau \neq 0$ and $\tau'^2 + \kappa'^2 \neq 0$. Assume that α is not a helix.

Using equations (4.2) and (4.4), we find

$$\kappa_d \sqrt{\tau'^2 + \kappa'^2}\, \langle N_d(t), d(t) \rangle = \tau \left(\frac{\tau'}{\sqrt{\tau'^2 + \kappa'^2}} \right)' + \kappa \left(\frac{\kappa'}{\sqrt{\tau'^2 + \kappa'^2}} \right)'. \tag{4.5}$$

Since α is not a helix, τ/κ is nonconstant. Thus, it follows from Proposition 4.1 that we have $\kappa_d > 0$ and as $\sqrt{\tau'^2 + \kappa'^2} \neq 0$. Hence equation (4.5) implies that

$$\tau \left(\frac{\tau'}{\sqrt{\tau'^2 + \kappa'^2}} \right)' + \kappa \left(\frac{\kappa'}{\sqrt{\tau'^2 + \kappa'^2}} \right)' = 0 \tag{4.6}$$

holds identically if and only if $\langle N_d(t), d(t) \rangle = 0$ holds identically. The latter condition means that the position vector field of the centrode always lies in the rectifying plane of the centrode. Consequently, the centrode $d(t)$ of $\alpha(t)$ is a rectifying curve if and only if equation (4.6) holds.

Now we shall solve differential equation (4.6). Since $\tau'^2 + \kappa'^2 \neq 0$, we have either $\kappa' \neq 0$ or $\tau' \neq 0$.

Case (1): $\kappa' \neq 0$. In this case we define a function g_1 by

$$g_1(t) = \tan^{-1}\left(\frac{\tau'}{\kappa'} \right). \tag{4.7}$$

From (4.7) we get

$$\sin g_1(t) = \frac{\tau'}{\sqrt{\tau'^2 + \kappa'^2}} \quad \text{and} \quad \cos g_1(t) = \frac{\kappa'}{\sqrt{\tau'^2 + \kappa'^2}}. \tag{4.8}$$

It is direct to verify that equation (4.6) is equivalent to

$$(\tau \cos g_1(t) - \kappa \sin g_1(t))\, g_1'(t) = 0. \tag{4.9}$$

If $\tau \cos g_1(t) - \kappa \sin g_1(t) = 0$ holds, then

$$\frac{\tau}{\kappa} = \tan g_1(t) = \frac{\tau'}{\kappa'},$$

which implies that τ/κ is a constant. However, this is impossible since the curve α is not a helix. Therefore, we obtain $g_1'(t) = 0$ from (4.9), and thus $g_1(t)$ is a constant. Consequently, $\tau' = c_1 \kappa'$ for some constant c_1. If we put $c_1 = a/b$ for constants a, b, then we obtain $a\kappa - b\tau = c$ for some constant c. Since τ/κ is nonconstant, we must have $c \neq 0$ and hence $a^2 + b^2 \neq 0$.

Case (2): $\tau' \neq 0$. In this case we define a function $g_2(t)$ by

$$g_2(t) = \tan^{-1}\left(\frac{\kappa'}{\tau'} \right). \tag{4.10}$$

From (4.10) we get

$$\sin g_2(t) = \frac{\kappa'}{\sqrt{\tau'^2 + \kappa'^2}} \quad \text{and} \quad \cos g_2(t) = \frac{\tau'}{\sqrt{\tau'^2 + \kappa'^2}}. \tag{4.11}$$

Similarly, we know that equation (4.6) is equivalent to

$$(\tau \cos g_2(t) - \kappa \sin g_2(t))\, g_2'(t) = 0. \tag{4.12}$$

Now, by applying a similar argument as Case (1), we obtain $a\kappa - b\tau = c$ for some constants a, b, c with $a^2 + b^2 \neq 0$ and $c \neq 0$.

Conversely, it is easy to verify that if κ and τ satisfy $a\kappa - b\tau = c$ for some constants a, b, c satisfying $a^2 + b^2 \neq 0$ and $c \neq 0$, then κ and τ satisfy the differential equation (4.6). This proves statement (a).

To prove statement (b) and statement (c), let us assume that the centrode $d(t)$ is a rectifying curve. Then statement (a) implies that the curvature and torsion of $\alpha(t)$ satisfy a nonhomogeneous linear equation

$$a\kappa - b\tau = c \tag{4.13}$$

for some constants a, b, c satisfying $a^2 + b^2 \neq 0$ and $c \neq 0$. In particular, we have either (i) $b \neq 0$ or (ii) $a \neq 0$.

Case (i): $b \neq 0$. We find from (4.13) that

$$\tau' = \hat{c}\kappa', \quad \hat{c} = \frac{a}{b}. \tag{4.14}$$

Therefore we find from (4.2) and (4.3) that

$$v_d = \kappa' \sqrt{1 + \hat{c}^2}, \quad T_d = \frac{\hat{c}}{\sqrt{1 + \hat{c}^2}} T + \frac{1}{\sqrt{1 + \hat{c}^2}} B. \tag{4.15}$$

Consequently, equation (4.4) reduces to

$$\kappa_d \kappa' \sqrt{1 + \hat{c}^2} N_d = \frac{\hat{c}\kappa - \tau}{\sqrt{1 + \hat{c}^2}} N,$$

which gives

$$\kappa_d = \frac{\hat{c}\kappa - \tau}{\kappa'(1 + \hat{c}^2)} \quad \text{and} \quad N_d = N. \tag{4.16}$$

By using equations (4.15), (4.16) and $B_d = T_d \times N_d$, we get

$$B_d = -\frac{1}{\sqrt{1 + c^2}} T + \frac{c}{\sqrt{1 + c^2}} B,$$

which on differentiation and using Frenet–Serret formulae gives

$$-\tau_d \kappa_d' \sqrt{1 + \hat{c}^2} N_d = -\frac{\hat{c}\tau + \kappa}{\sqrt{1 + \hat{c}^2}} N.$$

This proves statement (b).

Case (ii): $a \neq 0$. From (4.13) we have $\kappa' = \bar{c}\tau'$, $\bar{c} = b/a$. Thus we may apply a method similar to Case (i) to obtain statement (c). □

Remark 4.1 *The condition $a\kappa - b\tau = c$ with $a^2 + b^2 \neq 0$ and $c \neq 0$ given in Theorem 4.1(a) has been used by Lucas and Ortega-Yagües in [9] for their study of Betrand curves in the Euclidean 3-space $\mathbb{E}^3$ or Lorentz–Minkowski 3-space $\mathbb{L}^3$.*

Remark 4.2 *Let $\alpha(t)$ be the unit speed curve of Theorem 4.1 with $\kappa' \neq 0$ such that the centrode $d(t)$ of α is a rectifying curve. Then as $\hat{c}\kappa' = \tau'$, we get $\hat{c}\kappa = \tau + c_1$ for a constant $c_1 \neq 0$ (as τ/κ is nonconstant), that is, $\hat{c}\kappa - \tau = c_1$. Moreover, the arc-length parameter s of the rectifying curve $d(t)$ satisfies $\frac{ds}{dt} = \kappa'\sqrt{1+\hat{c}^2}$, which gives*

$$s = \kappa\sqrt{1+\hat{c}^2} + b$$

for a constant b. Thus, after using the expressions of curvature and torsion of $d(t)$, we obtain

$$\begin{aligned}
\frac{\tau_d}{\kappa_d} &= \frac{\hat{c}\tau + \kappa}{\hat{c}\kappa - \tau} = \frac{\hat{c}\tau + \kappa}{c_1} = \frac{\hat{c}}{c_1}(\hat{c}\kappa - c_1) + \frac{\kappa}{c_1} \\
&= \left(\frac{\hat{c}^2 + 1}{c_1}\right)\kappa - \hat{c} = \frac{\hat{c}^2 + 1}{c_1}\left(\frac{s - b}{\sqrt{1+\hat{c}^2}}\right) - \hat{c} \\
&= As + B,
\end{aligned}$$

where $A \neq 0$, B are constants. Thus, the ratio τ_d/κ_d is a linear function of the arc-length s as required by a rectifying curve (cf. [2, Theorem 2]).

Also observe that

$$\begin{aligned}
\tau^2 + \kappa^2 &= (\hat{c}\kappa - c_1)^2 + \kappa^2 = (1+\hat{c}^2)\kappa^2 - 2\hat{c}c_1\kappa + c_1^2 \\
&= (1+\hat{c}^2)\frac{(s-b)^2}{(1+\hat{c}^2)} - 2\hat{c}c_1\frac{s-b}{\sqrt{1+\hat{c}^2}} + c_1^2 \\
&= s^2 + \lambda_1 s + \lambda_2,
\end{aligned}$$

where λ_1, λ_2 are constants. Therefore the distance function $f(s) = ||d(s)||$ of the rectifying curve d satisfies $f(s) = \sqrt{s^2 + \lambda_1 s + \lambda_2}$, as required by a rectifying curve (cf. [2, Theorem 1]).

Note that similar arguments hold for a unit speed curve $\alpha(t)$ with $\tau' \neq 0$ (instead of $\kappa' \neq 0$) and with a rectifying centrode.

Finally, in this section we present the following corollaries of Theorem 4.1, which could be used to construct ample examples of rectifying curves.

Corollary 4.1 *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve whose curvature κ and torsion τ satisfy $\kappa' \neq 0$, τ/κ nonconstant, and $\tau' = \hat{c}\kappa'$, $\hat{c}$ being a constant. Then the centrode $d(t)$ of α is a rectifying curve with curvature $\kappa_d = \pm a/\kappa'$ and torsion $\tau_d = (b_1 s + c_1)/\kappa'$ for some constants $a \neq 0$, $b_1 \neq 0$, and c_1.*

Proof Under the hypothesis of the theorem, it follows from statement (a) of Theorem 4.1 that the centrode $d(t)$ of α is a rectifying curve satisfying

$$\frac{\tau_d}{\kappa_d} = \frac{\hat{c}\tau + \kappa}{\hat{c}\kappa - \tau},$$

which implies

$$1+\left(\frac{\tau_d}{\kappa_d}\right)^2=\frac{1+\hat{c}^2}{(\hat{c}\kappa-\tau)^2}\left(\tau^2+\kappa^2\right). \tag{4.17}$$

Now, as $\tau'=\hat{c}\kappa'$, we have $\hat{c}\kappa-\tau=c_0$, where c_0 is nonzero constant (as τ/κ is nonconstant), and equation (4.17) leads to

$$\frac{\tau_d^2+\kappa_d^2}{\tau^2+\kappa^2}=\frac{1+\hat{c}^2}{c_0^2}\kappa_d^2. \tag{4.18}$$

Furthermore, the expressions for κ_d and τ_d give

$$\tau_d^2+\kappa_d^2=\frac{(1+\hat{c}^2)}{\kappa'^2(1+\hat{c}^2)^2}(\tau^2+\kappa^2),$$

that is,

$$\frac{\tau_d^2+\kappa_d^2}{\tau^2+\kappa^2}=\frac{1}{\kappa'^2(1+\hat{c}^2)}.$$

After combining this equation with equation (4.18), we find

$$\kappa_d^2=\frac{c_0^2}{\kappa'^2(1+\hat{c}^2)^2}=\frac{c_2^2}{\kappa'^2}.$$

where c_2 is a nonzero constant. Therefore we obtain the required expression for the curvature κ_d of the rectifying curve $d(t)$.

The arc-length function s of the rectifying curve is given by $s=\kappa\sqrt{1+\hat{c}^2}+c_3$ for a constant c_3 and consequently, using the expression for torsion τ_d in statement (b) of Theorem 4.1, we derive that

$$\begin{aligned}\tau_d&=\frac{\hat{c}\tau+\kappa}{\kappa'(1+\hat{c}^2)}=\frac{\hat{c}(\hat{c}\kappa-c_0)+\kappa}{\kappa'(1+\hat{c}^2)}\\&=\frac{(1+\hat{c}^2)\kappa-\hat{c}c_0}{\kappa'(1+\hat{c}^2)}=\frac{1}{\kappa'}\left[\frac{s-c_3}{\sqrt{(1+\hat{c}^2)}}-\frac{\hat{c}c_0}{(1+\hat{c}^2)}\right]\\&=\frac{c_4s+c_5}{\kappa'},\end{aligned}$$

where $c_4\neq 0$ and c_5 are constants. □

Remark 4.3 *A result similar to Corollary 4.1 holds for a unit speed curve satisfying* $\tau'\neq 0$, τ/κ *nonconstant, and* $\kappa'=\bar{c}\tau'$ *with constant* $\bar{c}$.

5. Dilated centrodes as rectifying curves

In this last section, we study the dilated centrode of a unit speed twisted curve $\alpha: I\to\mathbb{E}^3$ with $\kappa>0$ and $\tau\neq 0$. The dilated centrode of $\alpha(t)$ is defined by

$$\beta(t)=\rho(t)d(t)=\frac{\tau(t)}{\kappa(t)}T+B, \tag{5.1}$$

where $d=\tau T+\kappa B$ is the centrode and $\rho=\kappa^{-1}$ is the radius of curvature of α.

Theorem 5.1 *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with curvature $\kappa > 0$. If α is neither a planar curve nor a helix, then the dilated centrode $\beta(t) = \rho(t)d(t)$ of α is a rectifying curve.*

Proof Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve of class C^5 with curvature $\kappa > 0$. Assume that α is neither a planar curve nor a helix. Consider the dilated centrode of α defined by (5.1). By differentiating equation (5.1) and using Frenet–Serret formulae, we get

$$\beta' = (\rho\tau)'T$$

and as α is not a helix we have $(\rho\tau)' \neq 0$. Thus the dilated centrode $\beta(t)$ is a regular curve whose speed v_β and unit tangent vector field T_β are given respectively by

$$v_\beta = |(\rho\tau)'|, \quad T_\beta = \pm T. \tag{5.2}$$

Let $\{\kappa_\beta, \tau_\beta, T_\beta, N_\beta, B_\beta\}$ be the Frenet–Serret apparatus of β. Then after differentiating equation (5.2) we find

$$\kappa_\beta\,|(\rho\tau)'|\,N_\beta = \pm\kappa N,$$

that is,

$$\kappa_\beta = \frac{\kappa}{(\rho\tau)'} \quad \text{and} \quad N_\beta = N. \tag{5.3}$$

Using equations (5.1) and (5.3), we arrive at $\langle\beta(t), N_\beta\rangle = 0$, that is, the dilated centrode β is a rectifying curve. This proves the theorem. □

Note that if follows from (5.2) and (5.3) that $B_\beta = \pm B$ and that

$$-\tau_\beta\,|(\rho\tau)'|\,N_\beta = \mp\tau N.$$

Therefore we also have

$$\tau_\beta = \frac{\tau}{(\rho\tau)'}. \tag{5.4}$$

Remark 5.1 *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$, $\kappa > 0$, $\tau \neq 0$, and $(\rho\tau)' \neq 0$. Then, according to Theorem 5.1, the curve $\beta(t) = \rho\tau T + B$ is a rectifying curve whose Frenet–Serret apparatus $\{\kappa_\beta, \tau_\beta, T_\beta, N_\beta, B_\beta\}$ is given by equations (5.2) and (5.3). If s is the arc-length of β, then by equation (5.2) we have $s = \pm\rho\tau + c$, for a constant c. The distance function $f(s) = \|\beta(s)\|$ in view of equation (5.1) is given by*

$$f(s) = \sqrt{1 + \rho^2\tau^2} = \sqrt{1 + (s - c)^2} = \sqrt{s^2 + c_1 s + c_2},$$

where $c_1 = -2c$, $c_2 = 1 + c^2$. Hence, the distance function $f(s)$ of the rectifying curve β has the form described in [2].

Similarly, using equations (5.3) and (5.4), we find

$$\frac{\tau_\beta}{\kappa_\beta} = \frac{\tau}{\kappa} = \mp(s - c) = as + b,$$

where $a \neq 0$, b are constants. Therefore the ratio of torsion and curvature of the rectifying curve β is the linear function of the arc-length (cf. e.g. [2]).

Remark 5.2 *Theorem 5.1 provides us a tool to construct many examples of rectifying curves. Indeed, if we select a unit speed curve* $\alpha : I \to \mathbb{E}^3$ *with* $\kappa > 0$ *that is neither a planar curve nor a helix, then the dilated centrode* $\rho(t)d(t)$ *of* $\alpha(t)$ *is a rectifying curve. For example, if we consider the curve* $\alpha : I \to \mathbb{E}^3$ *defined by*

$$\alpha(t) = \left(\cos t, \sin t, \frac{t^2}{2}\right),$$

then it has speed, unit tangent vector field, and curvature given respectively by

$$v_\alpha = \sqrt{1+t^2}, \quad T_\alpha = \frac{1}{\sqrt{1+t^2}}\left(-\sin t, \cos t, t\right), \quad \kappa_\alpha = \frac{\sqrt{2+t^2}}{(1+t^2)^{\frac{3}{2}}},$$

and with principal normal vector field

$$N_\alpha = \frac{1}{\sqrt{(1+t^2)(2+t^2)}}\left(t\sin t - (1+t^2)\cos t, -(1+t^2)\sin t - t\cos t, 1\right).$$

The binormal vector field and torsion of α *are*

$$B_\alpha = \frac{1}{\sqrt{2+t^2}}\left(\cos t + t\sin t, \sin t - t\cos t, 1\right), \quad \tau_\alpha = \frac{t}{2+t^2}.$$

Clearly, we have $\kappa_\alpha > 0$ *and that* α *is neither a planar curve nor a helix. Hence, by Theorem 5.1, the dilated centrode*

$$\beta(t) = \frac{1}{k_\alpha} d(t) = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right) T_\alpha + B_\alpha$$

of α *is a rectifying curve.*

Similarly, by taking unit speed curve $y(t)$ on the unit sphere S^2 with curvature κ_y and torsion $\tau_y \neq 0$ such that τ_y/κ_y is nonconstant, we could also show that the dilated centrode $\kappa_y^{-1} d_y(t)$ of $y(t)$ is a rectifying curve.

Remark 5.3 *The authors thank one of the referees, who points out to the authors that the dilated centrode of a curve* α *has been studied in [7] under the name of modified Darboux vector field. Moreover, rectifying curves are named conical geodesic curves in [7].*

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